Supplementary Online Appendix for
"How Should Performance Signals Affect Contracts?"

Pierre Chaigneau, Alex Edmans, and Daniel Gottlieb

E Proof of Lemma 4

For a given contract $w_s(q)$, the effort choice problem of the agent can be written successively as:

$$\max_e \mathbb{E}\left[u\left(\bar{W}+w_s(q)\right)\right]-C(e) \;\Leftrightarrow\; \max_e \sum_s \int_{\underline{q}}^{+\infty} u\left(\bar{W}+w_s(q)\right) f(q,s|e)dq - C(e)$$

The second derivative of the agent's objective function with respect to e is negative for any e if and only if:

$$\sum_s \int_{\underline{q}}^{+\infty} u\left(\bar{W}+w_s(q)\right) \frac{\partial^2 f(q,s|e)}{\partial e^2} dq < C''(e) \quad \forall e \in (0,\bar{e}). \tag{35}$$

Assume that u is bounded from above, with $\lim_{w\to\infty} u(w) \equiv u^+$. In addition, with limited liability, the minimum payment is $w_s(q)=0$; with an increasing utility function, this implies that the minimum value of u is $u(\bar{W})$. Therefore, for any $\{q,s\}$:

$$u\left(\bar{W}+w_s(q)\right) \in \left[u(\bar{W}),u^+\right]$$

Using notations K_e^+ and K_e^- defined in equations (29) and (30), the expression on the LHS of equation (35) can then be rewritten as:

$$\sum_s \int_{\underline{q}}^{+\infty} u\left(\bar{W}+w_s(q)\right) \min\left\{\frac{\partial^2 f(q,s|e)}{\partial e^2},0\right\}dq + \sum_s \int_{\underline{q}}^{+\infty} u\left(\bar{W}+w_s(q)\right) \max\left\{\frac{\partial^2 f(q,s|e)}{\partial e^2},0\right\}dq \tag{36}$$

As established above, we have $u\left(\bar{W}+w_s(q)\right) \geq u(\bar{W})$ for any q,s, and $u\left(\bar{W}+w_s(q)\right) \leq u^+$ for any q,s. Therefore, for any q,s such that $\frac{\partial^2 f(q,s|e)}{\partial e^2} \leq 0$ we have $u\left(\bar{W}+w_s(q)\right)\frac{\partial^2 f(q,s|e)}{\partial e^2} \leq u(\bar{W})\frac{\partial^2 f(q,s|e)}{\partial e^2}$; and for any q,s such that $\frac{\partial^2 f(q,s|e)}{\partial e^2} \geq 0$ we have $u\left(\bar{W}+w_s(q)\right)\frac{\partial^2 f(q,s|e)}{\partial e^2} \leq u^+\frac{\partial^2 f(q,s|e)}{\partial e^2}$. Integrating and summing over q and s, this implies that expression (36) is less than $K_e^- u(\bar{W}) + K_e^+ u^+$, which completes the proof. ■

F Proof of Proposition 3

From equation (8), since $\zeta_s > 0$, $\sigma_s > 0$, and $g(\cdot) > 0$, a distribution with location and scale parameters that satisfies MLRP is such that $g'(\cdot) > 0$ if q is lower than a threshold, and $g'(\cdot) < 0$ if q is higher than this threshold, i.e., the PDF g is single peaked – the output corresponding to the peak is denoted by q_s^P. For symmetric distributions, the single peak of the distribution, which is such that $g'\left(\frac{q_s^P - \xi_s}{\sigma_s}\right) = 0$, is at $q_s^P = \xi_s$ for $s \in \{s_i, s_j\}$. With CRRA utility, characterized by $u'(w) = w^{-\gamma}$ and $u'^{-1}(w) = w^{-\frac{1}{\gamma}}$, provided that the FOA holds we have:

$$w_s(q) = \begin{cases} \left(\lambda + \mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}} - \bar{W} & \text{if } \lambda + \mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right] \geq \frac{1}{u'(\bar{W})} \\ 0 & \text{if } \lambda + \mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right] < \frac{1}{u'(\bar{W})} \end{cases}. \tag{37}$$

For $\gamma > 1$, we use the condition for the FOA in Lemma 4. For $\gamma \leq 1$, we derive a condition for FOA to hold in this setting. The FOA holds if:

$$\sum_s \int_{\underline{q}}^{+\infty} u\left(\bar{W} + w_s(q)\right) \frac{\partial^2 f(q, s|e)}{\partial e^2} dq < C''(e) \quad \forall e \in (0, \bar{e}), \tag{38}$$

where $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ if $\gamma < 1$ and $\ln(w)$ if $\gamma = 1$, and $w_s\left(q\right)$ is defined by equation (37).

Part (i): Suppose that signals s_i and s_j differ only in their individual informativeness: $\frac{\partial\phi_{\hat{e}}^{s_i}/\partial e}{\phi_{\hat{e}}^{s_i}} > \frac{\partial\phi_{\hat{e}}^{s_j}/\partial e}{\phi_{\hat{e}}^{s_j}}$ (s_i and s_j are associated with the same output distribution). For notational convenience, let $\tilde{\phi}_s \equiv \frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s}$ and $w_s(q) \equiv W_q(\tilde{\phi}_s)$, which is a continuous function of $\tilde{\phi}_s$. For a given q, we can write:

$$w_{s_i}(q) - w_{s_j}(q) = W_q(\tilde{\phi}_i) - W_q(\tilde{\phi}_j) = \int_{\tilde{\phi}_j}^{\tilde{\phi}_i} \frac{\partial W_q(\tilde{\phi})}{\partial\tilde{\phi}} d\tilde{\phi}.$$

Holding all else constant including Lagrange multipliers (we are comparing two signal realizations, i.e. we do not change parameters of the contracting environment), for given q and s such that $w_s(q) > 0$:

$$\begin{aligned}\frac{\partial W_q(\tilde{\phi})}{\partial \tilde{\phi}} &= \frac{\partial}{\partial \tilde{\phi}}\left\{\left(\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}} - \bar{W}\right\} \\ &= \underbrace{\frac{\mu}{\gamma}}_{>0}\underbrace{\left(\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}}_{\geq 0 \text{ since } w_s(q)\geq 0}, \qquad (39)\end{aligned}$$

For given q and q_0, we have

$$\frac{w_{s_i}(q) - w_{s_i}(q_0)}{q - q_0} \geq \frac{w_{s_j}(q) - w_{s_j}(q_0)}{q - q_0} \quad \Leftrightarrow \quad w_{s_i}(q) - w_{s_j}(q) - \big(w_{s_i}(q_0) - w_{s_j}(q_0)\big) \geq 0. \qquad (40)$$

Thus, for given q and q_0 such that $q > q_0$ and $w_s(q_0) > 0$, we have:

$$\begin{aligned}& w_{s_i}(q) - w_{s_j}(q) - \big(w_{s_i}(q_0) - w_{s_j}(q_0)\big) \\ = \; & \frac{\mu}{\gamma}\int_{\tilde{\phi}_j}^{\tilde{\phi}_i}\left(\left(\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1} - \left(\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}\right) d\tilde{\phi}.\end{aligned}$$

We have: $q > q_0$ which implies $\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} > \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}$ by MLRP, so that, for a given $\tilde{\phi}_s$:

$$\begin{aligned}\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right] &> \lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}\right] \\ \left(\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1} &> \left(\lambda + \mu\left[\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1} \quad \text{iff } \gamma < 1. (41)\end{aligned}$$

If $\gamma < 1$, the condition in (41) holds, and so (40) also holds. If $\gamma < 1$, (41) does not hold, and so (40) does not either. If $\gamma = 1$, the equation in (41) is satisfied as an equality so that the expression on the right in (40) is equal to zero.

Part (ii): Suppose that signals s_i and s_j differ only in their equilibrium location parameter, with $\xi_i > \xi_j$. Let $w_s(q) \equiv W_q(\xi_s)$, which is a continuous function of ξ_s, since $\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}$ is by assumption continuously differentiable in the equilibrium location parameter ξ_s. For a given q, we have:

$$w_{s_i}(q) - w_{s_j}(q) = W_q(\xi_i) - W_q(\xi_j) = \int_{\xi_j}^{\xi_i}\frac{\partial W_q(\xi)}{\partial \xi}d\xi.$$

Holding all else constant including Lagrange multipliers, for given q and s such that $w_s(q) > 0$, we have:

$$\begin{aligned}\frac{\partial W_q(\xi)}{\partial \xi} &= \frac{\partial}{\partial \xi}\left\{\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}} - \bar{W}\right\} \\ &= \underbrace{\frac{\mu}{\gamma}}_{>0}\ \frac{\partial}{\partial \xi}\left\{\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right\}\underbrace{\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}}_{\geq 0 \text{ since } w_s(q) \geq 0},\end{aligned} \tag{42}$$

where:

$$\begin{aligned}\frac{\partial}{\partial \xi}\left\{\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right\} &= \frac{\partial}{\partial \xi}\left\{-\frac{\zeta_s}{\sigma_s}\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}\right\} \\ &= -\frac{\zeta_s}{\sigma_s^2}\underbrace{G(q)}_{\geq 0},\end{aligned} \tag{43}$$

where G is defined in equation (18). For given q and q_0, we have:

$$\frac{w_{s_i}(q) - w_{s_i}(q_0)}{q - q_0} \geq \frac{w_{s_j}(q) - w_{s_j}(q_0)}{q - q_0} \quad \Leftrightarrow \quad w_{s_i}(q) - w_{s_j}(q) - \big(w_{s_i}(q_0) - w_{s_j}(q_0)\big) \geq 0. \tag{44}$$

Thus, for given q and q_0 such that $q > q_0$ and $w_s(q_0) > 0$, we have:

$$\begin{aligned}& w_{s_i}(q) - w_{s_j}(q) - \big(w_{s_i}(q_0) - w_{s_j}(q_0)\big) \\ =\ & \frac{\mu}{\gamma}\frac{\zeta_s}{\sigma_s^2}\int_{\xi_j}^{\xi_i}\Bigg(-G(q)\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1} \\ & + G(q_0)\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}\Bigg)d\xi.\end{aligned} \tag{45}$$

From equation (17) and the definition of $G(q)$ in (18), the likelihood ratio of output is weakly concave in q if and only if $G'(q) \leq 0$ in which case $0 \leq G(q) \leq G(q_0)$ since $q > q_0$. In addition, $q > q_0$ implies $\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} > \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}$ by MLRP, so that, for a weakly concave likelihood ratio and

$\gamma \geq 1$:

$$\begin{aligned}
\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right] &> \lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)} \right] \\
\left(\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \right)^{\frac{1}{\gamma}-1} &\leq \left(\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)} \right] \right)^{\frac{1}{\gamma}-1} \\
G(q) \left(\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \right)^{\frac{1}{\gamma}-1} &\leq G(q_0) \left(\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)} \right] \right)^{\frac{1}{\gamma}-1} \quad (46)
\end{aligned}$$

We conclude that if the likelihood ratio is nonconvex and $\gamma \geq 1$, then (46) holds and, using equation (45), (44) holds too. Symmetrically, if the likelihood ratio is weakly convex (so that $G'(q) \geq 0$) and $\gamma \leq 1$, then the inequality in (46) is reversed, so that it is reversed in condition (44) too. Finally, if the likelihood ratio is linear (so that $G'(q) = 0$) and $\gamma = 1$, then (46) holds as an equality, and so does condition (44).

Part (iii): Suppose that signals s_i and s_j differ only in their impact parameter, with $\zeta_i > \zeta_j$. Let $w_s(q) \equiv W_q(\zeta_s)$, which is a continuous function of ζ_s, since $\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}$ is by assumption continuously differentiable in the parameter ζ_s. For a given q, we have:

$$w_{s_i}(q) - w_{s_j}(q) = W_q(\zeta_i) - W_q(\zeta_j) = \int_{\zeta_j}^{\zeta_i} \frac{\partial W_q(\zeta)}{\partial \zeta} d\zeta. \quad (47)$$

As above, holding all else constant including Lagrange multipliers, for given q and s such that $w_s(q) > 0$:

$$\begin{aligned}
\frac{\partial W_q(\zeta)}{\partial \zeta} &= \frac{\partial}{\partial \zeta} \left\{ \left(\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \right)^{\frac{1}{\gamma}} - \bar{W} \right\} \\
&= \frac{\mu}{\gamma} \frac{\partial}{\partial \zeta} \left\{ \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right\} \underbrace{\left(\lambda + \mu \left[\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \right)^{\frac{1}{\gamma}-1}}_{> 0 \text{ since } w_s(q) > 0}, \quad (48)
\end{aligned}$$

where

$$\frac{\partial}{\partial \zeta} \left\{ \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)} \right\} = \frac{\partial}{\partial \zeta} \left\{ -\frac{\zeta_s}{\sigma_s} \frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)} \right\} = \underbrace{-\frac{1}{\sigma_s}}_{<0} \underbrace{\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}}_{< 0 \text{ for } q > q_s^P}. \quad (49)$$

Thus, at output q, we have:

$$\int_{\zeta_j}^{\zeta_i} \frac{\partial W_q(\zeta)}{\partial \zeta} d\zeta = \int_{\zeta_j}^{\zeta_i} \frac{\mu}{\gamma}\frac{1}{\sigma_s} \underbrace{\left(-\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}\right)}_{>0 \text{ for } q>q_s^P} \underbrace{\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}}_{>0 \text{ since } w_s(q)>0} d\zeta.$$

In sum, with $q > q_0 > \max\{q_s^P, q_s^*, \xi_s\}$, we have:

$$\begin{aligned} & w_{s_i}(q) - w_{s_j}(q) - \left[w_{s_i}(q_0) - w_{s_j}(q_0)\right] = W_q(\zeta_i) - W_q(\zeta_j) - [W_{q_0}(\zeta_i) - W_{q_0}(\zeta_j)] \\ = \ & \frac{\mu}{\gamma}\frac{1}{\sigma_s}\int_{\zeta_j}^{\zeta_i}\left(\left(-\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}\right)\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}\right. \\ & \left. - \left(-\frac{g'\left(\frac{q_0-\xi_s}{\sigma_s}\right)}{g\left(\frac{q_0-\xi_s}{\sigma_s}\right)}\right)\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}\right) d\zeta. \end{aligned} \tag{50}$$

For $q > \max\{q_s^P, q_s^*, \xi_s\}$, both $-\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}$ and $\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}$ are positive and weakly increasing in q (by MLRP) if $\gamma \leq 1$. Therefore, if $\gamma \leq 1$, expression (50) is positive. Using (44), this means that, with $\zeta_i > \zeta_j$, the PPS measure $\frac{w_s(q)-w_s(q_0)}{q-q_0}$ is higher under s_i than under s_j.

Part (iv): Suppose that signals s_i and s_j differ only in their scale parameter, with $\sigma_i > \sigma_j$. Let $w_s(q) \equiv W_q(\sigma_s)$, which is a continuous function of σ_s, since $\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}$ is by assumption continuously differentiable in the scale parameter σ_s. For a given q, we have:

$$w_{s_i}(q) - w_{s_j}(q) = W_q(\sigma_i) - W_q(\sigma_j) = \int_{\sigma_j}^{\sigma_i} \frac{\partial W_q(\sigma)}{\partial \sigma} d\sigma. \tag{51}$$

Holding all else constant including Lagrange multipliers, for given q and s such that $w_s(q) > 0$:

$$\begin{aligned} \frac{\partial W_q(\sigma)}{\partial \sigma} &= \frac{\partial}{\partial \sigma}\left\{\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}} - \bar{W}\right\} \\ &= \underbrace{\frac{\mu}{\gamma}}_{>0} \frac{\partial}{\partial \sigma}\left\{\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right\} \underbrace{\left(\lambda + \mu\left[\frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}}_{>0 \text{ since } w_s(q)>0}, \end{aligned}$$

where:

$$\frac{\partial}{\partial\sigma}\left\{\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right\} = \frac{\partial}{\partial\sigma}\left\{-\frac{\zeta_s}{\sigma_s}\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}\right\}$$

$$= \underbrace{\frac{\zeta_s}{\sigma_s^2}}_{>0}\underbrace{\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}}_{<0\text{ for }q>q_s^P}+\underbrace{\frac{\zeta_s}{\sigma_s}}_{>0}\underbrace{G(q)}_{\geq 0}\underbrace{\frac{\xi_s-q}{\sigma_s^2}}_{<0\text{ for }q>\xi_s}. \tag{52}$$

So, at output q:

$$\int_{\sigma_j}^{\sigma_i}\frac{\partial W_q(\sigma)}{\partial\sigma}d\sigma = \int_{\sigma_j}^{\sigma_i}\frac{\mu}{\gamma}\frac{\zeta_s}{\sigma_s^2}\left(\underbrace{\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}+G(q)\frac{\xi_s-q}{\sigma_s}}_{<0\text{ for }q>\max\{\xi_s,q_s^P\}}\right)\underbrace{\left(\lambda+\mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s}+\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}}_{>0\text{ since }w_s(q)>0}d\sigma$$

In sum, with $q > q_0 > \max\{q_s^P, q_s^*, \xi_s\}$:

$$\begin{aligned}
&w_{s_i}(q)-w_{s_j}(q)-\big(w_{s_i}(q_0)-w_{s_j}(q_0)\big) = W_q(\sigma_i)-W_q(\sigma_j)-(W_{q_0}(\sigma_i)-W_{q_0}(\sigma_j))\\
&=\frac{\mu}{\gamma}\frac{\zeta_s}{\sigma_s^2}\int_{\sigma_j}^{\sigma_i}\left(\left(\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}+G(q)\frac{\xi_s-q}{\sigma_s}\right)\left(\lambda+\mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s}+\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}\right.\\
&\qquad\left.-\left(\frac{g'\left(\frac{q_0-\xi_s}{\sigma_s}\right)}{g\left(\frac{q_0-\xi_s}{\sigma_s}\right)}+G(q)\frac{\xi_s-q_0}{\sigma_s}\right)\left(\lambda+\mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s}+\frac{\frac{\partial f}{\partial e}(q_0|\hat{e},s)}{f(q_0|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}\right)d\sigma
\end{aligned} \tag{53}$$

For $q > \max\{q_s^P, q_s^*, \xi_s\}$, $\left(\frac{g'\left(\frac{q-\xi_s}{\sigma_s}\right)}{g\left(\frac{q-\xi_s}{\sigma_s}\right)}+G(q)\frac{\xi_s-q}{\sigma_s}\right)$ is negative and weakly decreasing in q if the likelihood ratio of output is weakly convex (by MLRP and since then $G'(q) \geq 0$, as per point (ii) above), while $\left(\lambda+\mu\left[\frac{\partial\phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s}+\frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)^{\frac{1}{\gamma}-1}$ is positive and weakly increasing in q (by MLRP) if $\gamma \leq 1$. Therefore, if $\gamma \leq 1$ and the likelihood ratio of output is weakly convex, expression (53) is negative. Using inequality (44), this means that, with $\sigma_i > \sigma_j$, the PPS measure $\frac{w_s(q)-w_s(q_0)}{q-q_0}$ is lower under s_i than under s_j if the likelihood ratio of output is weakly convex and $\gamma \leq 1$. ■

G The First-Order Approach with Limited Liability, Normally-Distributed Output, and Log Utility

This Appendix provides sufficient conditions for the FOA in the setting considered in Section 2.2, with limited liability, normally-distributed output, and log utility. We first derive the optimal contract and provide a sufficient condition for the FOA without an additional signal. Given effort $e \in [0, \bar{e}]$, output is determined by

$$q = e + \epsilon,$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

Proposition 4 *Suppose $C''(e) \geq \frac{\bar{e}}{\sigma^2}$ for all $e \in [0, \bar{e}]$. Let $\{w^*(\cdot), e^*\}$ be the optimal contract and the effort it implements. Then, there exist $\lambda > 0$ and $q^* \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}$ such that*

$$w^*(q) = \frac{\lambda}{\sigma^2} \cdot \max\left\{q - q^*,\ 0\right\}.$$

Moreover, $q^* = e^* + \frac{\sigma^2}{\lambda}\bar{W}$ if the IR does not bind.

For example, with a quadratic effort cost, $C(e) = \alpha e + \frac{\beta}{2}e^2$, for $\alpha > 0$ and $\beta > 0$, we have $C''(e) = \beta$ for all e, and the sufficient condition for the FOA is simply $\beta \geq \frac{\bar{e}}{\sigma^2}$.

Proof of Proposition 4:

As usual, let φ denote the PDF of the standard normal distribution. Let $W(q) \equiv w(q) + \bar{W}$ denote the manager's consumption. His IC, IR, and LL are, respectively:

$$\begin{aligned}
&e \in \arg\max_{\hat{e} \in [0,\bar{e}]} \int \ln\left(W\left(q\right)\right) \frac{1}{\sigma}\varphi\left(\frac{q - \hat{e}}{\sigma}\right) dq - C(\hat{e}), \\
&\int \ln\left(W\left(q\right)\right) \frac{1}{\sigma}\varphi\left(\frac{q - e}{\sigma}\right) dq - C(e) \geq 0, \\
&W\left(q\right) \geq \bar{W}\ \forall q.
\end{aligned}$$

To simplify notation, we will work with the manager's indirect utility, $u(q) \equiv \ln\left(W\left(q\right)\right)$, so that $W(q) = \exp\left[u(q)\right]$. This step is without loss of generality. The next step, which in general is not without loss of generality, is to replace the IC by its FOC:

$$\int u(q)\left(\frac{q-e}{\sigma^3}\right)\varphi\left(\frac{q-e}{\sigma}\right)dq - C'(e)\begin{cases}\geq 0 & \text{if } e=\bar{e}\\ =0 & \text{if } e\in(0,\bar{e})\\ \leq 0 & \text{if } e=0\end{cases}, \tag{54}$$

where we used the fact that $\varphi'(q) = -x\varphi(q)$, so that $\frac{d}{de}\left[\varphi\left(\frac{q-e}{\sigma}\right)\right] = \frac{q-e}{\sigma^2}\cdot\varphi\left(\frac{q-e}{\sigma}\right)$. Since replacing the IC by its FOC is not always valid, after solving the firm's relaxed program, we will need to verify that its solution satisfies the IC.

Writing in terms of the manager's indirect utility, the IR becomes

$$\int u(q)\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq - C(e) \geq 0. \tag{55}$$

It is convenient to multiply both sides of LL by $\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) > 0$, rewriting it as:

$$\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)u(q) \geq \frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)\ln\left(\bar{W}\right)\ \forall q. \tag{56}$$

The principal's *relaxed program* is:

$$\max_{u(\cdot),e}\int\{q-\exp[u(q)]\}\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq,$$

subject to (54), (55) and (56).

As in Grossman and Hart (1983), we break down this program in two parts. First, we consider the solution of the relaxed program holding each effort $e\in[0,\bar{e}]$ fixed:

$$\min_{u(\cdot)}\int\exp[u(q)]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq,$$

subject to (54), (55) and (56).

The optimal contract to implement the lowest effort ($e^* = 0$) pays a fixed wage. The utility given to the manager is set at the lowest level that still satisfies both LL and IR: $u(q) = \max\{\ln\left(\bar{W}\right),\ C(0)\}$ for all q. To see this, notice that a constant utility $u(q) = u^*$ always satisfies (54):

$$\int u^*\frac{q}{\sigma^3}\varphi\left(\frac{q}{\sigma}\right)dq - C'(0) = \frac{u^*}{\sigma^3}\times\int q\varphi\left(\frac{q}{\sigma}\right)dq - C'(0) = -C'(0)\leq 0.$$

Lemma 5 obtains the solution of the relaxed program for $e^* > 0$.

Lemma 5 *The optimal contract that implements $e^* > 0$ in the relaxed program is:*

$$w(q) = \frac{\lambda}{\sigma^2} \cdot \max\{q - q^*,\ 0\},$$

where $q^* \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}$ (with equality if the IR does not bind).

Proof. The (infinite-dimensional) Lagrangian gives the following FOC:

$$-\exp[u(q)]\frac{1}{\sigma}\varphi\left(\frac{q-e^*}{\sigma}\right)+\lambda\left(\frac{q-e^*}{\sigma^3}\right)\varphi\left(\frac{q-e^*}{\sigma}\right)+\mu_{IR}\frac{1}{\sigma}\varphi\left(\frac{q-e^*}{\sigma}\right)+\mu_{LL}(q)\frac{1}{\sigma}\varphi\left(\frac{q-e^*}{\sigma}\right)=0,$$

where λ is the multiplier associated with (54), and μ_{LL} and μ_{IR} are the multipliers associated with (55) and (56). Since the program corresponds to the minimization of a strictly convex function subject to linear constraints, the FOC above, along with the standard complementary slackness conditions and the constraints, are sufficient for an optimum. Substitute $\exp[u(q)] = W(q)$ and simplify the FOC above to obtain:

$$W(q) = \lambda\frac{q-e^*}{\sigma^2} + \mu_{IR} + \mu_{LL}(q).$$

Suppose first that the IR does not bind, so that $\mu_{IR} = 0$. Then, the FOC becomes

$$W(q) = \lambda\frac{q-e^*}{\sigma^2} + \mu_{LL}(q).$$

For $W(q) > \bar{W}$, complementary slackness gives $\mu_{LL}(q) = 0$, so that:

$$W(q) = \lambda \times \frac{q-e^*}{\sigma^2},$$

which exceeds $\bar{W}$ if and only if

$$\lambda \times \frac{q-e^*}{\sigma^2} > \bar{W} \iff q > e^* + \frac{\sigma^2\bar{W}}{\lambda} \equiv q^*.$$

For $W(q) = \bar{W}$, the FOC becomes:

$$\bar{W} = \lambda\frac{q-e^*}{\sigma^2} + \mu_{LL}(q) \therefore \mu_{LL}(q) = \bar{W} - \lambda\frac{q-e^*}{\sigma^2},$$

so that $\mu_{LL}(q) \geq 0$ if and only if

$$\bar{W} \geq \lambda \times \frac{q - e^*}{\sigma^2} \iff q \leq q^*.$$

Therefore, the optimal contract is

$$W(q) = \max\left\{\frac{\lambda\left(q - e^*\right)}{\sigma^2},\ \bar{W}\right\} = \begin{cases} \frac{\lambda(q-e^*)}{\sigma^2} & \text{if } q \geq q^* \\ \bar{W} & \text{if } q \leq q^* \end{cases}.$$

Writing in terms of the firm's payments, we have

$$w(q) = W(q) - \bar{W} = \frac{\lambda}{\sigma^2} \max\left\{q - q^*,\ 0\right\},$$

where the last equality uses the definition of q^*. The firm gives the manager an option with strike price $q^* = e^* + \frac{\sigma^2}{\lambda}\bar{W} > e^*$ and a slope $\frac{\lambda}{\sigma^2}$ chosen so that (54) holds (which can be shown to exist and be unique).

Next, suppose that the IR binds so that $\mu_{IR} \geq 0$. Then, for $W(q) > \bar{W}$, we must have

$$W(q) = \lambda \frac{q - e^*}{\sigma^2} + \mu_{IR},$$

so that

$$W(q) > \bar{W} \iff \mu_{IR} > \bar{W} - \lambda \frac{q - e^*}{\sigma^2}.$$

For $W(q) = \bar{W}$, we have:

$$\bar{W} = \lambda \frac{q - e^*}{\sigma^2} + \mu_{IR} + \mu_{LL}(q),$$

so that $\mu_{LL}(q) \geq 0$ if and only if

$$\mu_{LL}(q) = \bar{W} - \lambda \frac{q - e^*}{\sigma^2} - \mu_{IR} \geq 0$$

$$\iff \bar{W} - \lambda \frac{q - e^*}{\sigma^2} \geq \mu_{IR}.$$

Define the strike price q^* as the solution to

$$\bar{W} - \lambda \frac{q^* - e^*}{\sigma^2} = \mu_{IR},$$

that is,

$$q^* \equiv e^* + \frac{\sigma^2}{\lambda}\left(\bar{W} - \mu_{IR}\right) \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}.$$

Combining both conditions, we obtain

$$W(q) = \begin{cases} \frac{\lambda}{\sigma^2}(q - q^*) + \bar{W} & \text{if } q \geq q^* \\ \bar{W} & \text{if } q \leq q^* \end{cases},$$

which again corresponds to an option with strike price q^* and slope $\frac{\lambda}{\sigma^2}$. Here, λ and q^* are chosen so that both (54) and (55) hold with equality. ■

Lemma 6 gives an upper bound on λ:

Lemma 6 *Suppose $e^* > 0$ is the effort that solves the firm's relaxed program. Then the optimal contract is*

$$w(q) = \max\left\{\frac{\lambda}{\sigma^2}(q - q^*),\ 0\right\},$$

where $0 < \lambda < \sqrt{2\pi}\sigma e^*$ and $q^* \leq e^* + \frac{\sigma^2}{\lambda}\bar{W}$.

Proof. From Lemma 5, we need to show that $\lambda \leq \sqrt{2\pi}\sigma e^*$. Recall that the optimal contract that implements effort $e > 0$ is the option:

$$w(q) = \max\left\{\frac{\lambda}{\sigma^2}(q - q^*),\ 0\right\},$$

where $q^* \leq \frac{\sigma^2}{\lambda}\bar{W} + e$. Since the firm's net profits $q - w(q)$ are increasing in the strike price q^* (holding constant all other variables, including effort), its profits are bounded above by the profits from offering the option with the highest strike price ($\bar{q} = \frac{\sigma^2}{\lambda}\bar{W} + e \geq q^*$), which equal

$$e - \left[\frac{\lambda}{\sigma^2}\int_{\frac{\sigma^2}{\lambda}\bar{W}+e}^{\infty}\left(q - e - \frac{\sigma^2}{\lambda}\bar{W}\right)\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq\right].$$

Let $z \equiv q - e - \frac{\sigma^2}{\lambda}\bar{W}$, so that $q = z + e + \frac{\sigma^2}{\lambda}\bar{W}$. Note that $q \geq \frac{\sigma^2}{\lambda}\bar{W} + e$ if and only if $z \geq 0$. Thus, we can rewrite this expression as

$$e - \left[\frac{\lambda}{\sigma^2}\int_0^{\infty} z\frac{1}{\sigma}\varphi\left(\frac{z + \frac{\sigma^2}{\lambda}\bar{W}}{\sigma}\right)dz\right].$$

Moreover, since $\varphi(z)$ is decreasing in z for $z > 0$, it follows that

$$\varphi\left(\frac{z + \frac{\bar{W}}{\lambda}}{\sigma}\right) < \varphi\left(\frac{z}{\sigma}\right) \quad \forall z > 0.$$

Thus, the firm's profits are strictly less than

$$e - \frac{\lambda}{\sigma^2}\int_0^\infty \frac{z}{\sigma}\varphi\left(\frac{z}{\sigma}\right)dz. \tag{57}$$

Apply the following change of variables $y = \frac{z}{\sigma}$ (so that $z = \sigma y$, $dz = \sigma dy$) to write

$$\int_0^\infty \frac{z}{\sigma}\varphi\left(\frac{z}{\sigma}\right)dz = \sigma\int_0^\infty y\varphi\left(y\right)dy.$$

Integrating by parts gives

$$\int_0^\infty y\varphi(y)dy = [-\varphi(y)]_0^\infty = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting into (57), the firm's profits are strictly less than

$$e - \frac{1}{\sqrt{2\pi}} \cdot \frac{\lambda}{\sigma}.$$

Since the firm can always obtain a profit of zero by paying zero wages and implementing zero effort, we must have

$$e - \frac{1}{\sqrt{2\pi}} \cdot \frac{\lambda}{\sigma} > 0 \iff \lambda < \sqrt{2\pi}\sigma e.$$

■

Lemma 7 provides an additional upper bound:

Lemma 7 *For any $q^* \in \mathbb{R}$, $e \in [0, \bar{e}]$, $\sigma > 0$ and $\lambda > 0$, we have*

$$\int \ln\left(\bar{W} + \frac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right)\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq$$

$$\leq \int\left[\bar{W} + \frac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right]\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right)dq.$$

Proof. For notational simplicity, let $y \equiv \frac{q^*-e}{\sigma}$, apply the change of variables $z \equiv \frac{q-e}{\sigma}$, and let

$$g(z) \equiv \bar{W} + \frac{\lambda}{\sigma} \cdot \max\{z-y,\ 0\} - \ln\left(\bar{W} + \frac{\lambda}{\sigma} \cdot \max\{z-y,\ 0\}\right).$$

Then, the inequality in Lemma 7 can be written as

$$\int_{-\infty}^{\infty} g(z)\left(z^2-1\right)\varphi(z)\,dz \geq 0.$$

We claim that $g(\cdot)$ is non-decreasing. To see this, note that, for $z \leq y$, $g(z) = \bar{W} - \ln(\bar{W})$ (which is constant in z). For $z > y$, we have

$$g'(z) = \frac{\lambda}{\sigma}\left(\frac{\bar{W} - 1 + \frac{\lambda}{\sigma}(z-y)}{\bar{W} + \frac{\lambda}{\sigma}(z-y)}\right),$$

which is positive for all $z > y$ since $\bar{W} \geq 1$. Since g is non-decreasing, we have $g(q) \geq g(-q)$ for $q \geq 0$ and $\frac{d}{dq}[g(q) - g(-q)] \geq 0$. Note that, applying the change of variables $\tilde{z} = -z$ and using the symmetry of $(z^2-1)\varphi(z)$ around zero, we have:

$$\int_{-\infty}^{0} g(z)\left(z^2-1\right)\varphi(z)\,dz = -\int_{0}^{\infty} g(-z)\left(z^2-1\right)\varphi(z)\,dz. \tag{58}$$

Therefore,

$$\begin{aligned}
\textstyle\int g(z)\,(z^2-1)\,\varphi(z)\,dz &= \textstyle\int_{-\infty}^{0} g(z)\,(z^2-1)\,\varphi(z)\,dz + \int_0^{\infty} g(z)\,(z^2-1)\,\varphi(z)\,dz \\
&= \textstyle -\int_0^{\infty} g(-z)\,(z^2-1)\,\varphi(z)\,dz + \int_0^{\infty} g(z)\,(z^2-1)\,\varphi(z)\,dz \\
&= \textstyle\int_0^{\infty}[g(z)-g(-z)]\,(z^2-1)\,\varphi(z)\,dz \\
&= \textstyle\int_0^{1}[g(z)-g(-z)]\,(z^2-1)\,\varphi(z)\,dz + \int_1^{\infty}[g(z)-g(-z)]\,(z^2-1)\,\varphi(z)\,dz \\
&\geq \textstyle\int_0^{1}[g(1)-g(-1)]\,(z^2-1)\,\varphi(z)\,dz + \int_1^{\infty}[g(1)-g(-1)]\,(z^2-1)\,\varphi(z)\,dz \\
&= \textstyle[g(1)-g(-1)]\int_0^{\infty}(z^2-1)\,\varphi(z)\,dz = 0,
\end{aligned}$$

where the first line opens the integral between positive and negative values of z, the second line substitutes (58), the third line combines the terms from the two integrals, and the fourth line opens the integral between $z \leq 1$ and $z \geq 1$. The fifth line is the crucial step, which uses the following two facts: (i) $z^2 > (<) 1$ for $z > (<) 1$, and (ii) $g(z) - g(-z)$ is non-decreasing for all z. Therefore, substituting $g(z) - g(-z)$ by its upper bound where the term inside the integral is negative, and by its lower bound where it is positive, lowers the value of the integrand. The sixth line combines terms and uses the fact that

$$\int_0^{\infty} \left(z^2 - 1\right) \varphi\left(z\right) dz = \left[-z\varphi(z)\right]_0^{\infty} = 0.$$

■

Lemma 8 shows that the solution of the relaxed program also solves the firm's program if the effort cost is sufficiently convex, i.e. the FOA is valid.

Lemma 8 *Suppose $C''(e) \geq \frac{\bar{e}}{\sigma}$ for all $e \in [0, \bar{e}]$. Then, the solution of the firm's program coincides with the solution of the relaxed program.*

Proof. The manager's utility from choosing effort e is:

$$U(e;\ q^*, \lambda) \equiv \int \ln\left(\bar{W} + \frac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right) \frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq - C(e).$$

We know from previous results that $0 < \lambda < \sqrt{2\pi}\sigma e^*$. The FOA is valid if

$$\frac{\partial^2 U}{\partial e^2}(e;\ q^*, \lambda) \leq 0$$

for all $e \in [0, \bar{e}]$, all $q^* \in \mathbb{R}$, and $\lambda \in (0, \sqrt{2\pi}\sigma\bar{e})$. Differentiating gives

$$\begin{aligned} \frac{\partial^2 U}{\partial e^2} &= \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right) \tfrac{1}{\sigma} \tfrac{d^2}{de^2}\left[\varphi\left(\tfrac{q-e}{\sigma}\right)\right] dq - C''(e) \\ &= \tfrac{1}{\sigma^2} \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right) \tfrac{1}{\sigma}\left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right] \varphi\left(\tfrac{q-e}{\sigma}\right) dq - C''(e) \end{aligned}, \tag{59}$$

where the second line uses the fact that $\frac{d^2}{de^2}\left[\varphi\left(\frac{q-e}{\sigma}\right)\right] = \frac{1}{\sigma^2}\left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right]\varphi\left(\frac{q-e}{\sigma}\right)$. Note that

$$\begin{aligned} & \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right) \left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right] \tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq \\ \leq\ & \int \left[\bar{W} + \tfrac{\lambda}{\sigma^2} \cdot \max\left\{(q - q^*)\,,\ 0\right\}\right] \left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right] \tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq \\ =\ & \tfrac{\lambda}{\sigma^2} \cdot \int \left[\max\left\{(q - q^*)\,,\ 0\right\}\right] \left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right] \tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq \\ =\ & \tfrac{\lambda}{\sigma^2} \cdot \int_{q^*}^{\infty} (q - q^*) \left[\left(\tfrac{q-e}{\sigma}\right)^2 - 1\right] \tfrac{1}{\sigma}\varphi\left(\tfrac{q-e}{\sigma}\right) dq, \end{aligned}$$

where the inequality uses Lemma 7, the third line follows from $\int \left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right] \frac{1}{\sigma}\varphi\left(\frac{q-e}{\sigma}\right) dq = 0$ (a standard normal variable has variance 1), and the fourth line opens the max operator. Substituting in the expression from (59), we obtain the following sufficient condition for the validity of the FOA:

$$\frac{\lambda}{\sigma^4} \cdot \int_{q^*}^{\infty} (q-q^*) \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq \leq C''(e) \tag{60}$$

for all $e \in [0, \bar{e}]$, $q^* \in \mathbb{R}$, and $\lambda \in (0, \sqrt{2\pi}\sigma\bar{e})$.

Let $\xi(q^*) \equiv \int_{q^*}^{\infty} (q-q^*) \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq$. We claim that $\xi'(q^*) \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0 \iff q^* \left\{ \begin{matrix} < \\ > \end{matrix} \right\} e$. Differentiating yields:

$$\xi'(q^*) = -\int_{q^*}^{\infty} \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq. \tag{61}$$

Note that

$$\frac{d}{dq}\left[-\left(\frac{q-e}{\sigma} \right) \varphi \left(\frac{q-e}{\sigma} \right) \right] = -\frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) - \left(\frac{q-e}{\sigma} \right) \frac{1}{\sigma} \varphi' \left(\frac{q-e}{\sigma} \right) = \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right),$$

where the last equality uses the fact that $\varphi'(q) = -q\varphi(q)$. Therefore,

$$\int \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq = -\left(\frac{q-e}{\sigma} \right) \varphi \left(\frac{q-e}{\sigma} \right).$$

Substituting back into (61) gives:

$$\xi'(q^*) = -\left(\frac{q^*-e}{\sigma} \right) \varphi \left(\frac{q^*-e}{\sigma} \right) \left\{ \begin{matrix} > \\ < \end{matrix} \right\} 0 \iff q^* \left\{ \begin{matrix} < \\ > \end{matrix} \right\} e.$$

Therefore, $\xi(\cdot)$ is maximized at $q^* = e$, so that, by condition (60), it suffices to show that

$$\frac{\lambda}{\sigma^4} \cdot \xi(e) \leq C''(e). \tag{62}$$

Evaluating ξ at e gives:

$$\xi(e) = \int_{e}^{\infty} (q-e) \left[\left(\frac{q-e}{\sigma} \right)^2 - 1 \right] \frac{1}{\sigma} \varphi \left(\frac{q-e}{\sigma} \right) dq.$$

Performing the change of variables $z \equiv \frac{q-e}{\sigma}$, we obtain

$$\xi(e) = \int_e^{\infty} \left(\frac{q-e}{\sigma}\right) \left[\left(\frac{q-e}{\sigma}\right)^2 - 1\right] \varphi\left(\frac{q-e}{\sigma}\right) dq = \sigma \int_0^{\infty} z\left(z^2-1\right)\varphi\left(z\right) dz. \tag{63}$$

Integrating by parts gives

$$\int z\left(z^2-1\right)\varphi\left(z\right) dz = -z^2\varphi(z) + \int z\varphi\left(z\right) dz,$$

where we let $(z^2-1)\varphi(z)\,dz = dv$ so that $v = -z\varphi(z)$, and we let $u = z$, so that $du = dz$. Therefore

$$\int_0^{\infty} z\left(z^2-1\right)\varphi\left(z\right) dz = \int_0^{\infty} z\varphi\left(z\right) dz.$$

Using $\frac{d}{dz}\left[-\varphi(z)\right] = z\varphi\left(z\right)$, we have

$$\int_0^{\infty} z\left(z^2-1\right)\varphi\left(z\right) dz = \left[-\varphi(z)\right]_0^{+\infty} = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting into (63), yields

$$\xi(e) = \frac{\sigma}{\sqrt{2\pi}}.$$

Substituting into (62), we obtain the following sufficient condition:

$$\frac{\lambda}{\sqrt{2\pi}\sigma^3} \leq C''(e),$$

which is true for all $e \in [0, \bar{e}]$ and all $\lambda \in (0, \sqrt{2\pi}\sigma\bar{e})$ if and only if

$$C''(e) \geq \frac{\bar{e}}{\sigma^2} \quad \forall e \in [0, \bar{e}].$$

■

Proposition 5 provides a sufficient condition for the FOA with an additional performance signal, for a subset of signal distributions.

Proposition 5 *We consider the same setting as in Proposition 2, and a signal distribution such that: (i) $h_s''(e) \leq 0$ for all s; (ii) ϕ_e^s linear in e for all s; (iii) $h_{s_1}(e) \leq h_{s_2}(e)$, $h_{s_1}'(e) \leq h_{s_2}'(e)$, and $\sigma_{s_1} \geq \sigma_{s_2}$ for any s_1, s_2 with $\frac{d\phi_e^{s_1}}{de} > 0 > \frac{d\phi_e^{s_2}}{de}$ and any $e \in [0, \bar{e}]$. Then the FOA is valid if $C''(e) \geq \sum_s \phi_{e^*}^s h_s\left(\bar{e}\right) \frac{\sum_s \frac{\phi_e^s}{\sigma_s^3}\left(h_s'(e)\right)^2}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}$ for all $e \in [0, \bar{e}]$.*

Proof of Proposition 5: Let φ denote the PDF of the standard normal distribution. Let $W_s(q) := \bar{W} + w_s(q)$ denote the manager's consumption (i.e., the manager's initial wealth $\bar{W}$ plus his pay). His IC, IR, and LL are, respectively:

$$
\begin{aligned}
& e \in \arg\max_{\hat{e}\in[0,\bar{e}]} \sum_s \phi_e^s \int \ln\left[W_s\left(q\right)\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(\hat{e}\right)}{\sigma_s}\right) dq - C(\hat{e}), \\
& \sum_s \phi_e^s \int \ln\left(W_s\left(q\right)\right) \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right) dq - C(e) \geq 0, \\
& W_s\left(q\right) \geq \bar{W} \ \forall q, s.
\end{aligned}
$$

To simplify notation, we will work with the manager's indirect utility, $u_s(q) := \ln\left(W_s\left(q\right)\right)$, so that $W_s(q) = \exp[u_s(q)]$. This step is without loss of generality. The next step, which in general is not without loss of generality, is to replace the IC by its FOC:

$$
\sum_s \phi_e^s \int u_s\left(q\right) \frac{q-h_s\left(e\right)}{\sigma_s^3}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right) dq - C'(e) \begin{cases} \geq 0 & \text{if } \ e=\bar{e} \\ =0 & \text{if } \ e\in(0,\bar{e}) \\ \leq 0 & \text{if } \ e=0 \end{cases}. \tag{64}
$$

Since replacing the IC by its FOC is not always valid, after solving the firm's relaxed program, we will need to verify that its solution satisfies the IC. It is convenient to multiply both sides of LL by $\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\phi_e^s > 0$, rewriting it as:

$$
\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right)\phi_e^s u_s\left(q\right) \geq \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right)\phi_e^s \ln\left(\bar{W}\right) \ \ \forall q, s. \tag{65}
$$

The principal's *relaxed program* is:

$$
\max_{\{u_s(q)\}_{q,s},e} \sum_s \phi_e^s \int \left\{q - \exp\left[u_s(q)\right]\right\} \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right) dq
$$

subject to (64), (65), and

$$
\sum_s \phi_e^s \int u_s\left(q\right) \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right) dq - C(e) \geq 0. \tag{66}
$$

As in Grossman and Hart (1983), we break down this program in two parts. First, we consider the solution of the relaxed program holding each effort $e \in [0, \bar{e}]$ fixed:

$$\min_{u(\cdot)} \sum_s \phi_e^s \int \exp\left[u_s(q)\right] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq$$

subject to (64), (65), and (66).

The optimal contract to implement the lowest effort ($e^* = 0$) pays a fixed wage. The utility given to the manager is set at the lowest level that still satisfies both the LL and IR: $u_s(q) = \max\{\ln\left(\bar{W}\right),\ C(0)\}$ for all q, s.

Lemma 9 obtains the solution of the relaxed program for $e^* > 0$.

Lemma 9 *The optimal contract that implements $e^* > 0$ in the relaxed program is:*

$$w_s(q) = \frac{\lambda}{\sigma_s^2} \cdot \max\left\{q - q_s^*, 0\right\},$$

where $q_s^* \leq \sigma_s^2 \frac{\bar{W}}{\lambda} + h_s(e^*)$ (with equality if the IR does not bind).

Proof. The (infinite-dimensional) Lagrangian associated with this program is:

$$\begin{aligned}
&\sum_s \phi_e^s \int \exp\left[u_s(q)\right] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) dq \\
&+\lambda \left[\sum_s \phi_e^s \int u_s(q) \frac{q - h_s(e^*)}{\sigma_s^3} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) dq - C'(e^*)\right] \\
&+\mu_{IR} \left[\sum_s \phi_e^s \int u_s(q) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) dq - C(e^*)\right] \\
&+\mu_{LL}(q, s) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s u_s(q).
\end{aligned}$$

The FOC is:

$$-\exp\left[u_s(q)\right] \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s + \lambda \frac{q - h_s(e^*)}{\sigma_s^3} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s$$

$$+\mu_{IR} \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s + \mu_{LL}(q, s) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e^*)}{\sigma_s}\right) \phi_e^s = 0,$$

where λ is the multiplier associated with (64), and μ_{LL} and μ_{IR} are the multipliers associated with (65) and (66). Since the program corresponds to the minimization of a strictly convex function subject to linear constraints, the FOC above, along with the standard complementary slackness conditions and the constraints, are sufficient for an optimum. Substitute $\exp\left[u_s(q)\right] = W_s(q)$ and simplify the FOC above to obtain:

$$W_s(q) = \lambda \frac{q - h_s(e^*)}{\sigma_s^2} + \mu_{IR} + \mu_{LL}(q, s).$$

By complementary slackness, we must have $\mu_{IR} \geq 0$ (with $\mu_{IR} = 0$ if IR does not bind). Similarly, $\mu_{LL}(q) \geq 0$ with equality if $W_s(q) > \bar{W}$. Thus, for $W_s(q) > \bar{W}$, we must have

$$W_s(q) = \lambda \frac{q - h_s(e^*)}{\sigma_s^2} + \mu_{IR} > \bar{W},$$

which can be rearranged as

$$q > \sigma_s^2 \frac{\bar{W} - \mu_{IR}}{\lambda} + h_s(e^*) =: q_s^*.$$

For $W_s(q) = \bar{W}$, we must have

$$\mu_{LL}(q, s) = \bar{W} - \lambda \frac{q - h_s(e^*)}{\sigma_s^2} - \mu_{IR} \geq 0 \iff q \leq q_s^*.$$

Combining both conditions, we obtain

$$W_s(q) = \max\left\{\lambda \frac{q - h_s(e^*)}{\sigma_s^2} + \mu_{IR}, \bar{W}\right\} = \bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\left\{q - q_s^*, 0\right\}.$$

Thus,

$$w_s(q) = \frac{\lambda}{\sigma_s^2} \cdot \max\left\{q - q_s^*, 0\right\}.$$

Finally, since $\mu_{IR} \geq 0$,

$$q_s^* = \sigma_s^2 \frac{\bar{W} - \mu_{IR}}{\lambda} + h_s(e^*) \leq h_s(e^*) + \sigma_s^2 \frac{\bar{W}}{\lambda},$$

with equality if IR does not bind (in which case, we have $\mu_{IR} = 0$). ■

Lemma 10 gives an upper bound on λ:

Lemma 10 *Suppose $e^* > 0$ is the effort that solves the firm's relaxed program. Then the* *optimal contract is*

$$w_s(q) = \max\left\{\frac{\lambda}{\sigma_s^2}\left(q - q_s^*\right),\ 0\right\},$$

where $0 < \lambda < \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s(e^*)}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s^2}}$ *and* $q_s^* \leq h_s\left(e^*\right) + \sigma_s^2 \frac{\bar{W}}{\lambda}$.

Proof. From Lemma 10, we need to show that $\lambda \leq \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s(e^*)}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s^2}}$. Recall that the optimal contract that implements effort $e^* > 0$ is the option:

$$w_s(q) = \max\left\{\frac{\lambda}{\sigma_s^2}\left(q - q_s^*\right),\ 0\right\},$$

where $q_s^* \leq h_s\left(e^*\right) + \sigma_s^2 \frac{\bar{W}}{\lambda}$. Since the firm's net profits $q - w(q)$ are increasing in the strike price q_s^* (holding constant all other variables, including effort), its profits are bounded above by the profits from offering the option with the highest strike price for each signal s ($\bar{q}_s = h_s\left(e^*\right) + \sigma_s^2 \frac{\bar{W}}{\lambda} \geq q_s^*$), which equal

$$\sum_s \phi_{e^*}^s h_s\left(e^*\right) - \sum_s \phi_{e^*}^s \left[\frac{\lambda}{\sigma_s^2}\int_{h_s(e^*)+\sigma_s^2\frac{\bar{W}}{\lambda}}^{\infty}\left(q - h_s\left(e^*\right) - \sigma_s^2\frac{\bar{W}}{\lambda}\right)\frac{1}{\sigma_s}\varphi\left(\frac{q - h_s\left(e^*\right)}{\sigma_s}\right)dq\right].$$

For each s, let $z \equiv q - h_s\left(e^*\right) - \sigma_s^2 \frac{\bar{W}}{\lambda}$, so that $q = z + h_s\left(e^*\right) + \sigma_s^2 \frac{\bar{W}}{\lambda}$. Note that $q \geq h_s\left(e^*\right) + \sigma_s^2 \frac{\bar{W}}{\lambda}$ if and only if $z \geq 0$. Thus, we can rewrite this expression as

$$\sum_s \phi_{e^*}^s h_s\left(e^*\right) - \sum_s \phi_{e^*}^s \left[\frac{\lambda}{\sigma_s^2}\int_0^{\infty} z\frac{1}{\sigma_s}\varphi\left(\frac{z + \frac{\sigma_s^2}{\lambda}\bar{W}}{\sigma_s}\right)dz\right].$$

Moreover, since $\varphi(z)$ is decreasing in z for $z > 0$, it follows that, for any s,

$$\varphi\left(\frac{z + \sigma_s^2\frac{\bar{W}}{\lambda}}{\sigma_s}\right) < \varphi\left(\frac{z}{\sigma_s}\right) \quad \forall z > 0.$$

Thus, the firm's profits are strictly less than

$$\sum_s \phi_{e^*}^s h_s\left(e^*\right) - \sum_s \phi_{e^*}^s \frac{\lambda}{\sigma_s^2}\int_0^{\infty}\frac{z}{\sigma_s}\varphi\left(\frac{z}{\sigma_s}\right)dz. \tag{67}$$

Apply the following change of variables $y = \frac{z}{\sigma_s}$ (so that $z = \sigma_s y$, $dz = \sigma_s dy$) to write

$$\int_0^\infty \frac{z}{\sigma_s}\varphi\left(\frac{z}{\sigma_s}\right) dz = \sigma_s \int_0^\infty y\varphi\left(y\right) dy.$$

Integrating by parts gives

$$\int_0^\infty y\varphi(y)dy = \left[-\varphi(y)\right]_0^\infty = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting into (67), the firm's profits are strictly less than

$$\sum_s \phi_{e^*}^s h_s\left(e^*\right) - \sum_s \phi_{e^*}^s \frac{1}{\sqrt{2\pi}} \cdot \frac{\lambda}{\sigma_s}.$$

Since the firm can always obtain a profit of zero by paying zero wages and implementing zero effort, we must have

$$\sum_s \phi_{e^*}^s h_s\left(e^*\right) - \frac{\lambda}{\sqrt{2\pi}} \sum_s \frac{\phi_{e^*}^s}{\sigma_s} > 0 \iff \lambda < \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s\left(e^*\right)}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}.$$

■

Lemma 11 provides an additional upper bound:

Lemma 11 *For any* $q_s^* \in \mathbb{R}\,\forall s$, $e \in [0, \bar{e}]$, $e^* \in [0, \bar{e}]$, $\sigma_s > 0\,\forall s$, *and* $\lambda > 0$, *we have*

$$\begin{aligned} &\sum_s \phi_e^s \int \ln\left(\bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\left\{q - q_s^*,\ 0\right\}\right) \left(h_s'(e)\right)^2 \left[\frac{\left(q - h_s\left(e\right)\right)^2}{\sigma_s^4} - \frac{1}{\sigma_s^2}\right] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s\left(e\right)}{\sigma_s}\right) dq \\ \leq\ &\sum_s \phi_e^s \int \left[\bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\left\{q - q_s^*,\ 0\right\}\right] \left(h_s'(e)\right)^2 \left[\frac{\left(q - h_s\left(e\right)\right)^2}{\sigma_s^4} - \frac{1}{\sigma_s^2}\right] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s\left(e\right)}{\sigma_s}\right) dq. \end{aligned}$$

Proof. For notational simplicity, for each s, let $y_s := \frac{q_s^* - h_s(e)}{\sigma_s}$, apply the change of variables $z_s := \frac{q - h_s(e)}{\sigma_s}$, and let

$$g_s(z) := \bar{W} + \frac{\lambda}{\sigma_s} \cdot \max\left\{z - y_s,\ 0\right\} - \ln\left(\bar{W} + \frac{\lambda}{\sigma_s} \cdot \max\left\{z - y_s,\ 0\right\}\right).$$

Then, the inequality in Lemma 11 can be written as

$$\sum_s \frac{\phi_e^s}{\sigma_s^3}(h_s'(e))^2 \int_{-\infty}^{\infty} g_s(z)\left(z^2-1\right)\varphi(z)\,dz \geq 0. \tag{68}$$

The terms ϕ_e^s, σ_s, and $(h_s'(e))^2$ are positive, so it remains to prove that this integral is positive. We claim that, for each s, $g_s(\cdot)$ is non-decreasing. To see this, notice that, for $z_s \leq y_s$, $g_s(z) = \bar{W} - \ln(\bar{W})$ (which is constant in z_s). For $z_s > y_s$, we have

$$g_s'(z) = \frac{\lambda}{\sigma_s}\left(\frac{\bar{W} - 1 + \frac{\lambda}{\sigma_s}(z-y)}{\bar{W} + \frac{\lambda}{\sigma_s}(z-y)}\right),$$

which is positive for all $z_s > y_s$ since $\bar{W} \geq 1$. Since g is non-decreasing, we have $g_s(q) \geq g_s(-q)$ for $q \geq 0$ and $\frac{d}{dq}\left[g_s(q) - g_s(-q)\right] \geq 0$. Note that, applying the change of variables $\tilde{z} = -z$ and using the symmetry of $(z^2-1)\varphi(z)$ around zero, we have:

$$\int_{-\infty}^{0} g_s(z)\left(z^2-1\right)\varphi(z)\,dz = -\int_{0}^{\infty} g_s(-z)\left(z^2-1\right)\varphi(z)\,dz. \tag{69}$$

Therefore,

$$\begin{aligned}
\int g_s(z)(z^2-1)\varphi(z)\,dz &= \int_{-\infty}^{0} g_s(z)(z^2-1)\varphi(z)\,dz + \int_0^\infty g_s(z)(z^2-1)\varphi(z)\,dz \\
&= -\int_0^\infty g_s(-z)(z^2-1)\varphi(z)\,dz + \int_0^\infty g_s(z)(z^2-1)\varphi(z)\,dz \\
&= \int_0^\infty [g_s(z) - g_s(-z)](z^2-1)\varphi(z)\,dz \\
&= \int_0^1 [g_s(z) - g_s(-z)](z^2-1)\varphi(z)\,dz + \int_1^\infty [g_s(z) - g_s(-z)](z^2-1)\varphi(z)\,dz \\
&\geq \int_0^1 [g_s(1) - g_s(-1)](z^2-1)\varphi(z)\,dz + \int_1^\infty [g_s(1) - g_s(-1)](z^2-1)\varphi(z)\,dz \\
&= [g_s(1) - g_s(-1)]\int_0^\infty (z^2-1)\varphi(z)\,dz = 0,
\end{aligned}$$

where the first line opens the integral between positive and negative values of z, the second line substitutes (69), the third line combines the terms from the two integrals, and the fourth line opens the integral between $z \leq 1$ and $z \geq 1$. The fifth line is the crucial step, which uses the following two facts: (i) $z^2 > (<) 1$ for $z > (<) 1$, and (ii) $g_s(z) - g_s(-z)$ is non-decreasing for all z. Therefore, substituting $g_s(z) - g_s(-z)$ by its upper bound where the term inside the integral is negative, and by its lower bound where it is positive, lowers the value of the integrand. The sixth line combines terms and uses the fact that

$$\int_0^\infty \left(z^2-1\right)\varphi(z)\,dz = \left[-z\varphi(z)\right]_0^\infty = 0.$$

■

Lemma 12 shows that the solution of the relaxed program also solves the firm's program if the effort cost is sufficiently convex, i.e. the FOA is valid.

Lemma 12 *Suppose* $C''(e) \geq \sum_s \phi^s_{e^*} h_s(\bar{e}) \frac{\sum_s \frac{\phi^s_e}{\sigma^3_s}(h'_s(e))^2}{\sum_s \frac{\phi^s_{e^*}}{\sigma_s}}$ *for all* $e \in [0, \bar{e}]$*. Then, the solution of the firm's program coincides with the solution of the relaxed program.*

Proof. The manager's utility from choosing effort e is:

$$U(e; \{q^*_s\}, \lambda) := \sum_s \phi^s_e \int \ln\left(\bar{W} + \frac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \frac{1}{\sigma_s} \varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq - C(e).$$

We know from previous results that $0 < \lambda < \frac{\sqrt{2\pi} \sum_s \phi^s_{e^*} h_s(\bar{e})}{\sum_s \frac{\phi^s_{e^*}}{\sigma_s}}$. The FOA is valid if

$$\frac{\partial^2 U}{\partial e^2}(e; \{q^*_s\}, \lambda) \leq 0$$

for all $e \in [0, \bar{e}]$, all $q^*_s \in \mathbb{R}$, and $\lambda \in \left(0, \frac{\sqrt{2\pi} \sum_s \phi^s_{e^*} h_s(\bar{e})}{\sum_s \frac{\phi^s_{e^*}}{\sigma_s}}\right)$. Differentiating gives

$$\begin{aligned}
\frac{\partial^2 U}{\partial e^2} &= \sum_s \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \tfrac{1}{\sigma_s} \tfrac{d^2}{de^2}\left[\phi^s_e \varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right)\right] dq - C''(e) \\
&= \sum_s \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \tfrac{1}{\sigma_s} \tfrac{d}{de}\left[\tfrac{d\phi^s_e}{de} \varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right) + \phi^s_e \tfrac{d}{de}\varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right)\right] dq - C''(e) \\
&= \sum_s \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \tfrac{1}{\sigma_s}\left[\tfrac{d^2\phi^s_e}{de^2} \varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right) + 2\tfrac{d\phi^s_e}{de} \tfrac{d}{de}\varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right)\right. \\
&\quad \left. + \phi^s_e \tfrac{d^2}{de^2}\varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right)\right] dq - C''(e) \\
&= \sum_s \tfrac{d^2\phi^s_e}{de^2} \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \tfrac{1}{\sigma_s} \varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right) dq \\
&\quad + 2\sum_s \tfrac{d\phi^s_e}{de} \tfrac{h'_s(e)}{\sigma^2_s} \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \tfrac{q - h_s(e)}{\sigma_s} \varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right) dq \\
&\quad + \sum_s \tfrac{\phi^s_e}{\sigma^2_s} \int \ln\left(\bar{W} + \tfrac{\lambda}{\sigma^2_s} \cdot \max\{q - q^*_s,\ 0\}\right) \tfrac{1}{\sigma_s} \\
&\quad \times \left[(h'_s(e))^2 \left[\tfrac{(q - h_s(e))^2}{\sigma^2_s} - 1\right] + h''_s(e)(q - h_s(e))\right] \varphi\left(\tfrac{q - h_s(e)}{\sigma_s}\right) dq - C''(e)
\end{aligned}, \tag{70}$$

where the last equality uses the fact that

$$\frac{d^2}{de^2}\left[\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right)\right] = \frac{1}{\sigma^2_s}\left[(h'_s(e))^2 \left(\frac{(q - h_s(e))^2}{\sigma^2_s} - 1\right) + h''_s(e)(q - h_s(e))\right] \varphi\left(\frac{q - h_s(e)}{\sigma_s}\right).$$

First, with ϕ_e^s linear in e (assumption (ii) in Proposition 5), $\frac{d^2\phi_e^s}{de^2} = 0\,\forall s$, so that the first term on the RHS of (70) is zero.

Second, the second term on the RHS of (70) can be rewritten as:

$$\begin{aligned}
&2\sum_s \tfrac{d\phi_e^s}{de}\tfrac{h_s'(e)}{\sigma_s^2}\int \ln\left(\bar{W}+\tfrac{\lambda}{\sigma_s^2}\cdot\max\left\{q-q_s^*,\ 0\right\}\right)\tfrac{q-h_s(e)}{\sigma_s}\varphi\left(\tfrac{q-h_s(e)}{\sigma_s}\right)dq \\
&\quad= 2\sum_s \tfrac{d\phi_e^s}{de}\tfrac{h_s'(e)}{\sigma_s^2}\left[\int_{-\infty}^{q_s^*}\ln\left(\bar{W}\right)\tfrac{q-h_s(e)}{\sigma_s}\varphi\left(\tfrac{q-h_s(e)}{\sigma_s}\right)dq\right. \\
&\quad\left.+\int_{q_s^*}^{\infty}\ln\left(\bar{W}+\tfrac{\lambda}{\sigma_s^2}\left(q-q_s^*\right)\right)\tfrac{q-h_s(e)}{\sigma_s}\varphi\left(\tfrac{q-h_s(e)}{\sigma_s}\right)dq\right],
\end{aligned} \tag{71}$$

where $q_s^* = \sigma_s^2\frac{\bar{W}-\mu_{IR}}{\lambda} + h_s\left(e^*\right)$. For a given e, letting $\zeta_s := \frac{q-h_s(e)}{\sigma_s}$ and $\zeta_s^* := \frac{q_s^*-h_s(e)}{\sigma_s}$, we have:

$$\begin{aligned}
&\int \ln\left(\bar{W}+\frac{\lambda}{\sigma_s^2}\cdot\max\left\{q-q_s^*,\ 0\right\}\right)\frac{q-h_s\left(e\right)}{\sigma_s}\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right)dq \\
=\ &\int \ln\left(\bar{W}+\frac{\lambda}{\sigma_s}\cdot\max\left\{\zeta-\zeta_s^*,\ 0\right\}\right)\zeta\varphi\left(\zeta\right)d\zeta \\
=\ &\int_{-\infty}^{\zeta_s^*}\ln\left(\bar{W}\right)\zeta\varphi\left(\zeta\right)d\zeta + \int_{\zeta_s^*}^{\infty}\ln\left(\bar{W}+\frac{\lambda}{\sigma_s}\left(\zeta-\zeta_s^*\right)\right)\zeta\varphi\left(\zeta\right)d\zeta \geq 0,
\end{aligned} \tag{72}$$

where the inequality follows from $\bar{W} \geq 1$ and the symmetry of the normal distribution. This shows that, in equation (71), the term in brackets is increasing in $h_s(e)$ and in $h_s'(e)$, and decreasing in σ_s, all else equal. Note that, as $\sum_s \phi_e^s = 1\,\forall e$, we have $\sum_s \frac{d\phi_e^s}{de} = 0$, which implies

$$\sum_{s|\frac{d\phi_e^s}{de}>0}\frac{d\phi_e^s}{de} = -\sum_{s|\frac{d\phi_e^s}{de}<0}\frac{d\phi_e^s}{de}.$$

In sum, with assumption (iii), the expression in (71) is negative.

Third, we now show that the third term on the RHS of (70) is negative. With $\phi_e^s \geq 0$ and $\sigma_s > 0$ for all s, with $h_s''(e) \leq 0$ for all s (assumption (i)), and with equation (72), we have:

$$\sum_s \frac{\phi_e^s}{\sigma_s}h_s''(e)\int \ln\left(\bar{W}+\frac{\lambda}{\sigma_s^2}\cdot\max\left\{q-q_s^*,\ 0\right\}\right)\frac{q-h_s\left(e\right)}{\sigma_s}\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s\left(e\right)}{\sigma_s}\right)dq \leq 0. \tag{73}$$

Moreover:

$$
\begin{aligned}
& \sum_s \frac{\phi_e^s}{\sigma_s^2} \int \ln\left(\bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*,\ 0\}\right) \left[(h_s'(e))^2 \left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] + h_s''(e)\,(q - h_s(e))\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq \\
= & \sum_s \frac{\phi_e^s}{\sigma_s^2} \int \ln\left(\bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*,\ 0\}\right) (h_s'(e))^2 \left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq \\
& + \sum_s \frac{\phi_e^s}{\sigma_s} h_s''(e) \int \ln\left(\bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*,\ 0\}\right) \frac{q-h_s(e)}{\sigma_s} \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq \\
\leq & \sum_s \frac{\phi_e^s}{\sigma_s^2} (h_s'(e))^2 \int \left(\bar{W} + \frac{\lambda}{\sigma_s^2} \cdot \max\{q - q_s^*,\ 0\}\right) \left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq \\
= & \sum_s \lambda \frac{\phi_e^s}{\sigma_s^4} (h_s'(e))^2 \cdot \int \max\{q - q_s^*,\ 0\} \left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq \\
= & \sum_s \lambda \frac{\phi_e^s}{\sigma_s^4} (h_s'(e))^2 \cdot \int_{q_s^*}^{\infty} (q - q_s^*) \left[\frac{(q-h_s(e))^2}{\sigma_s^2} - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right) dq,
\end{aligned}
$$

where the first equality separates the sum into two components, the inequality that follows uses the result from Lemma 11 and equation (73), the next equality follows from

$$\int \left[\left(\frac{q - h_s(e)}{\sigma_s}\right)^2 - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq = 0,$$

(a standard normal variable has variance 1), and the last equality opens the max operator. Substituting in the expression from (70), we obtain the following sufficient condition for the validity of the FOA:

$$\lambda \sum_s \frac{\phi_e^s}{\sigma_s^4} (h_s'(e))^2 \cdot \int_{q_s^*}^{\infty} (q - q_s^*) \left[\left(\frac{q - h_s(e)}{\sigma_s}\right)^2 - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq \leq C''(e) \tag{74}$$

for all $e \in [0, \bar{e}]$, $q_s^* \in \mathbb{R}$, and $\lambda \in \left(0, \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s(\bar{e})}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}\right)$. Let

$$\xi_s(q_s^*) := \int_{q_s^*}^{\infty} (q - q_s^*) \left[\left(\frac{q - h_s(e)}{\sigma_s}\right)^2 - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq.$$

We claim that

$$\xi_s'(q_s^*) \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0 \iff q_s^* \left\{ \begin{array}{c} < \\ > \end{array} \right\} h_s(e). \tag{75}$$

Differentiating yields:

$$\xi_s'(q_s^*) = -\int_{q_s^*}^{\infty} \left[\left(\frac{q - h_s(e)}{\sigma_s}\right)^2 - 1\right] \frac{1}{\sigma_s}\varphi\left(\frac{q - h_s(e)}{\sigma_s}\right) dq. \tag{76}$$

Note that

$$\frac{d}{dq}\left[-\left(\frac{q-h_s(e)}{\sigma_s}\right)\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)\right] = -\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)-\left(\frac{q-h_s(e)}{\sigma_s}\right)\frac{1}{\sigma_s}\varphi'\left(\frac{q-h_s(e)}{\sigma_s}\right)$$
$$= \left[\left(\frac{q-h_s(e)}{\sigma_s}\right)^2-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right),$$

where the last equality uses the fact that $\varphi'(q) = -q\varphi(q)$. Therefore,

$$\int\left[\left(\frac{q-h_s(e)}{\sigma_s}\right)^2-1\right]\frac{1}{\sigma_s}\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq = -\left(\frac{q-h_s(e)}{\sigma_s}\right)\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right).$$

Substituting back into (76) gives:

$$\xi_s'(q_s^*) = -\left(\frac{q_s^*-h_s(e)}{\sigma_s}\right)\varphi\left(\frac{q_s^*-h_s(e)}{\sigma_s}\right)\left\{\begin{array}{c}>\\<\end{array}\right\}0 \iff q_s^*\left\{\begin{array}{c}<\\>\end{array}\right\}h_s(e).$$

Therefore, $\xi_s(\cdot)$ is maximized at $q_s^* = h_s(e)$, so that, by condition (74), it suffices to show that

$$\lambda\sum_s\frac{\phi_e^s}{\sigma_s^4}\left(h_s'(e)\right)^2\cdot\xi_s(h_s(e)) \le C''(e), \tag{77}$$

for all $e \in [0,\bar{e}]$ and $\lambda \in \left(0, \frac{\sqrt{2\pi}\sum_s\phi_{e^*}^s h_s(\bar{e})}{\sum_s\frac{\phi_{e^*}^s}{\sigma_s}}\right)$. Evaluating ξ_s at $h_s(e)$ gives:

$$\xi_s(h_s(e)) = \int_{h_s(e)}^{\infty}\frac{q-h_s(e)}{\sigma_s}\left[\left(\frac{q-h_s(e)}{\sigma_s}\right)^2-1\right]\varphi\left(\frac{q-h_s(e)}{\sigma_s}\right)dq.$$

Performing the change of variables $z_s \equiv \frac{q-h_s(e)}{\sigma_s}$, we obtain

$$\xi_s(h_s(e)) = \sigma_s\int_0^{\infty}z\left(z^2-1\right)\varphi(z)\,dz. \tag{78}$$

Integrating by parts gives

$$\int z\left(z^2-1\right)\varphi(z)\,dz = -z^2\varphi(z)+\int z\varphi(z)\,dz,$$

where we let $(z^2-1)\varphi(z)\,dz = dv$ so that $v = -z\varphi(z)$, and we let $u = z$, so that $du = dz$.

Therefore

$$\int_0^{\infty} z\left(z^2-1\right)\varphi\left(z\right)dz = \int_0^{\infty} z\varphi\left(z\right)dz.$$

Using $\frac{d}{dz}\left[-\varphi(z)\right] = z\varphi\left(z\right)$, we have

$$\int_0^{\infty} z\left(z^2-1\right)\varphi\left(z\right)dz = \left[-\varphi(z)\right]_0^{+\infty} = \varphi(0) = \frac{1}{\sqrt{2\pi}}.$$

Substituting into (78), yields

$$\xi_s(h_s(e)) = \frac{\sigma_s}{\sqrt{2\pi}}.$$

Substituting into (77), we obtain the following sufficient condition:

$$\frac{\lambda}{\sqrt{2\pi}}\sum_s \frac{\phi_e^s}{\sigma_s^3}\left(h_s'\left(e\right)\right)^2 \leq C''(e),$$

which is true for all $e \in [0, \bar{e}]$ and all $\lambda \in \left(0, \frac{\sqrt{2\pi}\sum_s \phi_{e^*}^s h_s(\bar{e})}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}}\right)$ if

$$\sum_s \phi_{e^*}^s h_s\left(\bar{e}\right)\frac{\sum_s \frac{\phi_e^s}{\sigma_s^3}\left(h_s'\left(e\right)\right)^2}{\sum_s \frac{\phi_{e^*}^s}{\sigma_s}} \leq C''(e) \;\; \forall e \in [0, \bar{e}].$$

■