Online Appendix

B.1. Proof of Proposition 4

B.1.1. Part i. We begin by establishing the preliminary result that if an equilibrium (p_e, λ_e) exists then $\frac{\Lambda}{\mu} \geq .75$, $v \geq 4\frac{w_s}{\mu}$ $\gamma \geq \frac{w_s}{v}$ implies that $\frac{\lambda_e}{\mu} \geq .75$ and $\gamma \geq \mu - \lambda_e$. Consider the case where customers only choose between balking or joining and being inflexible, then $\underline{\lambda} := \min\{\mu - \frac{w_s}{v}, \Lambda\}$ is the equilibrium arrival rate, see (Hassin and Haviv 2003, Chapter 3, Section 1.1) for the details. Given this, a little algebra shows that, in this case, if $\frac{\Lambda}{\mu} \geq .75$, then $v \geq 4\frac{w_s}{\mu}$ implies $\underline{\lambda} \geq .75\mu$ and $\gamma \geq \frac{w_s}{v}$ implies $\underline{\lambda} \geq \mu - \gamma$. Therefore, to establish this preliminary result, we must show that no fewer customers will join the system when customers can also join and be flexible in addition to the options to balk or join and be inflexible.

In the case when joining and being flexible is dominated, that is, $p_e = 0$, then $\lambda_e = \underline{\lambda}$ and $v - ws\bar{T}_{ss}(0, \underline{\lambda}) \geq 0$ where the inequality holds strictly only in cases where $\underline{\lambda} = \Lambda$. In the case when joining and being flexible is not dominated, that is, $p_e > 0$, we show $\lambda_e > \underline{\lambda}$ by contradiction. Assume by contradiction that for some $\lambda_e < \underline{\lambda}$ and $p_e \in (0, 1]$ is an equilibrium. Then by $\lambda < \underline{\lambda}$ we have that $v - w_s\bar{T}_{ss}(0, \lambda) > v - w_s\bar{T}_{ss}(0, \underline{\lambda}) \geq 0$ and by Proposition 3, specifically that $\bar{T}_{ss}$ is decreasing in p, we have that $v - w_s\bar{T}_{ss}(p, \lambda) > v - w_s\bar{T}_{ss}(0, \lambda) \geq 0$. Therefore, relative to the option to balk, the option to join and be inflexible has strictly higher utility than in the case when no one is flexible and customers join at rate $\underline{\lambda}$. This cannot be an equilibrium as customers have incentive to deviate, thus we have the contradiction we seek and it cannot be that strictly fewer customers join in equilibrium when customers are given the option to join and be flexible. Going forward we have that, for any potential equilibrium arrival rate λ_e, it is such that $\lambda_e \geq .75\mu$ and $\lambda_e \geq \mu - \gamma$.

There are six possible types of equilibrium strategies which are the combinations of $\lambda_e < \Lambda$ or $\lambda_e = \Lambda$ with $p_e = 0$ or $0 < p_e < 1$ or $p_e = 1$.

Case 1: $p_e = 0$ and $\lambda_e = \Lambda$. For this to be an equilibrium, it must be that $\Lambda < \mu$ so that the system is stable if all customers join, $v \geq w_s\bar{T}_{ss}(0, \Lambda)$ so that customers have no incentive to balk, and $h + w_r\bar{T}_{rr}(0, \Lambda) + w_s\bar{T}_{rs}(0, \Lambda) \geq w_s\bar{T}_{ss}(0, \Lambda)$ so that customers have no incentive to be flexible. These conditions are respectively equivalent to $\Lambda < \mu$, $v \geq \hat{v}_0 := \frac{w_s}{\mu - \Lambda}$ and $h \geq \hat{h}_\Lambda := \left(\frac{w_s}{\mu} - \frac{w_r}{\mu - \Lambda}\right)\frac{\Lambda^2}{\gamma\mu}(-\ln\frac{\Lambda}{\mu})$, which provides the region where $p_e = 0$ and $\lambda_e = \Lambda$ is an equilibrium strategy.

Case 2: $p_e = 0$ and $\lambda_e = \lambda_0 := \mu - \frac{w_s}{v} < \Lambda$. Where λ_0 is such that $0 = v - w_s\bar{T}_{ss}(0, \lambda_0)$ so that customers are indifferent between balking, and joining and being inflexible. In terms of model primitives, this indifference can be expressed as $v = \frac{w_s}{\mu - \lambda_0}$. For this to be an equilibrium, it must be either that $\Lambda \geq \mu$ or $v < w_s\bar{T}_{ss}(0, \Lambda)$ so that if all customers joined there would be incentive for some to balk, and $h + w_r\bar{T}_{rr}(0, \lambda_0) + w_s\bar{T}_{rs}(0, \lambda_0) \geq w_s\bar{T}_{ss}(0, \lambda_0)$ so that customers have no incentive to be flexible. These conditions are respectively equivalent to either $\Lambda \geq \mu$ or $v < \hat{v}_0$ and $h \geq \hat{h}_{\lambda_0} := \left(\frac{w_s}{\mu} - \frac{w_r}{\mu - \lambda_0}\right)\frac{\lambda_0^2}{\gamma\mu}(-\ln\frac{\lambda_0}{\mu})$, which provides the region where $p_e = 0$ and $\lambda_e = \lambda_0$ is an equilibrium strategy.

Case 3: $p_e = 1$ and $\lambda_e = \Lambda$. For this to be an equilibrium, it must be that $\Lambda < \mu$ so that the system is stable if all customers join, $h + w_r\bar{T}_{rr}(1, \Lambda) + w_s\bar{T}_{rs}(1, \Lambda) \leq v$ so that customers have no incentive to balk when all customers join and are flexible, and $w_s\bar{T}_{ss}(1, \Lambda) \geq h + w_r\bar{T}_{rr}(1, \Lambda) + w_s\bar{T}_{rs}(1, \Lambda)$ so that customers have no incentive to be inflexible. These conditions are respectively equivalent to $\Lambda < \mu$, $v \geq \hat{v}_1 := \frac{w_s}{\mu - \Lambda} + h - \frac{w_s - w_r}{\mu - \Lambda}\frac{\Lambda}{\mu}\left(1 - (\Lambda/\mu)^{\Lambda/\gamma}\right)$ and $h \leq \check{h}_\Lambda := \left(\frac{w_s}{\mu} - \frac{w_r}{\mu - \Lambda}\right)\frac{\Lambda}{\mu}\left(1 - (\Lambda/\mu)^{\Lambda/\gamma}\right)$, which provides the region where $p_e = 1$ and $\lambda_e = \Lambda$ is an equilibrium strategy.

Case 4: $p_e = 1$ and $\lambda_e = \lambda_1 < \Lambda$. Where λ_1 (see below for proof of existence and uniqueness) is such that $0 = v - h + w_r\bar{T}_{rr}(1, \lambda_1) + w_s\bar{T}_{rs}(1, \lambda_1)$ so that customers are indifferent between balking, and joining and being flexible. In terms of model primitives, this indifference can be expressed as $v = \frac{w_s}{\mu - \lambda_1} + h - \frac{w_s - w_r}{\mu - \lambda_1}\frac{\lambda_1}{\mu}\left(1 - (\lambda_1/\mu)^{\lambda_1/\gamma}\right)$. For this to be an equilibrium, it must be either that $\Lambda \geq \mu$ or $v < h + w_r\bar{T}_{rr}(1, \Lambda) + w_s\bar{T}_{rs}(1, \Lambda)$, so that if all customers join and are flexible there would be incentive for some to balk, and $h + w_r\bar{T}_{rr}(1, \lambda_1) + w_s\bar{T}_{rs}(1, \lambda_1) \leq w_s\bar{T}_{ss}(1, \lambda_1)$ so that customers have no incentive to be inflexible. These conditions are respectively equivalent to either $\Lambda \geq \mu$ or $v < \hat{v}_1$ and $h \leq \check{h}_{\lambda_1} := \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda_1}\right)\frac{\lambda_1}{\mu}\left(1 - (\lambda_1/\mu)^{\lambda_1/\gamma}\right)$, which provides the region where $p_e = 1$ and $\lambda_e = \lambda_1$ is an equilibrium strategy.

Existence and uniqueness of λ_1. To see that λ_1 exists and is unique, note that for any $\lambda < \mu$,

$$\frac{d}{d\lambda}\left[\frac{w_s}{\mu-\lambda} + h - \frac{w_s - w_r}{\mu-\lambda}\frac{\lambda}{\mu}\left(1 - \left(\frac{\lambda}{\mu}\right)^{\frac{\lambda}{\gamma}}\right)\right] =$$
$$\frac{w_s}{(\mu-\lambda)^2} - (w_s - w_r)\left(\frac{1}{(\mu-\lambda)^2}\left(1 - \left(\frac{\lambda}{\mu}\right)^{\frac{\lambda}{\gamma}}\right) - \frac{\lambda}{\mu(\mu-\lambda)}\left(\left(\frac{\lambda}{\mu}\right)^{\frac{\lambda}{\gamma}}\frac{1}{\gamma}\left(1 + \ln\frac{\lambda}{\mu}\right)\right)\right),$$

and that $\lambda/\mu > e^{-1} \approx .368$ is sufficient for this to be positive, which we have by assumption. Therefore, if $v \leq \check{v}_1$, then λ_1 must exist, and the monotonicity ensures that λ_1 is unique when it exists.

Case 5: $0 < p_e = \tilde{p} < 1$ and $\lambda_e = \Lambda$. Where $\tilde{p}$ (see below for proof of existence and uniqueness) is such that $v - w_s\bar{T}_{ss}(p,\Lambda) = v - h + w_r\bar{T}_{rr}(p,\Lambda) + w_s\bar{T}_{rs}(p,\Lambda)$ so that customers are indifferent between joining and choosing to be inflexible vs being flexible when all customers join. In terms of model primitives, this indifference can be expressed as $h = \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\Lambda}\right)\frac{\Lambda}{\tilde{p}\mu}\left(1 - (\Lambda/\mu)^{\tilde{p}\Lambda/\gamma}\right)$. For this to be an equilibrium, it must be that $\Lambda < \mu$ and $v \geq w_s\bar{T}_{ss}(\tilde{p},\Lambda)$ so that customers have no incentive to balk when everyone joins and a proportion $\tilde{p} \in (0,1)$ are flexible, and that $w_s\bar{T}_{ss}(p,\Lambda) = h + w_r\bar{T}_{rr}(p,\Lambda) + w_s\bar{T}_{rs}(p,\Lambda)$ so customers are indifferent between choosing to be flexible and inflexible. The region where $p_e = \tilde{p}$ and $\lambda_e = \Lambda$ is an equilibrium strategy, is when $\Lambda < \mu$, $v \geq \hat{v}_p := \frac{w_s}{\mu-\Lambda}\left(1 - (\Lambda/\mu)^2\left(1 - (\Lambda/\mu)^{\tilde{p}\Lambda/\gamma}\right)\right)$, and $\check{h}_\Lambda < h < \hat{h}_\Lambda$, where this last pair of inequalities is derived in the proof of existence and uniqueness below.

Existence and uniqueness of $\tilde{p}$. Such a $\tilde{p}$ will exist (and be unique) if $h \in [\hat{h}_\Lambda, \check{h}_\Lambda]$ because $\left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\Lambda}\right)\frac{\Lambda}{p\mu}\left(1 - (\Lambda/\mu)^{p\Lambda/\gamma}\right)$ is decreasing in p (see B.1.5), from $\hat{h}_\Lambda$ (when $p = 0$) to $\check{h}_\Lambda$ (when $p = 1$).

Case 6: $0 < p_e = \tilde{p} < 1$ and $\lambda_e = \tilde{\lambda} < \Lambda$. Where $(\tilde{p}, \tilde{\lambda})$ (see below for proof of existence and uniqueness) are the (p,λ) such that $0 = v - w_s\bar{T}_{ss}(p,\lambda)$ and $0 = v - (h + w_r\bar{T}_{rr}(p,\lambda) + w_s\bar{T}_{rs}(p,\lambda))$ so that customers are indifferent between balking, joining and being flexible, and joining and being inflexible. In terms of model primitives, this can be expressed as:

$$v = \frac{w_s}{\mu-\tilde{\lambda}}\left(1 - \left(\frac{\tilde{\lambda}}{\mu}\right)^2\left(1 - \left(\frac{\tilde{\lambda}}{\mu}\right)^{\frac{\tilde{p}\tilde{\lambda}}{\gamma}}\right)\right), \tag{1}$$

$$h = \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\tilde{\lambda}}\right)\frac{\tilde{\lambda}}{\tilde{p}\mu}\left(1 - \left(\frac{\tilde{\lambda}}{\mu}\right)^{\frac{p\tilde{\lambda}}{\gamma}}\right). \tag{2}$$

The region where $p_e = \tilde{p}$ and $\lambda_e = \tilde{\lambda}$ is an equilibrium strategy, is when either $\Lambda > \mu$ or $v < \hat{v}_p$, and $\check{h}_{\lambda_1} < h < \hat{h}_{\lambda_0}$, where these inequalities are derived in the proof of existence and uniqueness below.

Existence and uniqueness of $(\tilde{p}, \tilde{\lambda})$. To see that this system of equations has, at most, one solution when $\lambda/\mu \geq .75$ and $\gamma \geq (\mu - \lambda)$, assume by contradiction that two such solutions exist (p,λ) and (p',λ') such that $p \neq \tilde{p}$ and $\lambda \neq \tilde{\lambda}$; then,

$$\frac{w_s}{\mu-\lambda}\left(1 - \left(\frac{\lambda}{\mu}\right)^2\left(1 - \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right) = \frac{w_s}{\mu-\lambda'}\left(1 - \left(\frac{\lambda'}{\mu}\right)^2\left(1 - \left(\frac{\lambda'}{\mu}\right)^{\frac{p'\lambda'}{\gamma}}\right)\right), \tag{3}$$

$$\left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda}\right)\frac{\lambda}{p\mu}\left(1 - \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right) = \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda'}\right)\frac{\lambda'}{p'\mu}\left(1 - \left(\frac{\lambda'}{\mu}\right)^{\frac{p'\lambda'}{\gamma}}\right). \tag{4}$$

Without loss of generality, assume $\lambda < \lambda'$. Now we will come to a contradiction that (3) implies that $p' > p$ while (4) implies that $p' < p$. To see that (3) implies that $p' > p$, observe that the RHS of (1) is increasing in λ and decreasing in p (see B.1.2 and B.1.3). Therefore, since $\lambda < \lambda'$, for (3) to hold it must be that $p' > p$. To see that (4) implies that $p' < p$, observe that when $\frac{\lambda}{\mu} > .75$ and $\gamma > \mu - \lambda$ the RHS of equation (2) is decreasing in λ and decreasing in p (see B.1.4 and B.1.5). Therefore, since $\lambda < \lambda'$, for (4) to hold, it must be that $p' < p$. Hence we have contradiction, and therefore, at most one solution exists.

To see that the inequalities $v < \hat{v}_p$ and $\check{h}_{\lambda_1} < h < \hat{h}_{\lambda_0}$ implies that a solution exists, observe that $v < \hat{v}_p$ implies that, if everyone joins and customers are indifferent between choosing to be flexible and inflexible, then customers have incentive to balk. Since the RHS of (1) is increasing in λ, for any $p \in (0,1)$ there exists a unique $\lambda_p < \Lambda$ such that $v = \frac{w_s}{\mu-\lambda_p}\left(1 - (\lambda_p/\mu)^2\left(1 - (\lambda_p/\mu)^{p\lambda_p/\gamma}\right)\right)$. Note, this implies λ_p is increasing in p because the RHS is decreasing in p. Lastly, observing that $\left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda_p}\right)\frac{\lambda_p}{p\mu}\left(1 - (\lambda_p/\mu)^{p\lambda_p/\gamma}\right)$ is decreasing in both p and λ_p from $\hat{h}_{\lambda_0}$ (when $p = 0$ and $\lambda = \lambda_0$) to $\check{h}_{\lambda_1}$ (when $p = 1$ and $\lambda = \lambda_1$), the fact that $\check{h}_{\lambda_1} < h < \hat{h}_{\lambda_0}$ ensures the existence of a unique solution for the double $(\tilde{p}, \tilde{\lambda})$.

Cases 1 − 6 are mutually exclusive and collectively exhaustive. By the fact that the regions given in cases 1 − 6 are collectively exhaustive in the space of v and h, it must be that at least one case applies. By the fact that the regions given in cases 1 − 6 are also mutually exclusive, it must be that one, and only one, case applies. Therefore, by the fact that each case has a unique potential equilibrium, and that one, and only one case applies, we have that the equilibrium exists and is unique.

The supporting monotonicity results for Parts i and ii.

B.1.2. Monotonic non-decreasing behavior of the RHS of (1) in λ

$$\frac{d}{d\lambda}\left[\frac{w_s}{\mu-\lambda}\left(1-\left(\frac{\lambda}{\mu}\right)^2\left(1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right)\right] = w_s\frac{d}{d\lambda}\left[\frac{\mu+\lambda}{\mu^2} + \frac{1}{\mu-\lambda}\left(\frac{\lambda}{\mu}\right)^2\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right]$$
$$= w_s\left[\frac{1}{\mu^2} + \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\left(\frac{\lambda(2\mu-\lambda)}{\mu^2(\mu-\lambda)^2} + \frac{\lambda^2}{\mu^2(\mu-\lambda)}\left[\frac{p}{\gamma}\left(1+\ln\frac{\lambda}{\mu}\right)\right]\right)\right]$$

A sufficient condition for this to be positive is $1 + \ln\frac{\lambda}{\mu} > 0$ or $\frac{\lambda}{\mu} > e^{-1} \approx .368$, which we have by assumption.

B.1.3. Monotonic non-increasing behavior of the RHS of (1) in p

$$\frac{d}{dp}\left[\frac{w_s}{\mu-\lambda}\left(1-\left(\frac{\lambda}{\mu}\right)^2\left(1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right)\right] = \frac{w_s}{\mu-\lambda}\left(\frac{\lambda}{\mu}\right)^2\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\frac{\lambda}{\gamma}\left(\ln\frac{\lambda}{\mu}\right) \tag{5}$$

Which is negative because $\left(\ln\frac{\lambda}{\mu}\right) < 0$.

B.1.4. Monotonic non-increasing behavior of the RHS of (2) in λ

$$\frac{d}{d\lambda}\left[\left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda}\right)\frac{\lambda}{p\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right] =$$
$$-\frac{w_r}{(\mu-\lambda)^2}\frac{\lambda}{p\mu}\left(1-\rho^{\frac{p\lambda}{\gamma}}\right) + \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda}\right)\frac{1}{p\mu}\left[1-\rho^{\frac{p\lambda}{\gamma}} - \frac{p\lambda}{\gamma}\rho^{\frac{p\lambda}{\gamma}}(\ln\rho+1)\right] \tag{6}$$

Note, we are only interested in the case where (2) holds, therefore we consider the case where $\left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda}\right) > 0$. The first term of the derivative is non-positive and a sufficient (but not necessary) condition for the second term to also be non-positive is that the sub-term in square brackets is non-positive. The term in square brackets is non-positive if $\frac{\gamma}{p\lambda}\left(\rho^{-\frac{p\lambda}{\gamma}}-1\right)\le \ln\rho+1$. Note $\frac{\gamma}{p\lambda}\left(\rho^{-\frac{p\lambda}{\gamma}}-1\right)$ is decreasing in γ and increasing in p, the relevant derivatives are:

$$\frac{d}{d\gamma}\left[\frac{\gamma}{p\lambda}\left(\rho^{-\frac{p\lambda}{\gamma}}-1\right)\right]=\frac{1}{p\lambda}\left[\rho^{-\frac{p\lambda}{\gamma}}\left(1+\frac{p\lambda}{\gamma}\ln\rho\right)-1\right],$$

and

$$\frac{d}{dp}\left[\frac{\gamma}{p\lambda}\left(\rho^{-\frac{p\lambda}{\gamma}}-1\right)\right]=-\frac{\gamma}{p^2\mu}\left[\rho^{-\frac{p\lambda}{\gamma}}\left(1+\frac{p\lambda}{\gamma}\ln\rho\right)-1\right].$$

To establish the sign of these derivatives, we need to establish the sign of the term in brackets. The term in brackets is non-positive if $\ln\rho^{\frac{p\lambda}{\gamma}}\le\rho^{\frac{p\lambda}{\gamma}}-1$, or equivalently $\ln x\le x-1$, which is true for all $x\in[0,1]$. Therefore, in the domain $p\le 1$ and $(\mu-\lambda)/\gamma\le 1$, the left-hand side (LHS) is maximized at $p=1$ and $\gamma=\mu-\lambda$.

Given this, we establish that $\left[1-\rho^{\frac{p\lambda}{\gamma}}-\frac{p\lambda}{\gamma}\rho^{\frac{p\lambda}{\gamma}}(\ln\rho+1)\right]$ is non-positive if $\frac{1-\rho}{\rho}\left(\rho^{-\frac{\rho}{1-\rho}}-1\right)\le$ $\ln\rho+1$, which is true for all $\rho\in(.676,1)$, yielding the result.

B.1.5. Monotonic non-increasing behavior of the RHS of (2) in p

$$\frac{d}{dp}\left[\left(\frac{w_s}{\mu}-\frac{w_r}{\mu-\lambda}\right)\frac{\lambda}{p\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right]=-\left(\frac{w_s}{\mu}-\frac{w_r}{\mu-\lambda}\right)\frac{1}{p^2}\frac{\lambda}{\mu}\left[1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}+\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\ln\left(\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right]\tag{7}$$

Note, we are only interested in the case where (2) holds, therefore we consider the case where $\left(\frac{w_s}{\mu}-\frac{w_r}{\mu-\lambda}\right)>0$. Substituting $x=(\lambda/\mu)^{p\lambda/\gamma}$ into the term in brackets, this term is positive because $1-x+x\ln x\ge 0$ for all $x\in[0,1]$. Therefore, (7) is negative.

B.1.6. Part ii. Note that in cases 1 and 3, p_e and λ_e are fixed, thus it suffices to check cases 2, 4, 5, and 6. For notational convenience, let the RHS of (1) be denoted as α, the RHS of (2) be denoted as β, and the right-hand side of $v=\left[\frac{w_s}{\mu-\lambda_e}+h-\frac{w_s-w_r}{\mu-\lambda_e}\frac{\lambda_e}{\mu}\left(1-\left(\frac{\lambda_e}{\mu}\right)^{\frac{\lambda_e}{\gamma}}\right)\right]$ be denoted as δ. Note that $\frac{\partial\alpha}{\partial\lambda},\frac{\partial\delta}{\partial\lambda}$ are positive, by B.1.2 and the analysis in case 4 used to show λ_1 is unique, respectively. Also, $\frac{\partial\alpha}{\partial p},\frac{\partial\beta}{\partial\lambda},\frac{\partial\beta}{\partial p}$ are all negative by B.1.3,B.1.4 and B.1.5, respectively.

B.1.7. Part ii.a Comparative statics with respect to h.

Case 2: $p_e=0$ and $\lambda_e<\Lambda$. Then, $v=\alpha$ which is independent of h when $p=0$, thus $\frac{d\lambda}{dh}$ equals zero.

Case 4: $p_e=1$ and $\lambda_e<\Lambda$. Then, $v=\delta$, and taking the derivative with respect to h yields, $0=1+\frac{\partial\delta}{\partial\lambda}\frac{d\lambda}{dh}$. This implies that $\frac{d\lambda}{dh}$ is negative because $\frac{\partial\delta}{\partial\lambda}$ is positive.

Case 5: $p_e=\in(0,1)$ and $\lambda_e=\Lambda$. Then $h=\beta$, and taking the derivative with respect to h yields, $1=\frac{\partial\beta}{\partial h}+\frac{\partial\beta}{\partial p}\frac{dp}{dh}$. This implies that $\frac{dp}{dh}$ is negative because $\frac{\partial\beta}{\partial h}$ equals zero, and $\frac{\partial\beta}{\partial p}$ is negative.

Case 6: $p_e=\in(0,1)$ and $\lambda_e<\Lambda$. Then $v=\alpha$ and $h=\beta$, and taking the derivative with respect to h yields,

$$0=\frac{\partial\alpha}{\partial h}+\frac{\partial\alpha}{\partial\lambda}\frac{d\lambda}{dh}+\frac{\partial\alpha}{\partial p}\frac{dp}{dh},\tag{8}$$

$$1=\frac{\partial\beta}{\partial h}+\frac{\partial\beta}{\partial\lambda}\frac{d\lambda}{dh}+\frac{\partial\beta}{\partial p}\frac{dp}{dh}.\tag{9}$$

Equation (8) implies that $\frac{d\lambda}{dh}$ and $\frac{dp}{dh}$ are of the same sign (both are positive or both are negative) because $\frac{\partial\alpha}{\partial h}$ equals zero, so $\frac{\partial\alpha}{\partial\lambda}\frac{d\lambda}{dh}=-\frac{\partial\alpha}{\partial p}\frac{dp}{dh}$, and $\frac{\partial\alpha}{\partial p}$ is negative, and $\frac{\partial\alpha}{\partial\lambda}$ is positive. Equation (9) implies both $\frac{dp}{dh}$ and $\frac{d\lambda}{dh}$ must be negative because $\frac{\partial\beta}{\partial h}=0$ and both $\frac{\partial\beta}{\partial\lambda}$ and $\frac{\partial\beta}{\partial p}$ are negative.

Comparative statics with respect to w_r.

Case 2: $p_e = 0$ and $\lambda_e < \Lambda$. Then, $v = \alpha$ which is independent of w_r when $p = 0$, thus $\frac{d\lambda}{dw_r}$ equals zero.

Case 4: $p_e = 1$ and $\lambda_e < \Lambda$. Then, $v = \delta$, and taking the derivative with respect to w_r yields, $0 = \frac{\partial \delta}{\partial w_r} + \frac{\partial \delta}{\partial \lambda}\frac{d\lambda}{dw_r}$. This implies that $\frac{d\lambda}{dw_r}$ is negative because $\frac{\partial \delta}{\partial w_r}$ and $\frac{\partial \delta}{\partial \lambda}$ are positive.

Case 5: $p_e \in (0,1)$ and $\lambda_e = \Lambda$. Then $h = \beta$, and taking the derivative with respect to w_r yields, $0 = \frac{\partial \beta}{\partial w_r} + \frac{\partial \beta}{\partial p}\frac{dp}{dw_r}$. This implies that $\frac{dp}{dw_r}$ is negative because $\frac{\partial \beta}{\partial wr}$ and $\frac{\partial \beta}{\partial p}$ are negative.

Case 6: $p_e \in (0,1)$ and $\lambda_e < \Lambda$. Then $v = \alpha$ and $h = \beta$, and taking the derivative with respect to w_r yields,

$$0 = \frac{\partial \alpha}{\partial w_r} + \frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dw_r} + \frac{\partial \alpha}{\partial p}\frac{dp}{dw_r}, \tag{10}$$

$$0 = \frac{\partial \beta}{\partial w_r} + \frac{\partial \beta}{\partial \lambda}\frac{d\lambda}{dw_r} + \frac{\partial \beta}{\partial p}\frac{dp}{dw_r}. \tag{11}$$

Equation (10) implies that $\frac{d\lambda}{dw_r}$ and $\frac{dp}{dw_r}$ are of the same sign (both are positive or both are negative) because $\frac{\partial \alpha}{\partial w_r}$ equals zero, so $\frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dw_r} = -\frac{\partial \alpha}{\partial p}\frac{dp}{dw_r}$, and $\frac{\partial \alpha}{\partial \lambda}$ is positive and $\frac{\partial \alpha}{\partial p}$ is negative. Equation (11) implies both $\frac{dp}{dw_r}$ and $\frac{d\lambda}{dw_r}$ must be negative because $\frac{\partial \beta}{\partial w_r}$, $\frac{\partial \beta}{\partial \lambda}$, and $\frac{\partial \beta}{\partial p}$ are all negative.

B.1.8. Part ii.b Comparative statics with respect to v.

Case 2: $p_e = 0$ and $\lambda_e < \Lambda$. Then, $v = \alpha$, and taking the derivative with respect to v yields $1 = \frac{\partial \alpha}{\partial v} + \frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dv}$. This implies that $\frac{d\lambda}{dv}$ is positive because $\frac{\partial \alpha}{\partial \lambda}$ is positive and $\frac{\partial \alpha}{\partial v}$ equals zero.

Case 4: $p_e = 1$ and $\lambda_e < \Lambda$. Then, $v = \delta$, and taking the derivative with respect to v yields $1 = \frac{\partial \delta}{\partial v} + \frac{\partial \delta}{\partial \lambda}\frac{d\lambda}{dv}$. This implies that $\frac{d\lambda}{dv}$ is positive because $\frac{\partial \delta}{\partial \lambda}$ is positive (as shown in analysis of case 4) and $\frac{\partial \delta}{\partial v}$ equals zero.

Case 5: $p_e = \in (0,1)$ and $\lambda_e = \Lambda$. Then $h = \beta$, and taking the derivative with respect to v yields $0 = \frac{\partial \beta}{\partial v} + \frac{\partial \beta}{\partial p}\frac{dp}{dv}$. This implies that $\frac{dp}{dv} = 0$ because $\frac{\partial \beta}{\partial v} = 0$ and $\frac{\partial \beta}{\partial p} < 0$.

Case 6: $p_e = \in (0,1)$ and $\lambda_e < \Lambda$. Then $v = \alpha$ and $h = \beta$, and taking the derivative with respect to v yields,

$$1 = \frac{\partial \alpha}{\partial v} + \frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dv} + \frac{\partial \alpha}{\partial p}\frac{dp}{dv}, \tag{12}$$

$$0 = \frac{\partial \beta}{\partial v} + \frac{\partial \beta}{\partial \lambda}\frac{d\lambda}{dv} + \frac{\partial \beta}{\partial p}\frac{dp}{dv}. \tag{13}$$

Noting that $\frac{\partial \alpha}{\partial v} = \frac{\partial \beta}{\partial v} = 0$, solving (13) for $\frac{d\lambda}{dv}$ and substituting into the first equation yields:

$$1 = \left(-\frac{\partial \alpha}{\partial \lambda}\frac{\frac{\partial \beta}{\partial p}}{\frac{\partial \beta}{\partial \lambda}} + \frac{\partial \alpha}{\partial p} \right)\frac{dp}{dv}. \tag{14}$$

Since $\frac{\partial \alpha}{\partial \lambda}$ is positive, $\frac{\partial \beta}{\partial p}/\frac{\partial \beta}{\partial \lambda}$ is positive (negative divided by a negative), and $\frac{\partial \alpha}{\partial p}$ is negative, it must be that $\frac{dp}{dv}$ is negative. Given that $\frac{dp}{dv}$ is negative, (12) then implies that $\frac{d\lambda}{dv}$ is positive because $\frac{d\beta}{dv}$ equals zero, $\frac{\partial \beta}{\partial p}\frac{dp}{dv}$ is positive, and $\frac{\partial \beta}{\partial \lambda}$ is negative.

B.1.9. Part ii.c Comparative statics with respect to w_s.

Case 2: $p_e = 0$ and $\lambda_e < \Lambda$. Then, $v = \alpha$, and taking the derivative with respect to w_s yields $0 = \frac{\partial \alpha}{\partial w_s} + \frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dw_s}$. This implies that $\frac{d\lambda}{dw_s}$ is negative because, $\frac{\partial \alpha}{\partial w_s}$ and $\frac{\partial \alpha}{\partial \lambda}$ are positive.

Case 4: $p_e = 1$ and $\lambda_e < \Lambda$. Then, $v = \delta$, and taking the derivative with respect to w_s yields, $0 = \frac{\partial \delta}{\partial w_s} + \frac{\partial \delta}{\partial \lambda}\frac{d\lambda}{dw_s}$. This implies that $\frac{d\lambda}{dw_s}$ is negative because $\frac{\partial \delta}{\partial w_s}$ and $\frac{\partial \delta}{\partial \lambda}$ are positive.

Case 5: $p_e \in (0,1)$ and $\lambda_e = \Lambda$. Then $h = \beta$, and taking the derivative with respect to w_s yields, $0 = \frac{\partial \beta}{\partial w_s} + \frac{\partial \beta}{\partial p}\frac{dp}{dw_s}$. This implies that $\frac{dp}{dw_s}$ is positive because $\frac{\partial \beta}{\partial w_s}$ is positive and $\frac{\partial \beta}{\partial p}$ are negative.

Case 6: $p_e \in (0,1)$ and $\lambda_e < \Lambda$. Then $v = \alpha$ and $h = \beta$, and taking the derivative with respect to w_s yields,

$$0 = \frac{\partial \alpha}{\partial w_s} + \frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dw_s} + \frac{\partial \alpha}{\partial p}\frac{dp}{dw_s}, \tag{15}$$

$$0 = \frac{\partial \beta}{\partial w_s} + \frac{\partial \beta}{\partial \lambda}\frac{d\lambda}{dw_s} + \frac{\partial \beta}{\partial p}\frac{dp}{dw_s}. \tag{16}$$

Solving (15) for $\frac{dp}{dw_s}$ and substituting into (16) yields,

$$\frac{\partial \beta}{\partial p}\frac{\frac{\partial \alpha}{\partial w_s}}{\frac{\partial \alpha}{\partial p}} - \frac{\partial \beta}{\partial w_s} = \left(\frac{\partial \beta}{\partial \lambda} - \frac{\partial \beta}{\partial p}\frac{\frac{\partial \alpha}{\partial \lambda}}{\frac{\partial \alpha}{\partial p}}\right)\frac{d\lambda}{dw_s}. \tag{17}$$

The term in parentheses on the RHS is negative (negative minus a positive) because $\frac{\partial \beta}{\partial \lambda}$, $\frac{\partial \beta}{\partial p}$, and $\frac{\partial \alpha}{\partial p}$ are negative and $\frac{\partial \alpha}{\partial \lambda}$ is positive. Hence, given the sign of the LHS, then $\frac{d\lambda}{dw_s}$ has the oppositive sign.

The LHS (positive minus a positive) is a negative if $\frac{\partial \beta}{\partial w_s} > \frac{\partial \beta}{\partial p}\frac{\frac{\partial \alpha}{\partial w_s}}{\frac{\partial \alpha}{\partial p}}$ which, expressed in terms of model primitives is equivalent to:

$$\frac{1}{\mu}\left(1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right) \geq \left(\frac{1}{\mu} - \frac{w_r}{w_s(\mu-\lambda)}\right)\left[1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}} + \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\ln\left(\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right]\frac{\left(1-\left(\frac{\lambda}{\mu}\right)^2\left(1-\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right)}{\left(\frac{\lambda}{\mu}\right)^2\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\left(-\ln\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)}. \tag{18}$$

Observe that, if this inequality holds for the case when $w_r = 0$, it is true for all $w_r > 0$ because the RHS is decreasing in w_r. To see this note that all the terms multiplied by w_r are positive, to see the term in brackets is positive, let $x = (\lambda/\mu)^{p\lambda/\gamma}$ and note that $1 - x + x\ln x > 0,\ \forall x \in [0,1]$. Hence, letting $w_r = 0$ and substituting x for notational convenience the inequality reduces to

$$(\lambda/\mu)^2 \geq \frac{1 - x(1-\ln x)}{(1-x)(1-x\ln x)} \tag{19}$$

The RHS is decreasing in x, therefore if this inequality holds when $p = 1$ and $\gamma = \mu - \lambda$ it holds for all $p \in [0,1]$ and all $\gamma > \mu - \lambda$. Hence, letting $x = (\lambda/\mu)^{\frac{p\lambda}{\gamma}} \to (\lambda/\mu)^{\frac{\lambda}{\mu-\lambda}} = \rho^{\frac{\rho}{1-\rho}}$ we have the result when

$$\rho^2 \geq \frac{1-\rho^{\frac{\rho}{1-\rho}}(1-\ln \rho^{\frac{\rho}{1-\rho}})}{(1-\rho^{\frac{\rho}{1-\rho}})(1-\rho^{\frac{\rho}{1-\rho}}\ln \rho^{\frac{\rho}{1-\rho}})}. \tag{20}$$

A sufficient condition for this inequality to hold is $\rho > .5$ which we have by assumption. Hence $\frac{d\lambda}{dw_s} > 0$.

Given that $\frac{d\lambda}{dw_s} > 0$, (15) implies that $\frac{dp}{dw_s}$ is positive. To see this, observe that $\left(\frac{\partial \alpha}{\partial w_s}\right)$ is positive, $\left(\frac{\partial \alpha}{\partial \lambda}\frac{d\lambda}{dw_s}\right)$ is the product of two positives, therefore the third term must be negative, and since $\frac{\partial \alpha}{\partial p}$ is negative, it must be that $\frac{dp}{dw_s}$ is positive.

B.2. Proof of Proposition 5

We first show that $\frac{\Lambda}{\mu} \in [.75, 1)$, $v > 16^{w_s}/\mu$, and $\gamma \geq \sqrt{\mu w_s/v}$ implies that $\lambda_{so} > .75\mu$, and $\lambda_{so} \geq \mu - \gamma$, which we use in the proof below. First, if no customers can be flexible ($p = 0$), the socially optimal arrival rate is $\lambda^0_{so} := \min\{\Lambda, \mu - \sqrt{\frac{w_s\mu}{v}}\}$ (see (Hassin and Haviv 2003, Chapter 3, Section 1.2) for the details). Given this, a little algebra shows that, in this case, if $\Lambda\mu \geq .75$ then, $v > 16^{w_s}/\mu$ implies that $\lambda_{so}^0 \geq 0.75\mu$ and $\gamma \geq \sqrt{\mu w_s/v}$ implies that $\lambda_{so}^0 \geq \mu - \gamma$. Therefore, as long as a central planner, given the option to dictate customer flexibility $p \in [0,1]$ in addition to the arrival rate λ, would not dictate that fewer customers join than in the case when p is restricted to zero, we have the preliminary result. This is obvious as proactive service enables a provider to reduce delays while serving the same amount of customers (because delays are decreasing in p), thus, given the option to dictate flexibility, it can never be optimal to serve fewer customers as it would be dominated by the case where the same number of customers are served as the benchmark case and some positive proportion of customers are flexible. Therefore, for any socially optimal proportion of flexible customers and a socially optimal arrival rate (p_{so}, λ_{so}) we have that $\lambda_{so} > .75\mu$, and $\lambda_{so} \geq \mu - \gamma$.

B.2.1. Part 1: We next prove that $\lambda_{so} \leq \lambda_e$. We do it by showing that $\lambda_{so} \leq \lambda_0 \leq \lambda_e$, where $\lambda_0 = \min\{\Lambda, \mu - \frac{w_s}{v}\}$ is the equilibrium arrival rate when the proportion of flexible customers is fixed at zero, and the second inequality follows from the fact that the equilibrium arrival rate (λ_e) is no smaller than λ_0 (see the preliminary result in the proof of Proposition 4). Note in the case where $\lambda_e = \lambda_0 = \Lambda$, the inequality is trivially satisfied, therefore we will focus on the case where $\lambda_0 = \mu - \frac{w_s}{v} < \Lambda$. To prove the first inequality, we show that, for any fixed p, the partial derivative of the welfare function with respect to λ (given below), is negative for all $\lambda \geq \lambda_0$.

$$\frac{\partial}{\partial \lambda} W(p,\lambda) = \overbrace{v - \frac{w_s}{\mu-\lambda}}^{A} - \overbrace{\left(ph + \frac{w_r}{(\mu-\lambda)^2} \frac{\lambda(2\mu-\lambda)}{\mu} \left(1 - \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right) + \frac{w_s - w_r}{\mu-\lambda} \frac{\lambda}{\mu} \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}} \frac{p\lambda}{\gamma} \left(1 + \ln \frac{\lambda}{\mu}\right)\right)}^{B} \tag{21}$$

$$\underbrace{- \frac{w_s \lambda}{(\mu-\lambda)^2} \left[1 - \frac{(2\mu-\lambda)}{\mu} \left(1 - \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\right)\right]}_{C} \tag{22}$$

To show that this derivative is negative if $\lambda \geq \lambda_0$ we show that A is non-positive, and B is non-negative (-B is non-positive) and C is positive (-C is negative). To see that A is non-positive, observe that $v - \frac{w_s}{\mu-\lambda_0} = 0$, therefore for $\lambda \geq \lambda_0$ this term is non-positive. To see that B is non-negative, note the first two terms within B are trivially non-negative and that $\lambda/\mu > e^{-1}$ is a sufficient condition for the third term to be non-negative which we have by assumption, therefore B is non-negative. To see that $-C$ is negative, observe that this is true when the term within the square brackets is positive. Noting that the term in brackets is decreasing in p, letting $p = 1$ minimizes this term and therefore, as long as it is positive when $p = 1$, it is positive for all $p \in [0,1]$. Further noting that the term in brackets is increasing in γ then letting $\gamma = \mu - \lambda$ minimizes this term (note from the assumption that $\gamma \geq \sqrt{\mu w_s/v}$, and $v > w_s/\mu$ we have that $\gamma \geq \sqrt{\mu w_s/v} \geq \sqrt{w_s^2/v^2} = w_s/v = \mu - \lambda_0$, therefore for all $\lambda \geq \lambda_0$ we have that $\gamma \geq \mu - \lambda$). With these substitutions for p and γ, the term in brackets is positive if $1 - (2-\rho)\left(1 - \rho^{\frac{\rho}{1-\rho}}\right)$ is positive. Observing that this is true for all $\rho \in [0,1]$ we have the result that $\lambda_{so} \leq \lambda_e$.

B.2.2. Part 2: Now we show that $p_{so} \geq p_e$. Fix $\lambda > 0$. The partial derivative of the welfare function with respect to p is

$$\frac{\partial}{\partial p} W(p,\lambda) = -\lambda(h - \xi(p,\lambda)) \tag{23}$$

where

$$\xi_s(p,\lambda) = \frac{w_s - w_r}{\mu - \lambda} \left(\frac{\lambda^2}{\gamma\mu} \left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}} \left(-\ln \frac{\lambda}{\mu}\right) \right), \tag{24}$$

The second partial derivative of $W(p,\lambda)$ with respect to p is $-\frac{w_s-w_r}{\mu-\lambda}\left(\frac{\lambda^4}{\gamma^2\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{p\lambda}{\gamma}}\left(\ln\frac{\lambda}{\mu}\right)^2\right)$, which is non-positive by the assumption that $w_r \le w_s$. Therefore, for fixed $\lambda \in (0,\Lambda]$, the welfare function is concave with respect to p. Let $p_{so}(\lambda)$ denote the proportion of flexible customers that maximizes the welfare function when the arrival rate is λ.

By concavity of the welfare function, for fixed λ if (23) is positive when $p=1$, that is,

$$h \le \xi_s(1,\lambda) \tag{25}$$

then $p_{so}(\lambda)=1$, that is, it is social optimal for everyone to be flexible.

Now consider unregulated customer equilibrium where $c_r = c_s = 0$ and λ is fixed. Taking the difference in the utility of inflexible and flexible customers (given by $v - c_s - w_s\bar{T}_{ss}$ and $v - c_r - h - w_r\bar{T}_{rr} - w_s\bar{T}_{rs}$ respectively), we have (after some algebra) that if

$$h > \xi_e(p,\lambda) := \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda}\right)\frac{1}{\mu-\lambda}\bar{T}_{rr}(p,\lambda), \tag{26}$$

then the utility of inflexible customers is greater than that of flexible customers. Since $\bar{T}_{rr}$ is decreasing in the proportion of flexible customers p (which follows from the approximation given by $\bar{T}_{rr} = \frac{\bar{N}_r}{p\lambda} \approx \frac{1}{p(\mu-\lambda)}\rho\left(1-\rho^{\frac{p\lambda}{\gamma}}\right)$), if $h \ge \xi_e(0,\lambda)$, then choosing to be flexible is a dominated strategy. Substituting the approximation of $\bar{T}_{rr}$ into 26 and applying L'Hôpital's Rule, flexibility is dominated in equilibrium (for fixed λ) if,

$$h \ge \xi_e(0,\lambda) \tag{27}$$

where

$$\xi_e(0,\lambda) = \left(\frac{w_s}{\mu} - \frac{w_r}{\mu-\lambda}\right)\frac{\lambda^2}{\gamma\mu}\left(-\ln\frac{\lambda}{\mu}\right). \tag{28}$$

We prove below that if $\gamma \ge \mu-\lambda$ and $\frac{\lambda}{\mu} \ge .75$,

$$\xi_s(1,\lambda) \ge \xi_e(0,\lambda). \tag{29}$$

This, together with the fact that ξ_e is decreasing in λ (see B.1.4) and the fact that $\lambda_{so} < \lambda_e$ (from Part 1 above), implies that

$$\xi_s(1,\lambda_{so}) \ge \xi_e(0,\lambda_{so}) \ge \xi_e(0,\lambda_e). \tag{30}$$

Then the desired result follows by setting $\underline{h} := \xi_e(0,\lambda_e)$ and $\bar{h} := \xi_s(1,\lambda_{so})$.

To complete the proof, we now prove (29). We note that (29) is (after some algebra) equivalent to

$$\rho^{\frac{\lambda}{\gamma}} > \frac{w_s(1-\rho) - w_r}{w_s - w_r}. \tag{31}$$

Because LHS of this inequality is increasing in γ, under the assumption $\gamma \ge \mu-\lambda$ we have that $\rho^{\frac{\lambda}{\gamma}} \ge \rho^{\frac{\lambda}{\mu-\lambda}}$. Additionally, because the RHS is decreasing in w_r, hence

$$\frac{w_s(1-\rho) - w_r}{w_s - w_r} \le \frac{w_s(1-\rho)}{w_s} \tag{32}$$

Therefore (31) holds if $\rho + \rho^{\frac{\rho}{1-\rho}} > 1$ and the latter holds for all $\rho > .5$. Since we assume that $\lambda/\mu \ge 0.75$, we have (29).

References

Hassin, R., M. Haviv. 2003. *To queue or not to queue: equilibrium behavior in queueing systems*, vol. 59. Kluwer Academic Publishers, Norwell MA, USA.