Asset Pricing with Imperfect Information

by

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Chapter 1

Introduction

Imperfect information of the participants in financial market has been regarded as one of the important frictions which form the equilibrium in financial market. At the heart of the discussions regarding information-related frictions, there lies the revelation of private information through the price system. Early literature such as Milgrom and Stokey (1982) shows that there would be no trade in financial market unless there exist some noises which prevent the full revelation of private information. Therefore, a financial model with imperfect information typically involves the existence of noise in the trading volume which may stem from liquidity shock, investor sentiment, or any other factors unrelated to fundamentals of traded assets. Such a concept has been elegantly modeled as noisy rational expectation equilibrium models in Walrasian auctions (e.g., Grossman and Stiglitz (1980)) or market maker models (e.g., Kyle (1985)). A large volume of literature has been developed by building on these canonical frameworks of financial equilibrium with imperfect information, and these frameworks have shed light on many economic problems in financial markets.

The history of financial markets including the recent turmoils in financial markets has shown the fragility as well as the tendency of instability of financial markets. The canonical models, however, generally do not generate any instability due to its stabilizing force even in the presence of noises in the trading volume. I argue that it is because some of the important aspects regarding the source of the frictions are taken as given or assumed away for the simplicity of analysis. In this thesis, I attempt to verify how financial equilibrium can be easily distorted from the ones resulting from the canonical setups if some of the existing assumptions are relaxed. In particular, I study various new aspects of financial instability with imperfect information by examining the source of information-related frictions such as learning ability, information acquisition and noise trading. To achieve this goal, I first start my analysis by reviewing the existing models of financial markets as benchmark models. Chapter 2 studies two canonical frameworks in financial equilibrium with imperfect information: (i) Grossman and Stiglitz model, (ii) Kyle model. In this chapter, I explain a unifying framework which incorporates both types of models, and compare the equilibrium properties of these two types of models.

Traditional models of informed trading typically assume the existence of noise trading activities which generate pure random noise in trading volumes. Chapter 3 studies a multi-period model of speculative trading in the presence of a systematic component of the noise trading activities which is privately observed by a monopolistic risk-averse informed trader. Because of the incentive to hide the magnitude of informed trading, the informed trader's trades may comove with the mispricing caused by the systematic component of noise trading instead of engaging in arbitrage. The result implies that an arbitrageur who has superior information on non-fundamentals such as investor sentiment may not always reduce the mispricing caused by them given private information on fundamentals. The result demonstrates that market manipulation could easily occur in a standard Kyle model with relatively mild assumptions if private information has more than two dimensions including nonfundamental factors.

Chapter 4 develops a model of self-reinforcing financial fads in which traders' observational learning is constrained by partitions (or discrete categories). The traders only understand average behavior of prices over the partitions, but they are fully aware of such limitations. I assume that the partitions of the informed traders are finer than those of the uninformed traders. The informed traders may simultaneously shift their trading strategies depending on the arrival of certain price paths while the uninformed traders only recognize the possibility of such events. Unable to know the exact shift in the informed traders' strategies, the uninformed traders excessively extrapolate past returns. Consequently, feedback loops of price changes emerge because the uninformed traders' positive feedback trading gets amplified by its own feedback effects over time. Bubble-like price patterns arise intermittently due to feedback loops of price changes.

Chapter 5 studies a finite-horizon overlapping generations model where agents endogenously acquire information on a risky asset of which fundamental value fluctuates due to new fundamental shocks. Since the short-lived agents can not carry the asset until the liquidation of the long-lived risky asset, an early generation of agents does not engage in costly information acquisition unless the later generations engage in information acquisition consecutively until the liquidation like a chain. The result shows that fundamental shocks of high magnitude in the future may eliminate the incentive of acquiring information in the earlier periods, thereby breaking the 'information acquisition chain' from the earliest generation. Therefore, I explain in which circumstances the prices reflect available information slowly.

Chapter 2

A Review on Financial Equilibrium with Information Asymmetries

2.1 Motivation

Informational aspect of financial equilibrium has received much attention in the past three decades. In particular, two main frameworks have become canonical models in information asymmetries of financial market. First is Grossman and Stiglitz model (GS model) which I refer to the variations of the competitive market model used in one of seminal paper such as Grossman and Stiglitz (1980) (e.g. Grossman and Stiglitz (1980), Hellwig (1980), Admati (1985), Wang (1993), Wang (1994) and He and Wang (1995)). The second is Kyle model which I refer to the variations of the market microstructure model initially used in Kyle (1985) (e.g. Kyle (1985), Holden and Subrahmanyam (1992), Holden and Subrahmanyam (1994), Back, Cao, and Willard (2000), Foster and Viswanathan (1996) and Bernhaedt and Miao (2004)). Although these two different streams of models are widely used in many financial applications, the comparison between these two classes of models has not yet been fully attempted. In this chapter, I explain a unifying framework which incorporates both types of models, and compare the equilibrium properties of these two types of models.

The main difference between GS model and Kyle model is that investors are price-takers in

GS model while they are not in Kyle model. Therefore, the impact of one's own trade on the equilibrium price is incorporated in the investor's optimization problem in Kyle model. One of the main assumption which enables GS model to have price taking investors is that investors are infinitesimal so that there is no effect on the price through each of investor's trading activity. On the other hand, investors in Kyle model are not infinitesimal, thus they realize their trading activities have a significant effect on the price.

While the investors in GS model observe the price before choosing their demand, the investors in Kyle model choose their demand before observing the price. More precisely, the investors in GS model submits their whole demand schedule to a Walrasian auctioneer who clears the market according to demand schedules from all the investors in the economy. The investors in Kyle model submits an order flow to competitive market makers, then market makers sets the price to an appropriate price, which is usually equal to their expectation of the liquidation value of the traded asset. Therefore, the investors in Kyle model assume execution risks in trading while the investors in GS model do not.

Common feature of these two models is that uninformed investors in the economy (market makers in Kyle model) learn private information of informed investors through trades. The investors in GS model learn it through equilibrium prices whereas the investors in Kyle model learn it through equilibrium order flows. Also, both models assume the existence of noise traders or other source of noise to prevent equilibrium price or order flow from fully revealing private information. Otherwise, price or order flow becomes perfectly informative, thus there will be no trade in equilibrium (e.g. Milgrom and Stokey (1982)).

The chapter is organized as follows: In Section 2.2, I describe the common setting of both Kyle and GS model regarding investment opportunities, participants in the trading, and information structure. In Section 2.3, I explain GS model in an abstract level, and solve a simple example using the framework. In Section 2.4, I explain Kyle model in an abstract level, and solve a simple example using the framework. In Section 2.5, I compare GS model and Kyle model.

2.2 Setting

Consider an economy with T periods, $t = 1, 2, \ldots, T$. The economy contains two types of assets: one riskless asset and one risky assets. I set the return of riskless asset equal to unity for simplicity. I further assume that the supply of riskless asset is perfectly elastic.

Denote the set of all investors Λ. Consider an investor $i \in \Lambda$. The investor i has an initial wealth of W_0^i, and does not have any share of the risky asset or riskless asset initially. he has a constant absolute risk aversion (CARA) utility function, and maximizes his wealth at the final date T, i.e. $U(W_T^i) = -e^{\gamma W_T^i}$ where γ is a risk aversion parameter. Information set of investor i at date t is denoted by $\mathcal{F}_t^i$.

Consider a unobserved vector of variables ξ_t, which follows an autoregressive process such that

$$\xi_{t+1} = a_\xi \xi_t + \epsilon_{\xi,t+1}.$$

where $\epsilon_{\xi,t+1}$ is shocks to ξ_t at date t. For example, ξ_t may include the liquidation value of the risky asset, demand from noise traders, etc.

Define Ψ_t to be a state vector at date t which determines the excess return, and further assume that ξ_t determines Ψ_t. Ψ_t includes all the necessary information of ξ_t as well as other information such as investors' belief on ξ_t to determine the excess return. Also, denote ε_{t+1} to be a innovations to state vectors. Denote Ψ_t^i to be investor i's expectation of Ψ_t given his information set at date t, i.e. $\Psi_t^i \equiv E[\Psi_t|\mathcal{F}_t^i]$. Also, denote ε_t^i to be errors of investor i's expectation on Ψ_t. Further assume that ε_t^i follows normal distributions such that $\varepsilon^i \sim \mathcal{N}(0, \Sigma_t^i)$.

Investor i uses a linear filter described in Appendix A to forecast ξ_t by observing a vector y_t, which includes equilibrium price P_t as well as any private and public signals available to him:

$$E[\xi_t|\mathcal{F}_t^i] = E[\xi_t|\mathcal{F}_{t-1}^i] + K_t^i(y_t - E[y_t|\mathcal{F}_{t-1}^i])$$

where K_t^i is a matrix of proper order. As was mentioned earlier, it is essential to have some noise in the vector of signals y_t in order to prevent y_t from fully revealing private information on ξ_t. For example, Grossman and Stiglitz (1980) assumes a supply shock to total demand for the risky asset. Wang (1994) assumes a nontradable asset owned only by informed investors. Since its return is unobservable to uninformed investors, informed investors' demand becomes a noisy signal about the liquidation value of the risky asset.

2.3 Grossman and Stiglitz Model

In this section, I will define a further settings specific to GS model, and demonstrate the solution technique, which is an adaptation from Wang (1994). Suppose there are two groups of investors: informed investors (I), and uninformed investors (U). There are λ portion of informed investors, and $1-\lambda$ portion of uninformed investors in $\Lambda = \{I, U\}$. All investors are identical other than their private information. Also, suppose there are n state variables, i.e. Ψ_t is a n-vector.

- **Step 1. Assume an equilibrium price**

  Assume a linear equilibrium price function such that

$$P_t = \eta_t \Psi_t$$

  where η_t is a n-vector. by denoting ε_t^i to be errors between true Ψ_t and its conditional expectation Ψ_t^i, it could be also shown that $P_t = L(\Psi_t^i, \varepsilon_t^i)$ for some ε_t^i and for some linear function $L(.)$.

- **Step 2. Investors' learning problem and excess return**

  First, informed investors learn more about ξ_t by observing private signals available only to them. After informed investors update their belief, uninformed investors extract private information of informed traders from the equilibrium price. Uninformed investors update their belief using the sufficient statistic of P_t as well as all other available information, which constitutes y_t.

  In GS model, an investor submits a whole demand schedule of the risky asset to a Walrasian auctioneer. That is, it is equivalent to observe the price before ordering any amount of the risky asset at date t. Therefore, the excess return can be represented as $Q_{t+1} \equiv P_{t+1} - P_t$. Since price is a function of state vector Ψ_t, we can express Q_{t+1} as a function of state vector Ψ_t and innovations of state vectors ε_{t+1}.

$$Q_{t+1} = a_{Q,t+1}\Psi_t + b_{Q,t+1}\varepsilon_{t+1}, \tag{2.1}$$

where $a_{Q,t+1}, b_{Q,t+1}$ are matrix of constants in proper order. Since each investor has different information set, (2.1) could be rewritten for each investor i like the following:

$$Q_{t+1} = a^i_{Q,t+1}\Psi^i_t + b^i_{Q,t+1}\varepsilon^i_{t+1}.$$

- **Step 3. State process**

  One should verify that the state process follows an autoregressive process:

$$\Psi^i_{t+1} = a_{\Psi,t+1}\Psi^i_t + b_{\Psi,t+1}\varepsilon^i_{t+1}. \tag{2.2}$$

  where $a_{\Psi,t+1}, b_{\Psi,t+1}$ are matrix of constants in proper order.

- **Step 4. Investors' optimization problem**

  Investor i's problem can be formulated as the following:

$$\begin{aligned} \underset{X^i_t}{max} \quad & E\left[-e^{-\gamma W^i_T}\middle|\mathcal{F}^i_t\right] \\ \text{subject to} \quad & W^i_{t+1} = W^i_t + Q_{t+1}X^i_t, \end{aligned} \tag{2.3}$$

  where Q_{t+1} is the excess return at date t. Then, we can write the following Bellman equation which is equivalent to (2.3) like the following:

$$\begin{aligned} 0 = \underset{X_t}{Max} \quad & \{E[J(W^i_{t+1};\Psi^i_{t+1};t+1)|\mathcal{F}^i_t] - J(W^i_t;\Psi^i_t;t)\} \\ \text{subject to} \quad & W^i_{t+1} = W^i_t + Q_{t+1}X^i_t \\ & J(W^i_T;\Psi^i_T;T) = -e^{-\gamma W^i_T}. \end{aligned}$$

  Guess the value function has the form $J(W^i_t;\Psi^i_t;t) = -e^{-\gamma W^i_t - \frac{1}{2}\Psi^i_t\Omega_t\Psi^{i\top}_t}$ where Ω_T is a matrix which has all the elements being zero. The following lemma proves that this conjecture is in fact correct:

Lemma 2.1 *A risk averse investor i's optimal demand for the risky asset at date t is given by a linear function of state vector at trade date t:*

$$X_t^i = \frac{1}{\gamma} F_t \Psi_t^i \tag{2.4}$$

where $F_t \equiv (b_{Q,t+1}\Xi_{t+1}b_{Q,t+1}^\top)^{-1}(a_{Q,t+1} - b_{Q,t+1}\Xi_{t+1}b_{\Psi,t+1}^\top\Omega_{t+1}a_{\Psi,t+1})$ *and* $\Xi_{t+1} \equiv (\Sigma_t^{i\,-1} + b_{\Psi,t+1}\Omega_{t+1}b_{\Psi,t+1}^\top$.

Proof See He and Wang (1995) for the proof. ∎

- **Step 5. Market clearing**

  Finally, we solve a market clearing condition:

$$\omega X_t^I + (1-\omega)X_t^U = x_t \tag{2.5}$$

  where x_t is the net supply of the risky asset in the market.

- **Step 6. Solving equations**

  We should find η_t which makes (2.5) true for any realization of Ψ_t. Since we have n unknowns and n equations from (2.5), solving this equation system yields the equilibrium price function.

2.3.1 Grossman and Stiglitz Model Example

Using the tools suggested above, we will solve a simple GS model example equivalent to the one in Grossman and Stiglitz (1980). Consider an economy with two periods, $t = 1, 2$. Investors can trade assets at period 1, and receive its dividend at period 2. The risky asset yields a return $V = \theta + \epsilon_V$ where θ is privately observable to informed traders, and ϵ_V is unobservable. Assume that θ is privately observed by λ portion of informed investors. Further assume that there exists supply shocks to the risky asset at date t such that $x \sim \mathcal{N}(E[x], \sigma_x^2)$. Assume a linear equilibrium price:

$$P = \alpha_1 + \alpha_2 w_\lambda$$

where $w_\lambda = \theta - \mu(x - E[x])$. Since uninformed investors know α_1 and α_2, w_λ is a sufficient statistic of price P. Using Lemma A.1[1], uninformed investors' conditional expectation of θ is given by

$$E[\theta|w_\lambda] = E[\theta] + K^U(w_\lambda - E[\theta]), \tag{2.6}$$

where $K^U = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \mu^2\sigma_U^2}$, and the conditional variance is also given by

$$Var[\theta|w_\lambda] = (1 - K^U)\sigma_\theta^2..$$

Define $\Psi_t^I \equiv (1, w_\lambda, x - E[x])^\top, \varepsilon^I \equiv \epsilon_V$, and $\Psi_t^U \equiv (1, w_\lambda)^\top, \varepsilon^U \equiv (\theta - E[\theta|w_\lambda] + \epsilon_V)^\top$. We can easily observe that $Var[V|w_\lambda] = Var[\varepsilon^U|w_\lambda] = \sigma_\theta^2 + \sigma_\epsilon^2 - \lambda\sigma_\theta^2$. Using this definition, we can show the following is true:

Lemma 2.2 *For both informed and uninformed investors, the excess return is given by a linear function of state vector and shocks:*

$$Q = a_Q^i \Psi^i + b_Q^i \varepsilon^i$$

for $i \in \{I, U\}$.

Proof Since the excess return is $Q \equiv (\theta + \epsilon_V) - P = \theta + \epsilon_V - \alpha_1 - \alpha_2 w_\lambda$, the informed investors' expectation of excess return is given by

$$E[Q|\theta] \quad = \quad -\alpha_1 + (1 - \alpha_2)w_\lambda + \mu(x - E[x]).$$

[1]Set $A_t = 1, B_t = 0, H_t = 1, R_t = \mu^2\sigma_x^2$ in Lemma A.1

Therefore, Q is given by a linear function of Ψ^I and ε^I such that

$$Q \;=\; -\alpha_1 + (1-\alpha_2)w_\lambda + \mu(x - E[x]) + \epsilon_V = a_Q^I \Psi + b_Q^I \varepsilon^I,$$

where $a_Q^I \equiv (-\alpha_1, 1-\alpha_2, \mu)$ and $b_Q^I \equiv 1$. From (2.6), uninformed investors' conditional expectation of excess return is given by

$$E[Q|w_\lambda] \;=\; (1-\lambda)E[\theta] - \alpha_1 + (\lambda - \alpha_2)w_\lambda.$$

Therefore, Q is given by a linear function of Ψ^U and ε^U such that

$$Q \;=\; (1-\lambda)E[\theta] - \alpha_1 + (\lambda-\alpha_2)w_\lambda + (\theta - E[\theta|w_\lambda]) + \epsilon_V = a_Q^U \Psi^U + b_Q^U \varepsilon^U,$$

where $a_Q^I \equiv ((1-\lambda)E[\theta] - \alpha_1, \lambda - \alpha_2)$ and $b_Q^I \equiv 1$. $\blacksquare$

Since this is only two period model, we do not need to define the stochastic process of Ψ.[2] Using Lemma 2.1, we can easily derive

$$X^I = \frac{-\alpha_1 + (1-\alpha_2)w_\lambda + \mu(x - E[x])}{\gamma\sigma_\epsilon^2}$$
$$X^U = \frac{(1-\lambda)E[\theta] - \alpha_1 + (\lambda-\alpha_2)w_\lambda}{\gamma Var[V|w_\lambda]}$$

Market clearing condition is given by

$$\omega X^I + (1-\omega)X^U = x \tag{2.7}$$

[2]Since Ω_2 is a matrix of zeros, the state vector at date 2 does not affect the equilibrium.

Finally, we look for α_1, α_2 and μ which makes (2.7) always true for any realization of random variables θ, x. Therefore, (2.7) holds if the following is true:

$$\begin{aligned} \omega\frac{-\alpha_1}{\gamma\sigma_\epsilon^2} + (1-\omega)\frac{(1-\lambda)E[\theta]-\alpha_1}{\gamma Var[V|w_\lambda]} &= E[x] \\ \omega\frac{1-\alpha_2}{\gamma\sigma_\epsilon^2} + (1-\omega)\frac{\lambda-\alpha_2}{\gamma Var[V|w_\lambda]} &= 0 \\ \omega\frac{\mu}{\gamma\sigma_\epsilon^2} &= 1 \end{aligned}$$

Solving equation systems for α_1, α_2 and μ yields

$$\begin{aligned} \alpha_1 &= \frac{\frac{(1-\omega)(1-\lambda)E[\theta]}{\gamma Var[u|w_\lambda]} - E[x]}{\frac{\omega}{\gamma\sigma_\epsilon^2} + \frac{1-\omega}{\gamma Var[u|w_\lambda]}} \\ \alpha_2 &= \frac{\frac{\omega}{\gamma\sigma_\epsilon^2} + \frac{(1-\omega)\lambda}{\gamma Var[u|w_\lambda]}}{\frac{\omega}{\gamma\sigma_\epsilon^2} + \frac{1-\omega}{\gamma Var[u|w_\lambda]}} \\ \mu &= \frac{\gamma\sigma_\epsilon^2}{\omega}, \end{aligned}$$

which gives a consistent result with Grossman and Stiglitz (1980).

2.4 Kyle Model

Consider a model with a single monopolistic informed trader I, competitive risk neutral market makers M, and uninformed noise traders. In Kyle model, the informed investor does not observe the price before they submit their order. If we assume that the informed trader has perfect information on the liquidation value V and there is no borrowing constraint, the excess return at trade date t is in effect equivalent to $Q_{t+1} \equiv V - P_t$.

Let ΔX_t denote the investor's order flow for the risky asset at date t, and $X_t \equiv \sum_{\tau=1}^{t-1} \Delta X_\tau$ denote the demand from the investor i at date t where $X_0 = 0$, which means the initial position held by the informed trader is zero. There are also noise traders who submits order flow ΔU_t at date t.

- **Step 1. Assume informed trader's equilibrium order flow**

Assume a linear equilibrium order flow of an informed trader:

$$\Delta X_t = \eta_t \Psi_t$$

where η_t is a n-vector.

- **Step 2. Solve market makers' learning problem and excess return**

  Since market makers are risk-neutral and competitive, market makers set the price equal to their expectation of the liquidation value at date t, i.e. $P_t = E[V|\mathcal{F}_t^M]$. Using Lemma A.1, we derive

$$P_t = P_{t-1} + K_t^M (y_t - E[y_t])$$

  where K_t^M is a matrix of constant in proper order. One of the element in y_t is the aggregate order flow at date t, which is $\Delta X_t + \Delta U_t$. Then, $y_t - E[y_t]$ could be shown to be a linear combination of ΔX_t and Ψ_t. Therefore, the excess return at date t, $Q_{t+1} \equiv V - P_t$, is given by

$$Q_{t+1} = a_{Q,t+1} \Delta X_t + b_{Q,t+1} \Psi_t^I + c_{Q,t} \varepsilon_{t+1}^I \tag{2.8}$$

  where $a_{Q,t+1}, b_{Q,t+1}, c_{Q,t+1}$ are matrix of constants in proper order. Note that Ψ_t is perfectly observed by the informed trader, i.e. $\Psi_t = E[\Psi_t|\mathcal{F}_t^I]$.

- **Step 3. Derive state process**

  One should verify the state process follows an autoregressive process with an exogenous input ΔX_t:

$$\Psi_{t+1} = a_{\Psi,t+1} \Delta X_t + b_{\Psi,t+1} \Psi_t^I + c_{\Psi,t} \varepsilon_{t+1}^I \tag{2.9}$$

  Step 4. Solve informed trader's optimization problem

Investor i's problem can be formulated as the following:

$$\begin{aligned} \max_{\Delta X_t^i} \quad & E\left[-e^{-\gamma W_T^i}\middle|\mathcal{F}_t^i\right] \\ \text{subject to} \quad & W_{t+1}^i = W_t^i + Q_{t+1}\Delta X_t^i \end{aligned} \tag{2.10}$$

Therefore, we can solve the informed trader's optimization problem using the excess return function (2.8) and the state process (2.9). Using the same technique as in GS model, we can formulate the Bellman equation for the optimization problem (2.10). The solution for the optimization problem is derived according to the following lemma:

Lemma 2.3 *The risk averse informed investor's optimal order flow for the risky asset at date t is given by a linear function of state vector at trade date t:*

$$\Delta X_t = F_t \Psi_t \tag{2.11}$$

where $F_t \equiv \delta^{-1}\beta$, and where

$$\begin{aligned} \delta &= \gamma_{t+1}^2 c_{Q,t+1}\Xi_{t+1}c_{Q,t+1}^\top - 2\gamma_{t+1}a_{Q,t+1} - a_{\Psi,t+1}^\top \Omega_{t+1}a_{\Psi,t+1} \\ &\quad +(c_{\Psi,t+1}^\top \Omega_{t+1}a_{\Psi,t+1})^\top \Xi_{t+1}c_{\Psi,t+1}^\top \Omega_{t+1}a_{\Psi,t+1} + \; 2\gamma_{t+1}c_{Q,t+1}\Xi_{t+1}c_{\Psi,t+1}^\top \Omega_{t+1}a_{\Psi,t+1} \\ \beta &= \gamma_{t+1}(b_{Q,t+1} - c_{Q,t+1}\Xi_{t+1}c_{\Psi,t+1}^\top \Omega_{t+1}b_{\Psi,t+1}) + a_{\Psi,t+1}^\top \Omega_{t+1}b_{\Psi,t+1} \\ &\quad -(c_{\Psi,t+1}^\top \Omega_{t+1}^\top a_{\Psi,t+1})^\top \Xi_{t+1}c_{\Psi,t+1}^\top \Omega_{t+1}b_{\Psi,t+1} \end{aligned}$$

Proof See Appendix D for the proof. ▮

- **Step 5. Solve equations**

  Solve $\eta_t = F_t$ for all t.

2.4.1 Kyle Model Example

Using the tools suggested above, we will solve a simple Kyle model example equivalent to the one in Holden and Subrahmanyam (1994) with a single informed trader case. Assume that the liquidation value V is privately known to a single informed trader. First, assume the informed investor's order flow at date $t < T$ is given by the following linear function of state variables:

$$\Delta X_t = \eta_t (V_t - P_{t-1}) \tag{2.12}$$

where η_t is a constant. Using Lemma A.1[3], it could be easily shown that

$$\begin{aligned} E[V|\mathcal{F}_t^M] &= E[V|\mathcal{F}_{t-1}^M] + K_t^M(\zeta_t - E[\zeta_t|\mathcal{F}_{t-1}^M]) \\ &\Leftrightarrow P_t = P_{t-1} + K_t^M(\Delta X_t + \Delta U_t), \end{aligned}$$

where $K_t^M = \frac{\eta_t O_{t-1}}{\eta_t^2 O_{t-1} + \sigma_U^2}$ and $O_t = \frac{\sigma_U^2 O_{t-1}}{\eta_t^2 O_{t-1} + \sigma_U^2}$. Therefore, it is straightforward to show that

$$\Psi_t^I = -K_t^M \Delta X_t + \Psi_{t-1}^I - K_t^M \epsilon_{U,t+1}^I$$

Moreover, we find $Q_{t+1} = -K_t^M \Delta X_t + \Psi_{t-1}^I - K_t^M \epsilon_{U,t+1}^I$.

Using Lemma 2.3, we find

$$\eta_t = \frac{\gamma - K_t^M \Omega_{t+1}}{K_t^M[2\gamma - K_t^M \Omega_{t+1} + \gamma^2 K_t^M \sigma_U^2]} \tag{2.13}$$

Also,

$$\Omega_t = \frac{\gamma}{K_t^M[2\gamma - K_t^M \Omega_{t+1} + \gamma^2 K_t^M \sigma_U^2]}$$

Finally, we solve equation (2.13) for η_t for each $t < T$. The result is consistent with Holden and Subrahmanyam (1994) when the time interval is an unity (i.e. $\Delta t = 1$) and there is a single informed trader. Note that the notation here is slightly different from Holden and Subrahmanyam (1994).

[3] Set $A_t = 1, B_t = 0, H_t = \eta_t, R_t = \sigma_U^2$ for all $t < T$.

2.5 Comparison between GS model and Kyle model

Unlike GS model, a trader in Kyle model is well aware of the impact of his own trading on the equilibrium price. Therefore, informed traders' optimization problem in Kyle model includes extra motives for controlling the market as well as the motive for seeking informational rent as in GS model.

An investor's optimization problems in GS (2.3), and in Kyle (2.10) are essentially the same except for two things: (i) the control variable is total demand (X_t) in case of GS, and order flow (ΔX_t) in case of Kyle, (ii) the excess return is a function of control variable in Kyle model as well as state variables and innovations, (iii) the state process is a function of control variable in Kyle model as well as state variables and innovations.

Comparing the law of motions reveals a striking difference between two models: The excess return and state process in GS model (2.1), (2.2) are given by

$$\begin{aligned} Q_{t+1} &= a_{Q,t+1}\Psi_t^i + b_{Q,t}\varepsilon_{t+1}^i, \\ \Psi_{t+1}^i &= a_{\Psi,t+1}\Psi_t^i + b_{\Psi,t+1}\varepsilon_{t+1}^i. \end{aligned}$$

The excess return and state process in Kyle model From (2.8), (2.9) are given by

$$\begin{aligned} Q_{t+1} &= a_{Q,t+1}\Delta X_t + b_{Q,t+1}\Psi_t^i + c_{Q,t}\varepsilon_{t+1}^i, \\ \Psi_{t+1}^i &= a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t^i + c_{\Psi,t+1}\varepsilon_{t+1}^i. \end{aligned}$$

Note that the approach for both GS and Kyle are essentially the same dynamic programming approach. The only difference is that the excess return and the law of motion of state variables are linear functions of control variable in case of Kyle model while it is not in GS model.

A further analysis shows that the informed trader's order could be decomposed into two separate components: (i) a corrective order flow which considers only informational rent as a price-taker, (ii) a manipulative order flow which considers only the price impact of his own trading activities as a price-manipulator. By looking at the result of of Lemma 2.3 in 2.4, one can observe that δ_1, β_1 are not related to any of the informed trader's impact to the excess return nor state process. On the other hand, δ_2, β_2 are directly linked to the informed trader's impact to the excess return and state process. Therefore, the following theorem is directly obtained from the result of Lemma 2.3.

Theorem 2.4 *The informed trader's demand consists of two separate parts: (i) a corrective demand, (ii) a manipulative demand:*

$$\Delta X_t = \underbrace{\left(\frac{1}{\delta_1}\beta_1 - \frac{\delta_2}{\delta_1(\delta_1+\delta_2)}\beta_1\right)\Psi_t}_{\text{corrective demand}} + \underbrace{\left(\frac{1}{\delta_2}\beta_2 - \frac{\delta_1}{\delta_2(\delta_1+\delta_2)}\beta_2\right)\Psi_t}_{\text{manipulative demand}} \tag{2.14}$$

where δ_1, δ_2 are scalars, and β_1, β_2 are 3-vectors.

The first component is the informed trader's demand as a price taker, which consists of (i) a typical mean-variance utility maximizer as a price-taker: $\frac{1}{\delta_1}\beta_1$, and (ii) an adjustment term due to the risk regarding his own price impact: $-\frac{\delta_2}{\delta_1(\delta_1+\delta_2)}\beta_1$. The first term of corrective demand is exactly the same as competitive investors' demand in Wang (1994) or He and Wang (1995).[4] That is, corrective demand is defined as the demand of an competitive investor who ignores his own price impact plus an adjustment term to the exposure to the risk by his own price impact.

Similarly, the second component is the informed trader's demand as a price manipulator, which consists of (i) a mean-variance utility maximizer as a price-manipulator: $\frac{1}{\delta_2}\beta_2$, and (ii) an adjustment term due to the risk regarding his own corrective demand: $-\frac{\delta_1}{\delta_2(\delta_1+\delta_2)}\beta_2$. One can also verify that the manipulative demand disappears at the final trade date $T-1$ since there is no more room for manipulating the state process.

Therefore, the demand in Kyle model could be considered as corrective demand in GS model plus demand due to extra motives for controlling prices.

[4]Note that risk aversion parameter γ is included in δ_1 unlike GS model for its notational difference.

Chapter 3

Informed Trading, Predictable Noise Trading and Market Manipulation

3.1 Motivation

Would a rational arbitrageur correct the mispricing caused by noises in trading volume? The arbitrageur might well trade against noise traders because he would benefit from engaging in arbitrage activities as long as there is no market frictions such as short-sale constraint, limited liability and limited investment horizon.[1] This chapter attempts to answer the question in a slight different situation where a certain portion of noise trading activities are systematic, hence it is predictable to some degree. I assume that an arbitrageur has superior information on both the liquidation value of a risky asset and the systematic component of noise trading activities. Since market makers attempt to predict the systematic portion of noise trading as well, market makers' forecasting error on the systematic component of noise trading causes mispricing. The results show that the arbitrageur may not always correct the mispricing caused by noise trading even in the absence of any market friction as long as he has the incentive to camouflage his informed trading to further exploit the private information on fundamentals. While the arbitrageur in limited arbitrage literature does not correct the mispricing due to the

[1] For example, Dow and Gorton (1994) show that an arbitrageur who has limited investment horizon refrains from arbitrage because of the cost-of-carry associated with holding an arbitrage portfolio over an extended period of time.

frictions while the arbitrageur in this model intentionally amplifies the mispricing.

This chapter develops a dynamic model of informed trading in the presence of an autoregressive component of noise trading and public information release. Kyle (1985) has shown that a monopolistic risk-neutral informed trader gradually reveals his private information through his trades over time when he holds private information on the risky asset. Holden and Subrahmanyam (1992) and Holden and Subrahmanyam (1994) extend this finding by incorporating competition among informed traders and risk aversion to the preference of informed traders. Their result shows that both competition among informed traders and risk aversion of informed traders make informed traders more aggressive in the initial stage, thus private information is revealed more quickly. This chapter extends Kyle model by adding a systematic component of noise trading which follows an autoregressive process irrelevant to any fundamentals or information. Furthermore, market makers in this model are allowed to collect public signals on the liquidation value of a risky asset beside the information from trading volumes.

The main difference between competitive models (e.g. Grossman and Stiglitz (1980), Wang (1993), Wang (1994) and He and Wang (1995)) and Kyle model (e.g. Kyle (1985), Holden and Subrahmanyam (1992), Holden and Subrahmanyam (1994), Back, Cao, and Willard (2000), Foster and Viswanathan (1996), and Bernhaedt and Miao (2004)) is that informed traders are not price-takers in Kyle model, and their impact on the equilibrium price is incorporated in informed traders' optimization problem. Unlike competitive models, the inclusion of traders' own impact on the equilibrium price leads to extra technical complication in solving equilibrium.

This chapter makes difference from the previous literature in the strain of Kyle model on the following points: (i) This chapter introduces a time-varying systematic component of noise trading as another dimension of private information to the informed trader. Therefore, the decision making of the informed trader is associated not only with his private information on fundamental factors but also with non-fundamental factors. (ii) It features public signals, which is observed by everyone in the economy. It enables us to explore the impact of public signals on informed trading in the presence of predictable noise trading activities. The result shows that it may force the informed trader to correct the mispricing instead of riding on it. (iii) To develop a more flexible version of Kyle model in discrete time, this chapter adopts a more generalized approach similar to Wang (1994), and He and Wang (1995). This reformulation of Kyle model is simply a mathematical reinterpretation of the model for achieving extra tractablity. Thus, it does not alter any assumption in Kyle model, thereby keeping the properties of Kyle model unchanged. One of the benefit of this formulation is that it enables the model to feature dynamic learning of stochastic process using a linear filtering technique. This new reinterpretation also reveals that that the informed trader's demand for the risky asset at each period could be decomposed into two separate components according to their trading motives: (1) a corrective demand which is coming from a typical mean-variance demand for the excess return as a price-taker, and (2) a manipulative demand which is driven by price-controlling motives. This decomposition of the informed trader's demand plays a key role in revealing why the informed trader comoves with market makers' forecasting errors on the systematic component of noise trading instead of gaining immediate profit by engaging in arbitrage. Particularly, the decomposition of the informed trader's demand reveals that the comovement with the mispricing is caused only by manipulative demand, i.e., the informed trader suffers short-term losses to gain long-term profits by increasing noise in the market, which is market manipulation.

There are some empirical evidence supporting that informed traders could have superior information on noise trading activities. Brunnermeier and Nagel (2004) finds informed hedge funds prefer to ride bubbles because of predictable investor sentiment and limits to arbitrage. Chen, Hanson, Hong, and Stein (2008) finds that hedge funds engage in front-running strategies that exploit the predictable trades of mutual funds. Besides empirical facts, it is natural to assume that informed traders know more about noise trading activities since they are more able to distinguish uninformed order flows from informed order flows due to their private information. In case of pure random noises, however, even informed traders would not be able to have better estimation of such irrational trading volumes simply because there is no systematic patterns. As long as some systematic component persists in the uninformed order flows over time, informed traders would be able to form better estimation of such systematic components of uninformed order flows than other less informed traders. For the simplicity of analysis, I assume that the informed trader can observe the systematic part of noise trading directly at the start of every trading date while market makers attempt to learn it from the aggregate order flows.

One of the potential interpretations of the systematic component of noise trading featured in this chapter is investor sentiment while other interpretations such as autocorrelated liquidity shocks are still valid. Investor sentiment refers to individual investors' irrational trading behavior which is uncorrelated to any fundamental factors. Empirical literature such as Lee, Shleifer, and Thaler (1991) suggests various phenomena unexplained by standard finance theory might be driven by investor sentiment. De Long, Shleifer, Summers, and Waldmann (1990a) studies a model where the unpredictability of investor sentiment deters rational arbitrageurs from correcting investor sentiment. Furthermore, De Long, Shleifer, Summers, and Waldmann (1990b) finds that rational speculation destabilizes the market when there exists noise trading in the form of positive feedback trading.

The informed trader in this chapter takes a bit different stance from the ones in De Long, Shleifer, Summers, and Waldmann (1990a) or De Long, Shleifer, Summers, and Waldmann (1990b). The informed trader correctly observes a systematic part of noise trading (or investor sentiment) while he does not observe an unsystematic part (or pure noise). Therefore, the informed trader attempts to exploit the situation where he holds more information about noise trading compared to uninformed market makers. The informed trader in this economy does not completely correct the systematic part of noise trading not because investor sentiment poses extra risk to him but because it provides him with an extra camouflage in his trading activities.

The idea of trading against one's own private information has been studied by a large volume of literature on market manipulation, which finds informed traders may trade in the wrong direction to increase the noise in the trading volume. (e.g., Jarrow (1992), Allen and Gale (1992), Allen and Gorton (1992), Chakraborty and Yilmaz (2004a))[2] Most of papers in this line of literature adopts other models than Kyle model.[3] It is well known that an equilibrium with manipulative trading in Kyle model is ruled out under standard assumptions because of the monotonicity of the informed trader's equilibrium trading strategy. There exist a few exceptions which obtains manipulative trading with some variations of Kyle model such as Chakraborty and Yilmaz (2004b) which assumes that market makers are not certain about the

[2]There are other types of market manipulation models such as Goldstein and Gumbel (2008), which studies the case of manipulating the prices without private information in the presence of feedback effect.

[3]For example, Chakraborty and Yilmaz (2004a) adopt variations of Glosten and Milgrom (1985) model.

existence of informed traders and possible trade sizes are finite, and Huddart, Hughes, and Levine (2001) which assumes that there exist mandatory disclosure laws. This chapter shows that manipulative trading strategy could still easily happen in a variation of Kyle (1985) with standard assumptions if private information has more than two dimensions: (i) fundamentals and (ii) non-fundamentals. Therefore, this chapter contributes to the literature of manipulative informed trading by showing it with a standard Kyle model in the presence of predictable noise trading activities. That is, this chapter proves that manipulative informed trading easily arises when private information includes non-fundamentals in a standard Kyle model, which originally rules out manipulative trading. Furthermore, this chapter shows that such manipulative trading could be mitigated by releasing public signals while such stabilizing impact of public signals could deteriorate when public signals are correlated with the same systematic component of noise trading.

The chapter is organized as follows: In Section 3.2, I describes the investment opportunities, participants in the trading, and information structure. In Section 3.3, I solve for the equilibrium order flow of the informed trader and show the existence of a linear equilibrium . In Section 3.4, I analyze the properties of the linear equilibrium using numerical analysis.

3.2 Model

Consider a multiperiod model of trading a risky asset where traders place market orders to competitive market makers. Trading occurs at trading dates $1, ..., T-1$, and the liquidation value of each share is paid to traders at the final date T. This could be considered as $T-1$ sequential auctions with unit time intervals in Kyle (1985)'s notation.

3.2.1 Investment Opportunities

There are two assets in the economy, which are traded at trading dates $1, ..., T-1$: a riskless asset yielding a return R with perfectly elastic supply, and a risky asset. Shares of assets are infinitely divisible. I normalize the gross return of the riskless asset R to one for simplicity, which makes holding each position of riskless asset equivalent to a cash position in the absence of inflation. The liquidation value of the risky asset at the final trade date T is given by V.

3.2.2 Traders in the Economy

There are three types of traders in the economy: market makers, a single informed trader, and uninformed noise traders. The informed trader and noise traders place market orders to market makers. That is, the informed trader and noise traders simultaneously choose the amount of shares they want to trade, then market makers set a price and trade the order flow to clear the market. Market makers observe the aggregate order flows submitted by the informed trader and noise traders, but do not observe the individual order flow submitted by each trader separately. Therefore, the existence of noise traders prevents the equilibrium order flow from fully revealing the informed trader's private information.

Informed Trader

The monopolistic informed trader can observe both private and public signals of the fundamental value of the risky asset. The informed trader has an initial wealth of W_0, and does not have any share of the risky asset initially. The informed trader has a constant absolute risk aversion (CARA) utility function, and maximizes his wealth at the final date T, i.e. $\mathcal{U}(W_T) = -e^{\gamma W_T}$ where γ is a risk aversion parameter. Let ΔX_t denote the informed trader's order flow for the risky asset at date t.

I keep the setting of a single informed trader for simplicity throughout the proof of the existence of a linear equilibrium and numerical analysis. In Section 3.3.6, I show that featuring extra informed traders would not change the nature of market manipulation problem. As it is shown in Holden and Subrahmanyam (1992) and Holden and Subrahmanyam (1994), a multiple informed trader assumption simply makes informed traders more aggressive in the initial stage due to competition. Therefore, increasing the number of informed trader does not change the prediction of this chapter in other way.

Noise Trader

Noise traders are uninformed, and trade for other reasons than information such as liquidity reasons. Let ΔU_t denote noise traders' order flow for the risky asset at date t. The order flows from noise traders consist of two components: (1) demand driven by a certain systematic factor (or investor sentiment), (2) idiosyncratic shocks to noise traders' demand.

The process of the systematic factor S_t is given by a first-order autoregressive process:

$$S_{t+1} = a_S S_t + \epsilon_{S,t+1} \tag{3.1}$$

where $-1 < a_S < 1$ and $\epsilon_{S,t+1}$ is a shock to the systematic factor at trade date t, which follows normal distributions: $\epsilon_{S,t+1} \sim \mathcal{N}(0, \sigma^2_{S,t+1})$. Therefore, S_t is fluctuating around zero, and mean-reverting to zero in the steady state. Since S_t is irrational demand which is independent of fundamentals or information, it could be potentially interpreted as investor sentiment. Although another interpretation of S_t is still possible, I will refer to S_t as investor sentiment from now on for convenience.

Finally, the total demand of noise traders at trade date t could be written that

$$\Delta U_t = a_U S_t + \epsilon_{U,t+1} \tag{3.2}$$

where a_U is a non-negative scaling parameter and $\epsilon_{U,t+1}$ is an idiosyncratic shock to noise traders' demand at date t, which follows normal distributions: $\epsilon_{U,t+1} \sim \mathcal{N}(0, \sigma^2_{U,t+1})$. Note that I will normalize it to one throughout the numerical analysis in Section 3.4.

Market Maker

Market makers are risk neutral and competitive as in typical Kyle model. Since the competition among market makers drive their profit to zero, the price is set to market makers' expected liquidation value of the risky asset at the final date T. Although market makers observe public signals, they are not able to observe private signals of the informed trader. Thus, they attempt to infer private information of the informed trader using the aggregate order flow as well as public signals.

Let ΔZ_t denote the aggregate order flow by the informed traders and noise traders at date t. i.e.

$$\Delta Z_t \equiv \Delta X_t + \Delta U_t \ .$$

3.2.3 Information Structure

The prior information of market makers about the liquidation value of the risky asset V and investor sentiment S_t before the first trade date is common knowledge, and assume that the prior distributions are given by a certain distribution:

$$\begin{pmatrix} V \\ S_0 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_V^2 & 0 \\ 0 & \frac{\sigma_S^2}{1-a_S^2} \end{pmatrix} \right),$$

where v is the mean of prior distribution of the liquidation value V. The prior of investor sentiment before the first trade date, S_0, is given as its steady state distribution, and is independent of the prior of the liquidation value of the risky asset.

The informed trader observes investor sentiment S_t privately at each trade date t. However, the informed trader is still not able to know investor sentiment in the future due its stochastic nature of the process. On the other hand, market makers are unable to observe the investor sentiment.[4] There are also public signals which both the informed trader and market makers receive at date t before they engage in any trading activities:

$$Y_t = V + a_Y S_t + \epsilon_{Y,t}, \tag{3.3}$$

where a_Y is a non-negative constant, and $\epsilon_{Y,t}$ is a shock to public signal at trade date t. For example, the public signal Y_t is not distorted by investor sentiment S_t when $a_Y = 0$. On the other hand, the public signal Y_t is distorted by investor sentiment S_t when $a_Y > 0$.

Both the informed trader and market makers observe the past history of prices of the risky

[4]Even when the informed trader is not assumed to observe S_t directly, the informed trader would have superior information on S_t compared to market makers because he can infer past noise traders' order flows correctly from past aggregate order flows.

asset. Since market makers set the equilibrium price after observing the aggregate demand, the informed trader does not observe the equilibrium price until the next period. Therefore, the informed trader (I)'s information set at date t is given by:

$$\mathcal{F}_t^I = \{\mathcal{F}_0, V, P_{\tau-1}, S_t, Y_\tau : 1 \leq \tau \leq t\}, \tag{3.4}$$

where $\mathcal{F}_0$ denotes the common knowledge in the initial stage. On the other hand, a market maker (M)'s information set is given by:

$$\mathcal{F}_t^M = \{\mathcal{F}_0, P_{\tau-1}, \Delta Z_\tau, Y_\tau : 1 \leq \tau \leq t\}, \tag{3.5}$$

I will use the notation $\hat{x}_t^i \equiv E[x_t|\mathcal{F}_t^i]$ for any $i \in \{I, M\}$ (e.g. $\hat{V}_t^M \equiv E[x_t|\mathcal{F}_t^M]$). Now, I define state variables and shocks to the economy: (i) Denote $\Psi_t \equiv (V_t - P_{t-1}, S_t - a_S\hat{S}_{t-1}^M, Y_t - P_{t-1} - \hat{S}_{t-1}^M)^\top$ to be the vector of state variables. Since the informed trader can perfectly infer market makers' belief at each date t, one can easily observe that the informed trader knows Ψ_t correctly given his information set $\mathcal{F}_t^I$ at trade date t, i.e. $\hat{\Psi}_t^I \equiv E[\Psi_t|\mathcal{F}_t^I] = \Psi_t$ (ii) Denote $\varepsilon_t = (\epsilon_{S,t+1}, \epsilon_{U,t+1}, \epsilon_{Y,t+1})^\top$ to be the vector of shocks to the economy which has not yet arrived at trade date t. They are jointly normal, independent of each other, and independent over time. That is, the distribution of ε_t is given by $\varepsilon_{t+1} \sim \mathcal{N}(0, \Sigma_{t+1})$ where Σ_{t+1} is the covariance matrix of the shocks in which diagonal elements are $\sigma^2_{S,t+1}, \sigma^2_{U,t+1}, \sigma^2_{Y,t+1}$ respectively and other elements are all zero. Further assume that ε_t is independent of $E[V|\mathcal{F}_t^M]$.

3.3 Equilibrium

3.3.1 Equilibrium Order Flow

Consider a linear equilibrium in the economy.[5] There are three state factors which determines the equilibrium in this economy: fundamental factor $(V - P_{t-1})$, investor sentiment factor $(S_t - a_S\hat{S}_{t-1}^M)$, and public announcement factor $(Y_t - P_{t-1} - a_Y a_S\hat{S}_{t-1}^M)$. Each factor represents

[5] First of all, a linear equilibrium in this model makes more economic sense than potential nonlinear equilibria if any. Past literature has conjectured that there is no other equilibrium than a linear equilibrium in Kyle model, but has not been successful in showing it.

the difference between the true value and market makers' expectation of the liquidation value, investor sentiment, and error in public signal, respectively. Note that the first factor exactly matches the one in Kyle (1985) or Holden and Subrahmanyam (1994). This model requires two more factors than typical Kyle models since it features investor sentiment and public signals. The next theorem states that the equilibrium order flow of the informed trader at each trade date are given as a linear function of state variables.

Theorem 3.1 *In a linear equilibrium, the informed trader's order flows for the risky asset at trade date* $1 \leq t < T$ *are given by a linear function of state variables:*

$$\Delta X_t \quad = \quad a_{X,t}(V - P_{t-1}) + b_{X,t}(S_t - a_S \hat{S}^M_{t-1}) + c_{X,t}(Y_t - P_{t-1} - a_Y a_S \hat{S}^M_{t-1}) \ . \tag{3.6}$$

Equivalently,

$$\Delta X_t \quad = \quad \eta_t \Psi_t \ , \tag{3.7}$$

where $\eta_t \equiv (a_{X,t}, b_{X,t}, c_{X,t})$ *and* $\Psi_t \equiv (V - P_{t-1}, S_t - a_S \hat{S}^M_{t-1}, Y_t - P_{t-1} - a_Y a_S \hat{S}^M_{t-1})^\top$.

I will prove this theorem by assuming the above order flow and finding the informed trader's optimal order flow is indeed the same in a linear equilibrium. First, I show the learning problem of market makers, which determines the equilibrium price. Second, I solve the informed trader's optimization problem given the price function derived by the learning problem of market makers. Third, I show that the informed trader's optimal demand in equilibrium is indeed equal to the initial assumption, which proves the existence of the linear equilibrium.

3.3.2 Conditional Expectation

As I have mentioned earlier in the previous section, market makers observe the aggregate order flows, $\Delta Z_t \equiv \Delta X_t + \Delta U_t$. Define

$$\zeta_t \equiv \Delta X_t + \Delta U_t + a_{X,t} P_{t-1} + b_{X,t} a_S \hat{S}^M_{t-1} - c_{X,t}(Y_t - P_{t-1} - a_Y a_S \hat{S}^M_{t-1}) \tag{3.8}$$

Note that P_{t-1}, $\hat{S}^M_{t-1}$ and Y_t are all known to market makers at trade date t. Hence, observing the current order flow is equivalent to observing ζ_t given market makers' information set from the past period, $\mathcal{F}^M_{t-1}$, and the public signal, Y_t. That is, ζ_t is a sufficient statistic for the aggregate order flow ΔZ_t in equilibrium.[6] It is straight forward to show that ζ_t is equivalent to

$$\zeta_t = a_{X,t}V_t + (b_{X,t} + a_U)S_t + \epsilon_{U,t+1} \tag{3.9}$$

Market makers' updating belief on V at trade date t using ζ_t and Y_t could be solved by a simple Kalman filter:

Theorem 3.2 *Given the aggregate order flow for the risky asset and public news, $\hat{V}^M_t, \hat{S}^M_t$ is determined by the following linear filter:*

$$\begin{pmatrix} \hat{V}^M_t \\ \hat{S}^M_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a_S \end{pmatrix} \begin{pmatrix} \hat{V}^M_{t-1} \\ \hat{S}^M_{t-1} \end{pmatrix} + \begin{pmatrix} k^\zeta_{V,t} & k^Y_{V,t} \\ k^\zeta_{S,t} & k^Y_{S,t} \end{pmatrix} \begin{pmatrix} \zeta_t - E[\zeta_t|\mathcal{F}^M_{t-1}] \\ Y_t - E[Y_t|\mathcal{F}^M_{t-1}] \end{pmatrix}$$

where $k^\zeta_{V,t}, k^Y_{V,t}, k^\zeta_{S,t}, k^Y_{S,t}$ are constants.

Proof See Appendix A. ∎

Since market makers are competitive and risk neutral, the equilibrium price is always equal to market makers' conditional expectation of the liquidation value at trade date t, i.e. $P_t = E[V|\mathcal{F}^M_t] = \hat{V}^M_t$. The informed trader's excess return at trade date t is given by $Q_{t+1} \equiv V - P_t$ assuming that he has a perfect knowledge on the risky asset, and does not have any borrowing constraints. The following lemma shows that the excess return from trading the risky asset at trade date t is determined by the informed trader's order flow, state variables, and shocks.

Lemma 3.3 *The equilibrium excess return at trade date t is represented by a linear function of ΔX_t, Ψ_t and ε_{t+1}.*

$$Q_{t+1} = a_{Q,t+1}\Delta X_t + b_{Q,t+1}\Psi_t + c_{Q,t+1}\varepsilon_{t+1} \tag{3.10}$$

[6]In equilibrium, the informed trader's trading strategy $\eta_t \equiv (a_{X,t}, b_{X,t}, c_{X,t})$ is a common knowledge.

where $a_{Q,t+1}$ is a constant, and $b_{Q,t+1}$, $c_{Q,t+1}$ are vectors of constants in proper order.

Proof See Appendix B. ∎

3.3.3 Informed Traders' Optimization problem

Since $Q_{t+1} \equiv V - P_t$, the informed trader's problem can be formulated as the following:

$$\begin{aligned} \underset{\Delta X_t}{max} \quad & E\left[-e^{-\gamma W_T}\middle|\mathcal{F}_t^I\right] \\ \text{subject to} \quad & W_{t+1} = W_t + Q_{t+1}\Delta X_t \end{aligned} \tag{3.11}$$

This formulation of the informed trader's problem is a generalized version of Kyle model, which provides more flexibility in analyzing the informed trader's dynamic decision making problem. The main difference from a competitive market setting such as He and Wang (1995) is that the excess return Q_{t+1} is given as a function of the control variable ΔX_t,

In the following lemma, I show that the law of motion for the state vector Ψ_t is an autoregressive process with exogenous input ΔX_t. Unlike in competitive models such as Wang (1994) and He and Wang (1995), the state process is also affected by the control variable. Therefore, the informed trader is in fact able to affect the state process using the control variable for his own benefit.

Lemma 3.4 *The state vector Ψ_t is an autoregressive process with an exogenous input ΔX_t:*

$$\Psi_{t+1} = a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1} \tag{3.12}$$

where $a_{\Psi,t+1}, b_{\Psi,t+1}, c_{\Psi,t+1}$ are matrix of constants in proper order.

Proof See Appendix C. ∎

Recall that the state vector at trade date t, Ψ_t, is perfectly observed by the informed trader, i.e. $\Psi_t = E[\Psi_t|\mathcal{F}_t^I]$. Therefore, we can solve the informed trader's optimization problem using the excess return from Lemma 3.3 and the state process from Lemma 3.4. The Bellman equation for the optimization problem (3.11) is given by

$$
\begin{aligned}
0 = \underset{\Delta X_t}{Max} \quad & \{E[J(W_{t+1};\Psi_{t+1};t+1)|\mathcal{F}_t^I] - J(W_t;\Psi_t;t)\} \\
\text{subject to} \quad & W_{t+1} = W_t + Q_{t+1}\Delta X_t \\
& J(W_T;\Psi_T;T) = -e^{-\gamma W_T}.
\end{aligned}
$$

The solution for the optimization problem is derived according to the following lemma:

Lemma 3.5 *Suppose Q_{t+1} and Ψ_t are given by the following Gauss-Markov processes:*

$$
\begin{aligned}
Q_{t+1} &= a_{Q,t+1}\Delta X_t + b_{Q,t+1}\Psi_t + c_{Q,t+1}\varepsilon_{t+1} \\
\Psi_{t+1} &= a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1}
\end{aligned}
$$

Then, the risk averse informed trader's optimal order flow for the risky asset at date t is given by a linear function of state vector at trade date t:

$$\Delta X_t = F_t\Psi_t \tag{3.13}$$

where F_t is a vector of proper order.

Proof See Appendix D. ∎

3.3.4 Solving the Equilibrium

Finally, the equilibrium is determined by solving the following equation system:

$$\eta_t = F_t \quad \text{for all } 0 < t < T \tag{3.14}$$

where η_t is a vector of unknowns at date t as is given in Theorem 3.1, and F_t is a solution derived using Lemma 3.3, Lemma 3.4, and Lemma 3.5 given η_t. Therefore, the existence of solution proves the existence of a linear equilibrium since it satisfies the assumption which I have made on the equilibrium order flow at the start of Section 3.3. Like other literature with

Kyle model, an analytical solution of the equation system cannot be obtained in general. The numerical procedure of solving Equation (3.14) is described in Appendix F

3.3.5 Components of the Informed Trader's Order Flows

A further analysis on the informed trader's order flow shows that it could be decomposed into two separate components: (i) corrective demand which attempts to gain profits from informational rent as a price-taker, (ii) manipulative demand which attempts to gain profits from the price changes over time due to his own trading.

The result of of Lemma 3.5 in Appendix D reveals that the informed trader's optimal order flow at trade date t is given by

$$\Delta X_t = \frac{1}{\delta_1 + \delta_2}(\beta_1 + \beta_2)\Psi_t$$

where δ_1, δ_2 are constants, and β_1, β_2 are 3-vectors. By looking at the solution in Appendix D, one can observe that δ_1, β_1 are not related to any of the informed trader's impact on the excess return nor the state process. On the other hand, δ_2, β_2 are directly linked to the informed trader's impact on the excess return and the state process. Therefore, the following theorem is directly obtained from the result of Lemma 3.5.

Theorem 3.6 *The informed trader's demand consists of two separate parts: (i) corrective demand, (ii) manipulative demand:*

$$\Delta X_t = \underbrace{\left(\frac{1}{\delta_1}\beta_1 - \frac{\delta_2}{\delta_1(\delta_1 + \delta_2)}\beta_1\right)\Psi_t}_{\textit{corrective demand}} + \underbrace{\left(\frac{1}{\delta_2}\beta_2 - \frac{\delta_1}{\delta_2(\delta_1 + \delta_2)}\beta_2\right)\Psi_t}_{\textit{manipulative demand}} \tag{3.15}$$

where δ_1, δ_2 are constants, and β_1, β_2 are 3-vectors.

The first component is the informed trader's demand as a price taker, which consists of (i) a typical mean-variance utility maximizer as a price-taker: $\frac{1}{\delta_1}\beta_1$, and (ii) an adjustment term due to the risk regarding his own price impact: $-\frac{\delta_2}{\delta_1(\delta_1+\delta_2)}\beta_1$. The first term of corrective demand is exactly the same as competitive investors' demand in Wang (1994) or He and Wang (1995).[7] That is, corrective demand is defined as pure corrective demand ignoring his own price impact plus an adjustment term to the exposure to the risk by his own price impact.

Similarly, the second component is the informed trader's demand as a price manipulator, which consists of (i) a mean-variance utility maximizer as a price-manipulator: $\frac{1}{\delta_2}\beta_2$, and (ii) an adjustment term due to the risk regarding his own corrective demand: $-\frac{\delta_1}{\delta_2(\delta_1+\delta_2)}\beta_2$. One can also verify that the manipulative demand disappears at the final trade date $T-1$ since there is no more room for manipulating the state process.

3.3.6 Multiple Informed Traders

In this section, I will briefly show that the existence of multiple informed trader would not fundamentally change the findings of this model except for changing the degree of the informed trader's incentive of manipulative trading. More competition among the informed traders accelerates the revelation of private information through prices, thereby reducing the gains from market manipulation.

Instead of a single informed trader, suppose there are N informed traders who observe the liquidation value of the risky asset as well as the investor sentiment at every period. I further assume that they have identical preference, common knowledge and initial wealth. I denote ΔX_t^i to be ith informed trader's order flow at date t, and $\Delta \mathbb{X}_t \equiv \sum_{i=1}^{N} \Delta X_t^i$ to be the aggregate order flow of N informed traders.

Lemma 3.7 *The equilibrium excess return Q_t and state vector Ψ_t is given by an autoregressive process with an exogenous input $\Delta \mathbb{X}_t$:*

$$\begin{aligned} Q_{t+1} &= a_{Q,t+1}\Delta \mathbb{X}_t + b_{Q,t+1}\Psi_t + c_{Q,t+1}\varepsilon_{t+1} \\ \Psi_{t+1} &= a_{\Psi,t+1}\Delta \mathbb{X}_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1} \end{aligned}$$

where $a_{Q,t+1}, b_{Q,t+1}, c_{Q,t+1}, a_{\Psi,t+1}, b_{\Psi,t+1}, c_{\Psi,t+1}$ are matrices of constants in proper order. In

[7] Note that risk aversion parameter γ is included in δ_1 unlike Wang (1994), He and Wang (1995) for its notational difference.

equilibrium, i'th informed trader's order flow at period t is given by

$$\Delta X_t^i = \frac{1}{\delta_1 + \delta_2'}(\beta_1 + \beta_2)\Psi_t$$

where δ_1, δ_2' are constants, and β_1, β_2 are 3-vectors. δ_1, β_1 are related to corrective motives of trading while δ_2, β_2 are related to manipulative motives of trading. Furthermore, $\delta_1, \beta_1, \beta_2$ are not functions of the number of informed traders N while δ_2' is a function of N.

Proof See Appendix E. ∎

The result together with Theorem 3.6 shows that given all the things the same only the manipulative demand would be changed as the number of informed traders N while the corrective demand is unchanged.[8] Holden and Subrahmanyam (1992) and Holden and Subrahmanyam (1994) show that a multiple informed trader assumption makes informed traders more aggressive in the initial stage due to competition. That is, with reasonable parameters δ_2' would be an increasing function of N. With increasing δ_2' informed traders' incentive of manipulative trading would decrease, thereby increasing the share of corrective trading relatively. Therefore, featuring the competition among multiple informed traders would not change the nature of market manipulation found by this chapter in the later section using numerical results while it would strictly decrease the informed traders' incentive of market manipulation.

3.4 Properties of Equilibrium

3.4.1 Informed Trading and Investor Sentiment

In this subsection, I assume that there is no public signal release in order to focus on studying the relationship between informed trading and investor sentiment.

Figure 3.1 shows the impact of investor sentiment on price efficiency by comparing market makers' uncertainty on the liquidation value of the risky asset with and without investor sentiment. The information revelation through aggregate order flows is slower in the presence of

[8] Although parameter values would change, the functional form of $\delta_1, \beta_1, \beta_2$ would remain the same regardless of the number of informed traders.

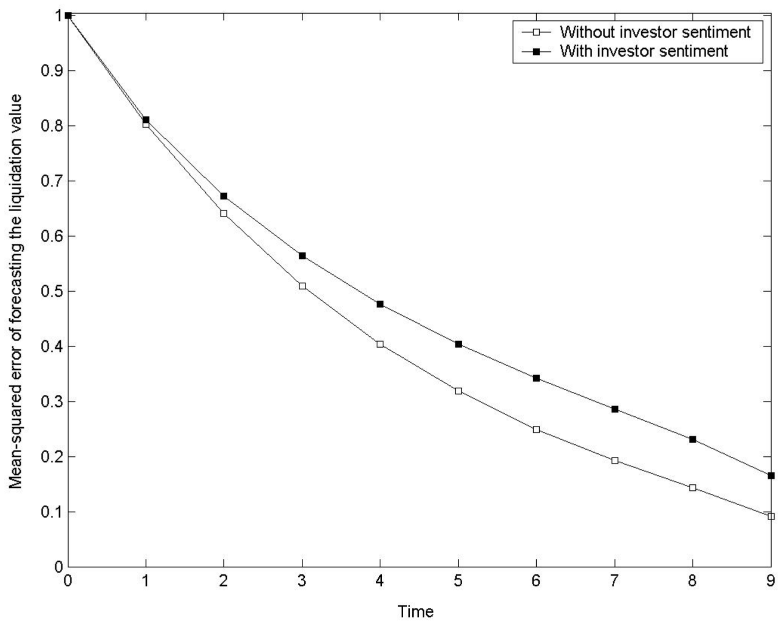

Figure 3.1: Mean-squared error of forecasting the liquidation value (o_t^V) over time with or without investor sentiment

Market makers' mean-squared error of forecasting the liquidation value at trade date t is given by $o_t^V = E[(V - \hat{V}_t^M)(V - \hat{V}_t^M)|\mathcal{F}_t^M]$. The line with black squares denotes o_t^V without investor sentiment, which corresponds to Holden and Subrahmanyam (1994). The line with white squares denotes o_t^V with investor sentiment ($\sigma_S = 0.2$). Public signals are not given. Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_U = 1$, $a_U = 1$.

investor sentiment. Since investor sentiment provides extra noise to the aggregate order flows, it naturally slows market makers' learning.

Table 3.1 compares the informed trader's equilibrium trading strategy between the case with and without investor sentiment. The informed trader becomes more aggressive when there exists investor sentiment. Even with the stronger intensity of informed trading, however, the informed trader is able to keep market makers less informed about the liquidation value using the camouflage of investor sentiment. By looking at the coefficients on investor sentiment factor, we can observe that the informed trader comove with market makers' forecasting errors on investor sentiment during earlier periods. Furthermore, such comovement with the mispricing due to investor sentiment gets more severe until trade date $t = 5$, then grows down afterwards. It can be also observed that the informed trader finally starts correcting the mispricing around the final trade date. Indeed, it could be easily shown that the informed trader always finds it optimal to correct investor sentiment at trade date $T-1$:

Corollary 3.8 *When the informed trader's optimization problem is a static problem instead of a dynamic problem (i.e. $t = T-1$), the informed trader's optimal order flow is given by the following:*

$$\Delta X_t = \frac{V - P_{t-1}}{\Gamma_t} - \frac{k^{\zeta}_{V,t}(S_t - a_S \hat{S}^M_{t-1})}{\Gamma_t},$$

where $\Gamma_t \equiv k^{\zeta}_{V,t}(2 + \gamma k^{\zeta}_{V,t}\sigma^2_{U,t+1})$.

Since $\Gamma_t, k^{\zeta}_{V,t}$ are positive constants[9], Corollary 3.8 implies that the informed trader always bets against market makers' error of forecasting investor sentiment, $S_t - a_S \hat{S}^M_{t-1}$ at the final trade date $T-1$. Thus, if the informed trader ever bets on investor sentiment, it is because of the dynamic property of his optimization problem. That is, the informed trader might find it profitable to comove with investor sentiment because the expected profit which he will achieves in the future by manipulating prices dominates the profit which he achieves by correcting investor sentiment at the current period. The next result in fact reveals that the informed trader comoves with investor sentiment out of price-controlling motives.

Table 3.2 reports the decomposition of the informed traders' order flow into two separate components defined in Section 3.3.5: (i) corrective demand, and (ii) manipulative demand. It reveals that the comovement of the informed trader with investor sentiment is driven by his manipulative demand rather than his corrective demand. We can clearly observe that the corrective demand corrects the error in market makers' forecasting investor sentiment in every period, however, the manipulative demand which comoves with investor sentiment dominates the corrective demand except for a few trade dates near the liquidation date T. That is, the manipulative demand overwhelms the corrective demand during early trading dates. While correcting the mispricing due to investor sentiment could give short-term profits, comoving with the mispricing give better long-term profits. The informed trader finds that the long-term

[9] Γ_t is positive due to the second order condition of the informed trader's optimization problem. $k^{\zeta}_{V,t}$ could also be shown to be positive at trade date $T-1$ in the linear equilibrium.

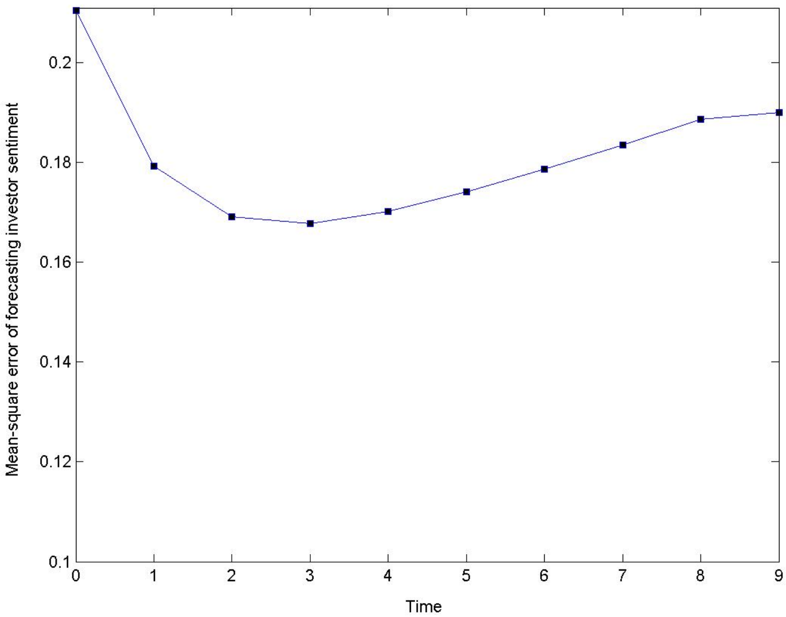

Figure 3.2: Mean-squared error of forecasting investor sentiment (o_t^S) over time

Market makers' mean-squared error of forecasting investor sentiment at trade date t is given by $o_t^S = E[(S_t - \hat{S}_t^M)(S - \hat{S}_t^M)|\mathcal{F}_t^M]$. The line with white squares denotes o_t^V with investor sentiment ($\sigma_S = 0.2$). Public signals are not given. Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_U = 1$, $\sigma_S = 0.2$, $a_U = 1$.

profits from comovement with investor sentiment factor overwhelms the short-term protifs from correcting the mispricing during early trading dates. As a result, the informed trader trades in the wrong direction regarding investor sentiment to increase the noise in the market during early trading dates. The result is in line with recent empirical observations such as Brunnermeier and Nagel (2004): the informed trader magnifies irrational demands of noise traders during early periods of trading, and corrects it near the liquidation of the risky asset.

Figure 3.2 shows market makers' uncertainty about investor sentiment. Since investor sentiment evolves over time, the uncertainty goes back to the steady state level unless market makers keep learning new information on it. The figure shows an interesting comparison with Figure 3.1, which shows monotone-decreasing market makers' uncertainty about the liquidation value. While the uncertainty about the liquidation value is rather gradually decreasing, the uncertainty about investor sentiment decreases rapidly at first, but picks up slowly afterwards.

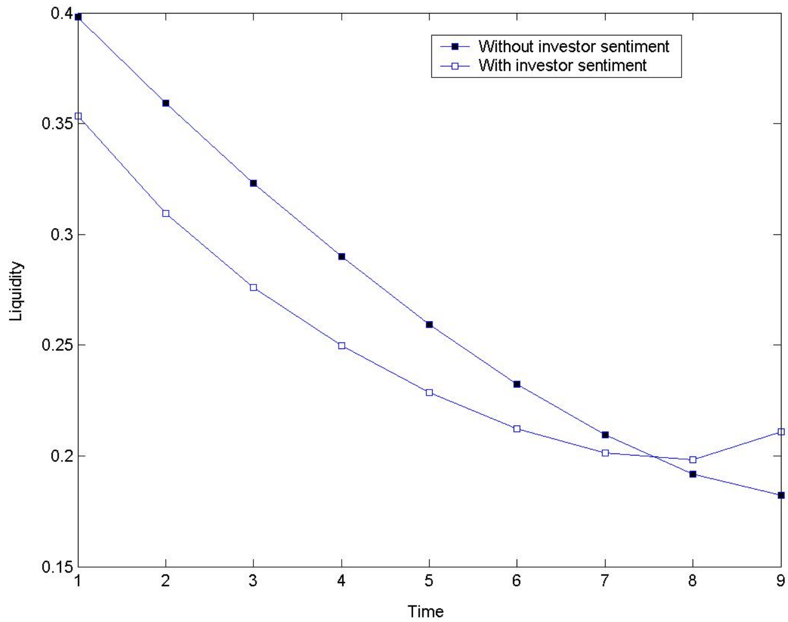

Figure 3.3: Liquidity over time with or without investor sentiment

Liquidity parameter is measured by the price sensitivity to the aggregate order flow ($k^{\zeta}_{V,t}$). The line with black squares denotes liquidity with investor sentiment. The line without white squares denotes liquidity without investor sentiment ($\sigma_S = 0.2$). Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_U = 1$, $a_U = 1$.

When the final trade date is far away, the informed trader deliberately chooses to reveal less about the liquidation value of the risky asset at the cost of revealing more about investor sentiment by comoving with it. This shows a case where informed traders may ride a bubble driven by investor sentiment for informational reason because informed traders are able to less reveal their private information by leaning toward investor sentiment.

Figure 3.3 also shows that the price sensitivity to the aggregate order flow which could be interpreted as the reverse of liquidity in the market. It reveals that investor sentiment provides higher liquidity to the informed trader over all. However, the liquidity becomes suddenly lower near the final trade date when there exists investor sentiment. It is because the aggregate order flow becomes suddenly very informative near the final trade date because the informed trader suddenly starts correcting investor sentiment. Since investor sentiment prevents private information from being revealed, private information which would have been revealed without investor sentiment gets accumulated over time without being revealed. Moreover, the informed trader starts correcting aggressively the mispricing due to investor sentiment near the liquidation of the risky asset. As a result, the revelation of these accumulated signals near the final trade date with dampened noises from investor sentiment makes market makers more sensitive to the aggregate order flows. Therefore, market depth would increase gradually with sharp decrease just near announcements if informed traders have been manipulating the market.

3.4.2 Informed Trading and Public News

In this section, I study the impact of public singals on informed trading. The release of public information weakens the informed trader's incentive to manipulate the market, however, such stabilizing effect of public information would deteriorate if public information is also distorted by investor sentiment. The result is robust whether information release is given as a single shock or sequential shocks.

Single Public Information Release

Consider the arrival of a single public signal at one specific date $t = 3$. The variance of the signal is given by $\sigma_{Y,3} = 1$, and $\sigma_{Y,t} = 10^6$ for all $t \neq 3$. i.e., the accuracy would be considered as $1/\sigma_{Y,3} = 1$, and $1/\sigma_{Y,t} = 1/10^6 \approx 0$ for all $t \neq 3$.

Figure 3.4 reports market makers' uncertainty about the liquidation value when there is a public announcement at $t = 3$. It shows that the uncertainty reduction due to the pubic announcement is bigger when the public signal is not affected by investor sentiment.

Table 3.3 and Table 3.4 show the informed trader's trading strategy when there is a relatively accurate public announcement at date $t = 3$. We can observe that the informed trader corrects the mispricing due to investor sentiment before the announcement date $t = 3$ unlike the case without any pubic announcement. Therefore, the release of public information stabilizes the market by mitigating the informed trader's incentive to manipulate the market. We can also observe that correction of the mispricing due to investor sentiment when the announcement is affected by investor sentiment (Table 3.4) is weaker than the correction of the mispricing

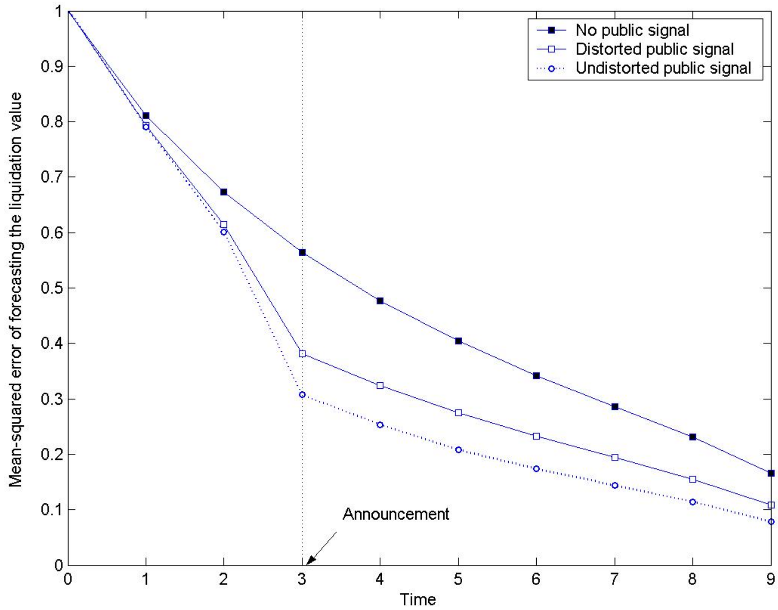

Figure 3.4: Price efficiency parameter o_t^V over time in the presence of public signals

Price efficiency parameter is measured by market makers' mean-squared error of forecasting at trade date t, i.e. $o_t^V = E[(V - \hat{V}_t^M)(V - \hat{V}_t^M)]$. Solid line denotes o_t^V when the public signal is not affected by investor sentiment, i.e. $a_Y = 0$. Dotted line denotes o_t^V when the public signal is affected by investor sentiment, i.e. $a_Y = 1$. Parameter values are given by $T = 10, \gamma = 4,\ \sigma_V = 1,\ \sigma_S = 0.2, \sigma_U = 1, \sigma_{Y,t} = 10^6$ for all $t \neq 3, \sigma_{Y,t} = 1$ for $t = 3, a_U = 1$.

when the announcement is not affected by investor sentiment (Table 3.3). Therefore, the price-stabilizing effect of the public signal is reduced when the signal is distorted by investor sentiment.

Sequential Public Information Release

Consider sequential arrivals of public signal at each date. Note that the signal is chosen to be relatively noisier compared to the single information shock case: $\sigma_{Y,t} = 5$ for all $0 < t < T$. i.e, the accuracy of signals could be considered as $1/\sigma_{Y,t} = 0.2$ for all $0 < t < T$.

Figure 3.5 reports a similar result as the single arrival of public signal about the price efficiency. It shows that the uncertainty reduction due to the pubic announcement is bigger when the public signal is not affected by investor sentiment.

Table 3.5 and Table 3.6 show the informed trader's trading strategy with public signal arriving at each trade date. As the case with single information shock, we can observe that the

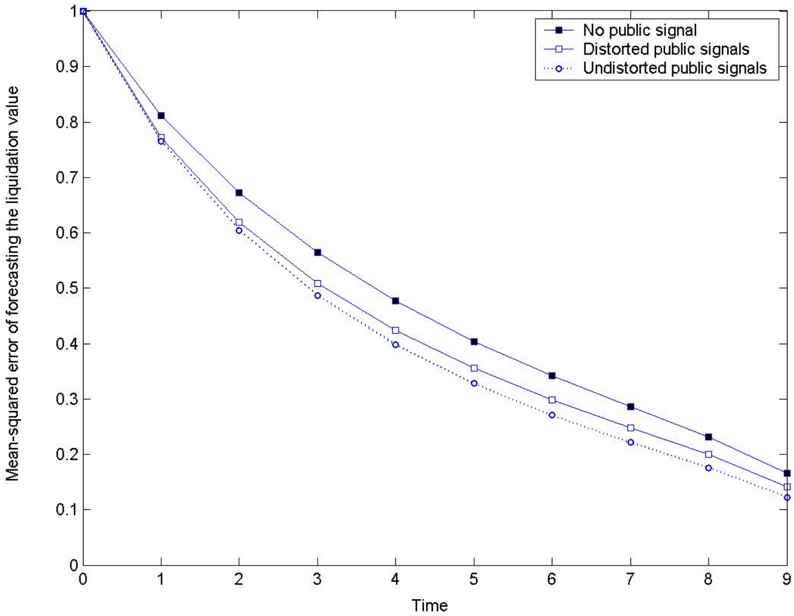

Figure 3.5: Mean-squared error of forecasting the liquidation value (o_t^V) over time with or without distortions in public signals

Market makers' mean-squared error of forecasting the liquidation value at trade date t is given by $o_t^V = E[(V - \hat{V}_t^M)(V - \hat{V}_t^M)|\mathcal{F}_t^M]$. The line with black squares denotes o_t^V without public signals. The line with white squares denotes o_t^V with distorted public signals ($a_Y = 0$). The dotted line with white circles denotes o_t^V with undistorted public signals ($a_Y = 1$). Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_S = 0.2$, $\sigma_U = 1$, $\sigma_Y = 5$, $a_U = 1$.

intensity of riding the mispricing due to investor sentiment becomes weaker compared to the case without any public signal. We can also observe that the correction of investor sentiment when public signal is affected by investor sentiment (Table 3.6) is weaker than the one when public signal is not affected by investor sentiment (Table 3.5). It also confirms that the price-stabilizing effect of public signal is reduced when the signal is distorted by the same source of noise.

Therefore, this section develops an implication for the policy of market stabilization. Not surprisingly, pubic information needs to be revealed as much as possible to mitigate the destabilizing impact of market manipulation. Furthermore, such public signal needs to be free from the source of noise in the market. For example, the release of news which are potentially affected market prevalent investor sentiment would not be very helpful for stabilizing the market according to the prediction of this chapter.

3.5 Conclusion

This chapter attempts to answer the question whether a rational arbitrageur who has superior information about non-fundamental factor such as noise trading activities would reduce the mispricing caused by such non-fundamentals in the market. I analyze a dynamic model of informed trading in the presence of an autoregressive component of noise trading which is privately observed by a monopolistic risk-averse informed trader. To develop a more flexible version of Kyle model in discrete time, this chapter adopts a more generalized approach similar to Wang (1994), and He and Wang (1995). Using the reinterpreted version of Kyle model, this chapter shows that the informed trader's demand for the risky asset at each period could be decomposed into two separate components according to their trading motives: (i) a corrective demand which is a typical price-taking mean-variance demand for the excess return, and (ii) a manipulative demand which is driven by price-controlling motives.

The result shows that the informed trader may ride on the mispricing caused by the systematic component of noise trading during early periods of trading, and only start correcting it near the liquidation of the risky asset. The decomposition of the informed trader's demand reveals that such comovement is caused by manipulative demand, i.e., the informed trader suffers short-term losses to gain long-term profits by increasing noise in the market, which is market manipulation. The informed trader chooses to comove with irrational demands because such comovement allows him to reveal less information through his trading volume, which leads to more profit in later periods. Furthermore, I show that the release of public signals help stabilize prices to some degree since it reduces the informed traders' incentive to manipulate the market. However, such price-stabilizing effect of public information is severely weakened when public information is also distorted by the same irrationality of noise traders.

In summary, my findings shows that an arbitrageur who has superior information on non-fundamentals such as investor sentiment may not always reduce the mispricing caused by non-fundamentals given private information on fundamentals. Therefore, manipulative trading could occur under standard Kyle model setting if private information includes both fundamentals and non-fundamentals.

	Without investor sentiment	With investor sentiment	
t	$a_{X,t}$	$a_{X,t}$	$b_{X,t}$
1	0.4958	0.5329	0.0308
2	0.5605	0.6439	0.0635
3	0.6341	0.7528	0.0913
4	0.7182	0.8622	0.1125
5	0.8153	0.9752	0.1238
6	0.9322	1.0948	0.1171
7	1.0877	1.2249	0.0742
8	1.3431	1.3765	-0.0485
9	2.0122	1.6678	-0.3517

Table 3.1: Coefficients of informed trading on fundamental factor $V - P_{t-1}$ and investor sentiment factor $S_t - a_S\hat{S}^M_{t-1}$ with or without investor sentiment

$a_{X,t}$ denotes the coefficient of the informed trader's order flow at trade date t on fundamental factor $V - P_{t-1}$. $b_{X,t}$ denotes the coefficient of the informed trader's order flow at trade date t on investor sentiment factor $S_t - a_S\hat{S}^M_{t-1}$. The first column reports the coefficients on order flow without investor sentiment, the next two columns report the coefficients on order flow with investor sentiment ($\sigma_S = 0.2$). Public signals are not given. Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_U = 1$, $a_U = 1$.

	Trading strategy		Corrective demand		Manipulative demand	
t	$a_{X,t}$	$b_{X,t}$	$a_{X,t}$	$b_{X,t}$	$a_{X,t}$	$b_{X,t}$
1	0.5329	0.0308	0.8469	**-0.1710**	-0.3140	**0.2018**
2	0.6439	0.0635	1.0563	**-0.1815**	-0.4124	**0.2451**
3	0.7528	0.0913	1.2680	**-0.1932**	-0.5153	**0.2844**
4	0.8622	0.1125	1.4815	**-0.2062**	-0.6193	**0.3187**
5	0.9752	0.1238	1.6912	**-0.2211**	-0.7160	**0.3448**
6	1.0948	0.1171	1.8829	**-0.2393**	-0.7881	**0.3564**
7	1.2249	0.0742	2.0249	**-0.2647**	-0.8000	**0.3389**
8	1.3765	-0.0485	2.0399	**-0.3051**	-0.6633	**0.2566**
9	1.6678	-0.3517	1.6678	**-0.3517**	0	**0**

Table 3.2: Informed trader's trading strategy and the decomposition

$a_{X,t}$ denotes the coefficient of the informed trader's order flow at trade date t on fundamental factor $V - P_{t-1}$. $b_{X,t}$ denotes the coefficient of the informed trader's order flow at trade date t on investor sentiment factor $S_t - a_S\hat{S}^M_{t-1}$. The first two columns report the informed trader's strategy which are the sums of competitive and manipulative demands, and the next four columns report the decomposition of it. Public signals are not given. Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_S = 0.2$, $\sigma_U = 1$, $a_U = 1$.

	η_t			Corrective demand			Manipulative demand		
t	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$
1	0.5594	**-0.0685**	0	0.7786	-0.2078	0	-0.2192	0.1393	0
2	0.7341	**-0.2025**	0	0.8244	-0.2603	0	-0.0902	0.0578	0
3	1.0107	0.0942	-0.3738	1.8068	-0.2442	-0.6917	-0.7961	0.3384	0.3179
4	1.1893	0.1052	0	2.1247	-0.2641	0	-0.9354	0.3692	0
5	1.3876	0.0813	0	2.3517	-0.2835	0	-0.9641	0.3648	0
6	1.5590	0.1532	0	2.9144	-0.2995	0	-1.3554	0.4528	0
7	1.7627	0.0984	0	3.1210	-0.3253	0	-1.3582	0.4237	0
8	2.0012	-0.0483	0	3.1116	-0.3615	0	-1.1105	0.3131	0
9	2.4984	-0.3827	0	2.4984	-0.3827	0	0	0	0

Table 3.3: Informed trader's trading strategy and the decomposition given public announcement at $t = 3$ without distortions by investor sentiment

	Trading strategy			Corrective demand			Manipulative demand		
t	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$
1	0.5590	**-0.0525**	0	0.7867	-0.2000	0	-0.2276	0.1475	0
2	0.7335	**-0.0556**	0	0.8558	-0.1807	0	-0.1223	0.1251	0
3	1.0069	0.1309	-0.3305	1.7814	-0.2275	-0.5606	-0.7745	0.3584	0.2301
4	1.1295	0.1365	0	1.9997	-0.2428	0	-0.8701	0.3793	0
5	1.2558	0.1502	0	2.2626	-0.2581	0	-1.0067	0.4083	0
6	1.3943	0.1313	0	2.4627	-0.2772	0	-1.0684	0.4085	0
7	1.5455	0.0846	0	2.6297	-0.3016	0	-1.0842	0.3863	0
8	1.7297	-0.0646	0	2.5726	-0.3391	0	-0.8429	0.2745	0
9	2.1041	-0.3699	0	2.1041	-0.3699	0	0	0	0

Table 3.4: Informed trader's trading strategy and the decomposition given public announcement at $t = 3$ with distortions by investor sentiment

$a_{X,t}$ denotes the coefficient of the informed trader's order flow at trade date t on fundamental factor $V - P_{t-1}$. $b_{X,t}$ denotes the coefficient on investor sentiment factor $S_t - a_S\hat{S}^M_{t-1}$. $c_{X,t}$ denotes the coefficient on public signal factor $Y_t - P_{t-1} - a_Y a_S \hat{S}^M_{t-1}$. The first three columns report the informed trader's strategy which are the sums of competitive and manipulative demands, and the next six columns report the decomposition of it. Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_S = 0.2$, $\sigma_U = 1$, $\sigma_{Y,t} = 10^6$ for all $t \neq 3, \sigma_{Y,t} = 1$ for $t = 3$, $a_Y = 0$ (Table 3.3.), $a_Y = 1$ (Table 3.4.), $a_U = 1$.

	Trading strategy			Corrective demand			Manipulative demand		
t	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$
1	0.5654	-0.0251	-0.0217	0.8256	-0.1922	-0.0318	-0.2602	0.1670	0.0100
2	0.6929	0.0116	-0.0206	1.0627	-0.2032	-0.0321	-0.3698	0.2148	0.0115
3	0.8214	0.0470	-0.0192	1.3165	-0.2149	-0.0319	-0.4950	0.2620	0.0127
4	0.9535	0.0782	-0.0179	1.5851	-0.2281	-0.0313	-0.6317	0.3063	0.0134
5	1.0917	0.1002	-0.0166	1.8604	-0.2434	-0.0303	-0.7687	0.3435	0.0136
6	1.2392	0.1034	-0.0156	2.1221	-0.2619	-0.0287	-0.8829	0.3654	0.0131
7	1.4005	0.0674	-0.0147	2.3277	-0.2874	-0.0262	-0.9272	0.3548	0.0115
8	1.5891	-0.0546	-0.0142	2.3780	-0.3265	-0.0223	-0.7889	0.2719	0.0081
9	1.9571	-0.3643	-0.0151	1.9571	-0.3643	-0.0151	0	0	0

Table 3.5: Informed trader's strategy and the decomposition given sequential public signals without distortions by investor sentiment

	Trading Strategy			Corrective demand			Manipulative demand		
t	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$	$a_{X,t}$	$b_{X,t}$	$c_{X,t}$
1	0.5644	0.0051	-0.0216	0.8416	-0.1781	-0.0307	-0.2772	0.1831	0.0091
2	0.6909	0.0488	-0.0189	1.0859	-0.1877	-0.0286	-0.3950	0.2365	0.0097
3	0.8147	0.0866	-0.0166	1.3369	-0.1991	-0.0262	-0.5222	0.2857	0.0097
4	0.9383	0.1159	-0.0145	1.5911	-0.2127	-0.0237	-0.6528	0.3286	0.0092
5	1.0645	0.1326	-0.0126	1.8393	-0.2286	-0.0209	-0.7748	0.3612	0.0083
6	1.1962	0.1281	-0.0110	2.0629	-0.2482	-0.0179	-0.8667	0.3762	0.0069
7	1.3373	0.0834	-0.0093	2.2245	-0.2748	-0.0145	-0.8872	0.3582	0.0052
8	1.4992	-0.0458	-0.0075	2.2369	-0.3157	-0.0105	-0.7377	0.2699	0.0030
9	1.8161	-0.3585	-0.0054	1.8161	-0.3585	-0.0054	0	0	0

Table 3.6: Informed trader's strategy and the decomposition given sequential public signals with distortions by investor sentiment

$a_{X,t}$ denotes the coefficient of the informed trader's order flow at trade date t on fundamental factor $V - P_{t-1}$. $b_{X,t}$ denotes the coefficient on investor sentiment factor $S_t - a_S\hat{S}^M_{t-1}$. $c_{X,t}$ denotes the coefficient on public signal factor $Y_t - P_{t-1} - a_Y a_S \hat{S}^M_{t-1}$. The first three columns report the informed trader's strategy which are the sums of competitive and manipulative demands, and the next six columns report the decomposition of it. Parameter values are given by $T = 10$, $\gamma = 4$, $\sigma_V = 1$, $\sigma_S = 0.2$, $\sigma_U = 1$, $\sigma_{Y,t} = 5$ for all $0 < t < 10$, $a_Y = 0$ (Table 3.5.), $a_Y = 1$ (Table 3.6.), $a_U = 1$.

Chapter 4

Limits to Observational Learning and Feedback Loops of Price Changes

4.1 Motivation

The self-reinforcing nature of price changes has been often considered to be one of the key factors that contribute to the occurrences of bubbles and crashes. A feedback loop of price changes may emerge due to wealth effects, consumption habits or credit expansions.[1] However, it has been suggested that this may also emerge due to learning from prices. For example, according to Shiller (2002), "The essence of a speculative bubble is the familiar feedback pattern - from price increases to increased investor enthusiasm to increased demand and, hence, to further price increases." Furthermore, feedback loops driven by observational learning may explain financial fads or manias occurring in a short period better than wealth effects, consumption habits or credit expansions do. This is because other feedback mechanisms take a relatively longer time to be built up and unwound. There have not been enough theoretical attempts to incorporate feedback loops of price changes driven by observational learning into financial

[1] See, for example, Kindleberger (2000) for general surveys on this issue.

equilibrium, despite its intuitive appeal and analytical usefulness.

This chapter attempts to explain why and how self-reinforcing fads may occur in financial equilibrium by analyzing a multiperiod trading model in which traders' observational learning is constrained by partitions (or discrete categories). I argue that seemingly 'excessive' extrapolation of past returns, which leads to self-reinforcing fads, is rather a consequence of natural responses to scarce cognitive resources than irrationality. Therefore, the equilibrium with bubble-like price patterns is stable even in the long run.

Traders may have some limitations to their observational learning for many reasons such as scarce cognitive resources (e.g., attention and memory) and physical constraints (e.g., time).[2] As a result, the traders who are constrained by scarce cognitive resources may have to use heuristic simplifications or a 'rule of thumb' in their observational inference. However, these heuristics are by no means irrational because they are based on sensible estimation procedures.[3] I assume that the traders are boundedly rational in the sense that they know their limits to observational learning, and the learning is rational under such limitations. Therefore, the traders are fully aware of their limitations and maximize their expected utility considering the limitations. The traders only understand the average behavior of prices over the partitions of price history. Their observational learning is coarse in the sense that the extraction of private information from the price history is limited by the partitions. For example, a trader whose observational learning is constrained by the partitions would learn the same informational content from observing price changes which only differ by a few basis points. Nevertheless, the result of observational learning will be correct on average with any given partition.

In this chapter, I focus on 'fads', which are bubble-like price patterns typically caused by increasing optimism of some portion of investors.[4] I define a bubble to be a situation in which prices deviate from the fundamental value of the traded asset for consecutive periods due to self-reinforcing belief updates.[5] Suppose that the price goes up at time t. The traders at

[2]See Kahneman (1973) for a more detailed discussion on the cognitive limits to the ability of reasoning.

[3]See Gilovich and Griffin (2002).

[4]Fads arise in the events in which some investors' valuation becomes increasingly higher than the true value of the traded asset.

[5]Bubbles could be defined in many different ways depending on the economic situations. Note that the bubble in this chapter is different from the class of bubbles, which are often called 'growth bubbles', where the price is higher than the valuation of all agents (e.g., Tirole (1982) and Allen, Morris, and Postlewaite (1993)). See

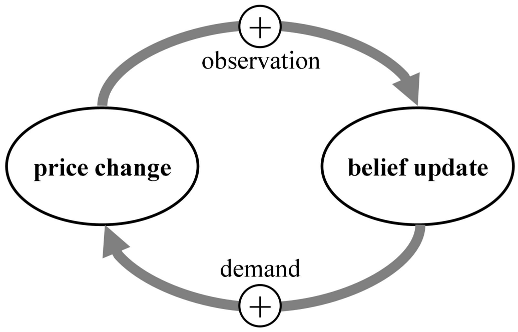

Figure 4.1: Self-reinforcing cycle of a learning bubble

time $t + 1$ who observe the previous return become more optimistic because they interpret the price movement as a good signal. As they increase their demand accordingly, the price at time $t + 1$ goes up again. The upward price movement makes the traders at time $t + 2$ more optimistic, so they increase their demand again. Consequently, the price at time $t + 3$ goes up again, and induces more optimistic beliefs in the subsequent future, and so on. Therefore, price changes generate a self-feeding upward momentum through the channel of observational learning even though price changes in this case do not contain any informational content regarding fundamentals. I call this feedback loop of price changes that are purely driven by excessive extrapolation of past returns a 'learning bubble'. It is a manifestation of self-reinforcing financial fads in which positive feedback trading is gradually amplified by its own feedback effects. Figure 4.1 provides a schematic illustration of the feedback mechanism.

The possibility of a learning bubble is crucially dependent on whether the informed traders are equipped with finer observational partitions relative to the uninformed traders. In the case of a learning bubble, the competitive informed traders would prefer to ride predictable upward price movements despite the divergence of prices from fundamentals. Suppose that a simultaneous shift of the informed traders' trading strategies is triggered by a particular path of the price history (or a 'technical signal') which predicts the occurrence of a learning bubble. If both the informed and uninformed traders are equipped with the same partitions, the uninformed traders also utilize the technical signal as much as the informed traders do. Therefore, any

Camerer (1989) or Chapter 2 of Brunnermeier (2001) for more discussion on the definition of bubbles.

upward price movement following the technical signal does not make the uninformed traders more optimistic about fundamentals. That is, a learning bubble does not occur.

Now suppose that the informed traders' observational partitions are finer than those of the uninformed traders. Even though the uninformed traders are not able to fully utilize each path of the price history for observational learning, they correctly weight the probability of the realization of each price path given the observed partition.[6] If a technical signal realizes with a small probability, the arrival of the technical signal would trigger a simultaneous shift of the informed traders' strategies without making too much impact on the belief updates of the uninformed traders. Unable to know the exact shift in the informed traders' strategies, the uninformed traders excessively extrapolate past returns. Consequently, feedback loops of price changes emerge because the uninformed traders' positive feedback trading gets amplified over time by its own feedback effects. In summary, learning bubbles may occur because the uninformed traders are unable to precisely distinguish between the price path which leads to a learning bubble and other regular price paths which do not. This chapter further shows that crashes may follow bubbles through the same mechanism as that creating learning bubbles.

Many financial anomalies can potentially be explained by excessive extrapolation of past returns.[7] This chapter argues that such seemingly 'excessive' extrapolation may not be naive after all, and is rather a consequence of natural responses to scarce cognitive resources. Therefore, the equilibrium outcome driven by excessive extrapolation is stable even in the long run. In this chapter, positive feedback trading is on average aligned towards the right direction. However, positive feedback trading can be intermittently amplified through feedback loops due to excessive extrapolation of past returns. This explains why positive feedback trading may cause momentum in general and why it does not necessarily cause reversal afterwards. That is, some incidents of momentum would be always followed by reversal because they are the consequences of feedback loops, which are driven by excessive extrapolation of past returns.[8]

[6] Strictly speaking, the traders observe an element (or price path) which belongs to a certain partition. In that case, the traders remember the partition rather than the element itself.

[7] See, for example, Lakonishok, Shleifer, and Vishny (1994) and Benartzi (2001) for evidence of excessive extrapolation of past returns.

[8] Momentum without reversal is the consequence of underreaction to observed information due to coarse observational learning.

It is a puzzle why practitioners continue to rely on technical analysis even though academics have long been skeptical about it. However, recent empirical evidence such as Brock, Lakonishok, and LeBaron (1992) and Lo, Mamaysky, and Wang (2000) finds a significant forecasting power of technical analysis. The result of this chapter implies that technical signals may naturally emerge from the equilibrium over the space of observation, and could possess significant power for forecasting future prices due to its self-fulfilling nature. This is because technical signals do not necessarily deliver information regarding fundamentals, and in fact they deliver non-fundamental information such as the timing of rallies driven by some traders who understand the signals. This confirms Jegadeesh (2000) who describes "my discussions with practitioners suggested that chartists tend to use signals from patterns in the price history in conjunction with other information to time their trades."

This chapter is organized as follows: In Section 4.2, I review the related literature. In Section 4.3, I describe the investment opportunities, traders and information structure. In Section 4.4, I give the formal definition of the limits to observational learning and an equilibrium concept under such limits. In Section 4.5, I first solve a benchmark equilibrium when the informed traders are equipped with observational partitions as coarse as the uninformed traders. By introducing strictly finer observational partitions to the informed traders, I show that a learning bubble equilibrium emerges under certain conditions. In Section 4.6, I further extend the model to the case of (i) a learning bubble with a crash, and (ii) a slow-growing learning bubble. In Section 4.7, I discuss empirical implications of the model regarding (i) bubbles and crashes, (ii) price behaviors such as momentum, reversal and excess volatility, and (iii) technical analysis.

4.2 Related Literature

This chapter is related to four strands of literature. The first strand studies the feedback effects of price changes. Most of the literature in this category focuses on the interaction between their price changes and their impact on fundamentals (e.g., Dow and Rahi (2003), Hirshleifer, Subrahmanyam, and Titman (2006) and Ozdenoren and Yuan (2007)). This chapter is closest to De Long, Shleifer, Summers, and Waldmann (1990b) who focus on the feedback effects of price changes caused by positive feedback traders. De Long, Shleifer, Summers, and Waldmann (1990b) show that informed speculators may jump onto rising prices when there are positive feedback traders. This chapter extends the result of De Long, Shleifer, Summers, and Waldmann (1990b) in the following directions: First, this chapter demonstrates that positive feedback trading could be generated by a belief system which is consistent with equilibrium while De Long, Shleifer, Summers, and Waldmann (1990b) are silent about why positive feedback trading arises.[9] Second, this chapter focuses on why and when positive feedback trading gets amplified by feedback loops of price changes. Third, this chapter studies the impact of positive feedback trading over a longer horizon, enabling the model to explain more empirical implications which include momentum, reversal and excess volatility. In particular, the findings of this chapter explain why reversal may or may not always follow momentum.

The second strand of literature studies how financial fads may arise due to observational learning. For example, Avery and Zemsky (1998) find that some specific structures of signals could lead to information cascades. This strand typically studies how information cascades arise in sequential trading models as shown in Glosten and Milgrom (1985) as a result of herd behavior driven by observational learning (e.g., Bikchandani, Hirshleifer, and Welch (1992), Lee (1998), Avery and Zemsky (1998), Decamps and Lovo (2006) and Park and Sabourian (2008)). Due to the existence of risk-neutral market makers with infinite liquidity, the price is usually set to be the market makers' conditional expectation of the fundamental value of an asset. Herd behavior of informed traders typically leads to information cascades. Consequently, prices which are equal to the valuation of the market makers do not change any more once herd behavior starts. Although the sequential trading setup successfully highlights why herd behavior may occur due to the failure of observational learning, it does not explain how herd behavior would lead to further abnormal price behavior afterwards. On the contrary, this chapter studies trading behavior driven by observational learning in a competitive market setup. One advantage of adopting a competitive market setup is that the price is set by a Walrasian auctioneer at a market clearing price. Therefore, the price can still respond to increased demand even after

[9] Consistent belief means the belief system is on average correct in equilibrium.

herd behavior of informed traders. It enables us to study feedback effects of price changes, which may be an essential feature of bubbles.

The third strand of literature explains financial anomalies such bubbles, momentum, and excess volatilities by adopting behavioral or psychological biases (e.g., De Long, Shleifer, Summers, and Waldmann (1990a), Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subramanyam (1998), Odean (1998) and Scheinkman and Xiong (2003)). For example, De Long, Shleifer, Summers, and Waldmann (1990a) shows that the existence of investor sentiment leads to mispricing due to limited arbitrage. Scheinkman and Xiong (2003) show that overconfident traders with heterogenous belief may get into speculative trades, thereby creating bubbles. The literature in this strand is usually silent about whether those belief systems would survive in the long run. This chapter complements this strand of literature by showing why there may exist subjective beliefs departing from objective beliefs by endogenizing the belief system into the model under a bounded rationality framework. Furthermore, this strand of literature usually assumes that information is exogenously given rather than endogenously learned from prices. This chapter contributes to an emerging literature by adding a new mechanism of generating financial instability through observational learning.

The fourth strand of literature focuses on how bubbles arise or are sustained due to the coordination failure of competitive informed traders (e.g., De Long, Shleifer, Summers, and Waldmann (1990a), Shleifer and Vishny (1997) and Abreu and Brunnermeier (2003)). For example, Abreu and Brunnermeier (2003) show that a bubble may persist because informed traders are unable to burst bubbles due to the dispersion of opinions and the lack of temporary coordination. This chapter complements this line of literature as follows: First, this chapter tracks the formation and collapse of bubbles, which are triggered by the simultaneous actions of the informed traders. Second, the coordination mechanism considered in this chapter is given by technical signals which are emerging in the price history. Third, this chapter contributes to an emerging literature in this strand by linking bubble phenomena with observational learning from prices.

In canonical noisy rational expectation equilibrium (NREE) models such as Grossman and Stiglitz (1980), Wang (1993) and Wang (1994), the price may depart from the fundamentals due to adverse liquidity shocks. In case similar kinds of shocks arrive consecutively, mispricing may become larger and the uninformed traders' beliefs may further deviate from the fundamentals. However, the feedback loops of price changes during a learning bubble fundamentally differ from the mispricing caused by a series of adverse liquidity shocks because a liquidity shock does not create any feedback effects by itself. That is, the initial mispricing is not magnified in the subsequent period unless similar kinds of liquidity shocks happen to arrive again. Furthermore, the feedback loops arise because the occurrence of a learning bubble is private information as well as the fundamental value of the risky asset in this model. The information regarding the occurrence of the feedback loops is only known to the speculators who have superior ability to interpret the price history. While canonical NREE models only consider information asymmetries on fundamentals, this chapter adds an extra dimension of information asymmetries on non-fundamental information about the shift of other informed traders' strategies.

Finally, the equilibrium concept in this chapter is related to Analogy-Based Expectations Equilibrium introduced in Jehiel (2005), who studies a game theoretic equilibrium with players who only understand the average behavior of their opponent over bundles of states. Players in the game defined in Jehiel (2005) use analogy-based inference on other traders' equilibrium behavioral strategy where players bundle states into analogy classes, and play best-response to their opponent's average strategy in those analogy classes. In this chapter, traders perform observational inference on equilibrium prices instead of other players' actions. Furthermore, the traders in this model may not have to know other traders' strategy themselves as long as they are aware of the equilibrium behavior of prices.[10] In their observational learning, the traders bundle observed price changes into disjoint partitions (or analogy classes in terms of Jehiel (2005)) and forms their average belief given the observation. Mullainathan, Schwartzstein, and Shleifer (2008) also study categorical thinking related to this line of literature.

[10] In the case of a large population game such a trading game in financial market among infinitely many traders, it is also difficult for each trader to follow every action of other traders in the market.

4.3 Basic Model

I consider a multiperiod trading model in a finite horizon where trading occurs at each time $t \in \mathcal{T} \equiv \{0, 1, ..., T\}$. There exists a riskless asset yielding a fixed return in every period with perfectly elastic supply. I normalize the gross return of the riskless asset to one for simplicity. There also exists a risky asset which yields the liquidation value V at time $T+1$. There are two states of the world where the state space is given by $\Omega = \{G, B\}$. Each unit of the share of the risky asset yields the payoff of $V = 1$ in the good state $\omega = G$ and $V = 0$ in the bad state $\omega = B$. The prior distribution of the state of the world, which is common knowledge among all traders, is given by

$$\omega \in \begin{cases} G, & \text{with probability } \pi; \\ B, & \text{with probability } 1 - \pi; \end{cases}$$

where $0 < \pi < 1$.

All traders are infinitesimal. Thus, they are all price takers. There are two types of traders in the economy: (i) uninformed long-term traders (denoted by U) and (ii) informed speculators (denoted by I). The long-term traders are long-lived, risk-averse and uninformed about the true state of the world. They have an identical CARA utility function with a risk aversion parameter γ. They also have an identical initial wealth W_0 at time $t = 0$. I assume that they form their demand based only on the price relative to the fundamental value of the risky asset similarly to 'passive investors' in De Long, Shleifer, Summers, and Waldmann (1990b) or 'newswatchers' in Hong and Stein (1999).[11] That is, they only consider the long-term prospect of the risky asset rather than the short-term speculative gains. On the other hand, the speculators are short-lived, risk-neutral and perfectly informed about the true state of the world. They have no initial wealth, and trade on margin to finance their investment. For simplicity, I assume that

[11] While the long-term traders trade dynamically, their demand is based on the static-optimization. That is, the long-term traders have time-inconsistency in their dynamic portfolio choice as the passive investors in De Long, Shleifer, Summers, and Waldmann (1990b) or the newswatchers in Hong and Stein (1999). The role of the time-inconsistent investors in the model is setting the price towards their valuation of the risky asset including the risk premium regardless of the speculative gains. Therefore, they work as a stabilizing force in the model as long as their belief is well aligned with the fundamentals.

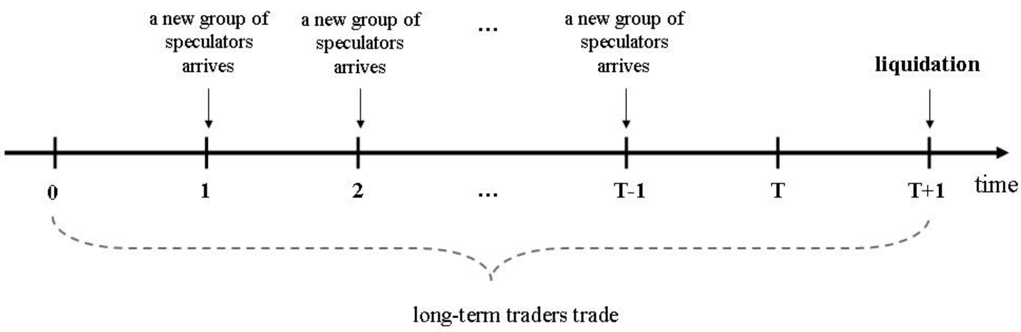

Figure 4.2: Timeline for the model

the margin constraint is given as a form of the maximum number of long or short positions, i.e., $x_t \in [-M, M]$ for all $t = 1, 2, \ldots, T-1$ where x_t is the position held by a speculator.[12]

The long-term traders arrive at the market at time $t = 0$ and trade until the liquidation at time $t = T + 1$. On the other hand, a new group of speculators arrives at the market at each time $t = 1, 2, \ldots, T-1$, and leaves the market in the subsequent future. I normalize the mass of the long-term traders and the new group of speculators who arrive at the market at each trading time $t = 1, 2, \ldots, T-1$ to be one, respectively. The timing of the model is summarized in Figure 4.2.

I assume that the net supply of shares y_t is initially set to zero (i.e., $y_0 = 0$), and follows an autoregressive process:

$$y_t = y_{t-1} + \epsilon_t,$$

where ϵ_t is a new innovation to the process of y_t. The random variable ϵ_t is independent over

[12]This is a very simplified assumption regarding the margin requirement. See Shleifer and Vishny (1997), Gromb and Vayanos (2002), Attari, Mello, and Ruckes (2005) and Brunnermeier and Pedersen (2009) for the more specialized setup.

time, and has a realization such that

$$\epsilon_t \in \begin{cases} \bar{y}+\rho, & \text{with probability } \eta; \\ \bar{y}, & \text{with probability } \frac{1-\theta}{2}-\eta; \\ 0, & \text{with probability } \theta; \\ -\bar{y}, & \text{with probability } \frac{1-\theta}{2}-\eta; \\ -\bar{y}-\rho, & \text{with probability } \eta; \end{cases}$$

where η, ρ are non-negative numbers. I index the innovation to the supply of shares by 'more positive' (++), 'positive' (+), 'null' ($\emptyset$), 'negative' (−), 'more negative' (−−) for $\bar{y}+\rho, \bar{y}, 0, -\bar{y}, -\bar{y}-\rho$, respectively. I assume that the magnitude of the innovation to the supply of shares is always smaller than the maximum aggregate demand of the speculators, i.e., $0 < \bar{y}+\rho < M$. I further assume that ρ is an increasing function of η, e.g., $\rho(\eta) = \eta$. Therefore, one could interpret the innovation ++ (or −−) as a small deviation from + (or −) when η is small.

4.4 Equilibrium Concept under Coarse Observational Learning

4.4.1 Coarse Observational Learning

I suppose that the traders receive the data of the price history in the form of returns.[13] Let $\mathcal{R}_t = \mathbb{R}$ be the space of returns at time t. Then, the space of the price history at time t is given by a product of the spaces of returns until time t: $\Lambda_t \equiv \mathcal{R}_1 \times \mathcal{R}_2 \times \ldots \times \mathcal{R}_t$. For example, the traders at time t would receive the data of the price history such that $\lambda_t \in \Lambda_t$ where $\lambda_t \equiv (r_1, r_2, \ldots, r_t)$ and $r_z \in \mathcal{R}_z$ is the realized return at each time $z = 1, 2, \ldots, t$. The long-term traders attempt to infer the true state of the world from the observation of λ_t. Therefore, observational learning of the long-term traders at time t could be represented by the following

[13]The form of the reported return could be any form such as logarithmic or arithmetic return or the difference of prices. For example, suppose p_{t-1}, p_t are the prices at time $t-1$ and time t, respectively. Then, the traders at time t can observe the return data $r_t \equiv \log(\frac{p_t}{p_{t-1}})$.

mapping:[14]

$$\mu_t^U(\cdot|\lambda_t \in \Lambda_t) : \Omega_t \to [0,1]. \tag{4.1}$$

On the other hand, the speculators attempt to predict the price in the subsequent future conditional on ω and the observation of λ_t. Therefore, observational learning of the speculators at time t could be represented by the following mapping:[15]

$$\mu_t^I(\cdot|\lambda_t \in \Lambda_t, \omega \in \Omega) : \mathcal{P}_{t+1} \to [0,1], \tag{4.2}$$

where $\mathcal{P}_{t+1}$ is the set of feasible prices at time $t+1$.

Due to the limitations to cognitive resources, agents may prefer using a discrete scale even when the underlying state is on a continuous scale. Dow (1991) shows that such discretization endogenously arises due to memory constraints in optimal consumption choices. Categorical ranking systems of analysts' recommendations are an example of transferring information on a partitioned signal space when the state space is continuous. Another example could be the perception of prices which are constrained by decimal numbers. It may have contributed to well-documented empirical phenomena of price clustering or resistance points of stock prices around decimal levels (e.g., Niederhoffer (1965), Niederhoffer (1966), Harris (1991), Donaldson and Kim (1993) and Sonnemans (2006)). Such partitions which compress the amount of information could be considered as heuristic simplifications which are natural responses to cognitive constraints.[16] In this chapter, I assume that the traders are equipped with discrete partitions which constrain their observational learning. I give a formal definition of coarse observational partitions as follows:

Definition The **return partition** of trader a at time t is a member of $\mathcal{R}_t^a \equiv \{R_{j,t}^a\}_{j=1}^N$, which is a collection of intervals such that $R_{1,t}^a \equiv (-\infty, r_1), R_{2,t}^a \equiv [r_1, r_2), \ldots, R_{N,t}^a \equiv [r_N, \infty)$ for some

[14]For example, if a long-term trader assigns 0.8 to the posterior probability of $\omega = G$ given (r_1, r_2), I say $\mu_2^U(G|(r_1, r_2)) = .8$.

[15]For example, if a speculator assigns 0.2 to the posterior probability of $p_{t+1} = 0.5$ given (r_1, r_2) and $\omega = G$, I say $\mu_2^I(0.5|(r_1, r_2), G) = .2$.

[16]See Hirshleifer (2001) for more discussion on heuristic simplifications.

sequence $-\infty < r_1 < r_2 < \ldots < r_N < \infty$.

Definition The **history partition** of trader a at time t is a member of $\mathcal{J}_t^a \equiv \{J_{j,t}^a\}_{j=1}^{\mathcal{N}}$, which is a collection of non-empty subsets of Λ_t such that

(i) $J_{j,t}^a \equiv R_{j_1,1}^a \times R_{j_2,2}^a \times \ldots \times R_{j_t,t}^a$ where $R_{j_z,z}^a \subset \mathcal{R}_z^a$ for all $z = 1, 2, \ldots, t$,

(ii) $J_{j,t}^a$'s are mutually disjoint,

(iii) $\cup_{j=1}^{\mathcal{N}} J_{j,t}^a = \Lambda_t$.

For notational convenience, I denote $J_t^a(\lambda_t)$ to be the history partition of trader a at time t which includes λ_t. Finally, I define coarse observational learning which is constrained by $\mathcal{J}_t^U, \mathcal{J}_t^U$, respectively:

Definition (i) The long-term traders at time t has **coarse observational learning** if $\mu_t^U(\omega|\lambda_t) = \mu_t^U(\omega|\lambda_t')$ for all $\omega \in \Omega$ and all $\lambda_t, \lambda_t' \in J_{j,t}^U$ given any $J_{j,t}^U \in \mathcal{J}_t^U$.

(ii) The speculators at time t has **coarse observational learning** if $\mu^I(p_{t+1}|\lambda_t, \omega) = \mu_t^I(p_{t+1}|\lambda_t', \omega)$ for all $p_{t+1} \in \mathcal{P}_{t+1}, \omega \in \Omega$ and all $\lambda_t, \lambda_t' \in J_{j,t}^I$ given any $J_{j,t}^I \in \mathcal{J}_t^I$.

Therefore, a trader who performs coarse observational learning deduce the same information content from two different price paths if they belong to the same history partition. Coarse observational learning is consistent with the equilibrium if and only if the learning is correct on average in each history partition.

Definition (i) μ_t^U is **consistent** with the history λ_t if

$$\mu_t^U(\omega|\lambda_t) \equiv \sum_{\lambda_t' \in J_t^U(\lambda_t)} \frac{Pr(\lambda_t')Pr(\omega|\lambda_t')}{\sum_{\lambda_t' \in J_t^U(\lambda_t)} Pr(\lambda_t')} \tag{4.3}$$

(ii) μ_t^I is **consistent** with the history λ_t if

$$\mu_t^I(p_{t+1}|\lambda_t, \omega) \equiv \sum_{\lambda_t' \in J_t^I(\lambda_t)} \frac{Pr(\lambda_t'|\omega)Pr(p_{t+1}|\lambda_t', \omega)}{\sum_{\lambda_t' \in J_t^I(\lambda_t)} Pr(\lambda_t'|\omega)}. \tag{4.4}$$

A trader with consistent coarse observational learning puts the correct weighting on each possible price path in the given history partition. This condition is equivalent to the 'weak consistency' in Jehiel (2005). Similarly to Jehiel (2005), the interpretation of this condition goes as follows: If a trader repeatedly participates in trading, the trader will eventually understand the correct weighting which coincides with the true frequency of each price path in the given history partition.

4.4.2 The Evolution of the Long-term Traders' Beliefs

For simplicity, I call $p_t \geq p_{t-1}$ an 'upward' price movement, and $p_t < p_{t-1}$ a 'downward' price movement. I assume that the long-term traders' return partitions at time t are given by

$$\mathcal{R}_z^U = \{(-\infty, 0), [0, \infty)\} \text{ for } z = 1, 2, \ldots, t-1,$$

and

$$\mathcal{R}_t^U = \{(-\infty, \infty)\}.$$

Therefore, the long-term traders' observational learning is 'binary' and 'delayed'. That is, the long-term traders only learn from either upward or downward price movements in the past. Since their learning is delayed, they do not infer the state of the world from the Walrasian auctioneer's price offers at the current period.[17] A similar delay in observational learning in a Walrasian auction setup has been considered in Hellwig (1982).[18] On the contrary, I assume that the speculators' observational learning is not delayed. I say that the speculators' observational partitions are 'finer than' those of the long-term traders if the speculators have finer return partitions than binary partitions in every period. I say that the speculators' observational partitions are 'as coarse as' those of the long-term traders if the speculators also have binary return partitions. For example, the long-term traders in the current period and the speculators in the previous period would observe the same signal from the price history if the speculators'

[17]Note that their demand is still elastic to price changes regardless of the coarseness of their observational partitions.

[18]There are also other reasons to create delayed response to information; for example, Vayanos and Woolley (2008) considers gradual adjustment of the demand to observed information due to contractual restrictions or institutional lags in a continuous time framework.

observational partitions are as coarse as those of the long-term traders.

The set of observable price changes at time t can be represented by $S_t \equiv \{u_t, d_t\}$ where u_t, d_t denotes the upward and downward price movement at time t, respectively. I index the price change at time t as $s_t \in S_t$. For notational convenience, I will denote the long-term traders' history partition at time t by suppressing the superscript U as follows:

$$J_{t-1}(\lambda_t) \equiv J_t^U(\lambda_t) = \{s_1, s_2, \ldots, s_{t-1}\}.$$

The long-term traders' beliefs about $\omega = G$ conditional on λ_t, of which derivation is relegated to Appendix G, is given by

$$\mu_t^U(\omega|\lambda_t) = \frac{Pr(J_{t-1}(\lambda_t)|\omega)\pi}{Pr(J_{t-1}(\lambda_t)|G)\pi + Pr(J_{t-1}(\lambda_t)|B)(1-\pi)}. \tag{4.5}$$

I define l_t to be the log-likelihood ratio of the long-term traders' beliefs at time t, which measures the odds of being in the state G versus B given the long-term traders' beliefs based on the observed history λ_t:

$$l_t \equiv \log\left(\frac{\mu_t^U(G|\lambda_t)}{\mu_t^U(B|\lambda_t)}\right) = \log\left(\frac{Pr(G|J_{t-1}(\lambda_t))}{Pr(B|J_{t-1}(\lambda_t))}\right).$$

Then, the initial condition is given by $l_1 = l_0 = \log\left(\frac{\pi}{1-\pi}\right)$ due to delayed observational learning. The following lemma is derived immediately from (4.5) and the definition of l_t:

Lemma 4.1 *The evolution of the long-term traders' beliefs about ω is governed by the following linear equation with the initial condition $l_1 = \log\left(\frac{\pi}{1-\pi}\right)$:*

$$l_t = l_{t-1} + \Delta l_{s,t},$$

where

$$\Delta l_{s,t} = \begin{cases} 0, & \text{if } Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}), G) = Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}), B) = 0; \\ \log\left(\frac{Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}),G)}{Pr(s_{t-1}|J_{t-2}(\lambda_{t-1}),B)}\right), & \text{otherwise}; \end{cases}$$

4.4.3 Coarse Learning Expectations Equilibrium

I denote the conditional expectation of the long-term traders and the speculators at time t under coarse observational learning using a superscript U and I, respectively:

$$\begin{aligned} E_t^U\big[\mathcal{U}\big(W_0+(V-p_t)x\big)\big] &= -\sum_{\omega'\in\Omega}\mu_t^U(\omega'|\lambda_t)\exp\Big(-\gamma\big(W_0+(V(\omega')-p_t)x\big)\Big); \\ E_t^I\big[(p_{t+1}-p_t)x\big] &= \sum_{p'_{t+1}\in\mathcal{P}_{t+1}}\mu_t^I(p'_{t+1}|\lambda_t,\omega)(p'_{t+1}-p_t)x, \end{aligned}$$

where $V(\omega)$ is the liquidation value of the risky asset in the state ω. Finally, a symmetric equilibrium under coarse observational learning is formalized as follows:[19]

Definition The sequence $\big(p_t, x_t^U, x_t^I\big)_{t=0}^T$ is a **Coarse Learning Expectations Equilibrium (CLEE)** under observational partitions $\mathcal{J}_t^U, \mathcal{J}_t^I$ if there exists beliefs μ_t^U, μ_t^I such that

(i) $x_t^U \in \underset{x\in(-\infty,\infty)}{\text{argmax}}\ E_t^U[\mathcal{U}(W_0+(V-p_t)x]$ for all $t=0,1,\ldots,T$,

(ii) $x_t^I \in \underset{x\in[-M,M]}{\text{argmax}}\ E_t^I[(p_{t+1}-p_t)x]$ for all $t=1,2,\ldots,T-1$ and $x_t^I=0$ for all $t=0,T$,

(iii) μ_t^U, μ_t^I are consistent with λ_t,

(iv) $x_t^U + x_t^I = y_t$ for all $t=0,1,\ldots,T$.

4.4.4 Learning Bubble Equilibrium

I call a feedback loop of upward price changes that is driven by excessive extrapolation of past returns a 'learning bubble'. A learning bubble equilibrium is a CLEE in which a learning bubble occurs with a positive probability in the bad state. There may exist other types of equilibria in which a feedback loop of downward price changes occur in the good state. For simplicity of analysis, I only consider the CLEE with a feedback loop of upward price changes in the bad state.

I formalize the concept of a learning bubble equilibrium. First, I define $\Delta\bar{l}$ to be an endogenous parameter which does not depend on time or sample path.[20] I also define Σ to be the set of all possible realizations of $\{\epsilon_t\}_{t=1}^T$, and $\hat{\Sigma}$ to be a nonempty subset of Σ.

[19] Since the expected utilities given all observations within the same partition are identical, the set of optimal strategies is also identical. I focus on the case where the equilibrium strategy is identical for each type.

[20] $\Delta\bar{l}$ is an endogenous parameter because it is computed in the equilibrium.

Definition A CLEE is a **learning bubble equilibrium** if a learning bubble arises with a positive probability in the bad state, i.e., there exists $\mathcal{T}^* \subset \mathcal{T}$ which satisfies the followings when $\omega = B, \{\epsilon_t\}_{t=1}^T \in \hat{\Sigma}$ with $Pr\big(\omega = B, \{\epsilon_t\}_{t=1}^T \in \hat{\Sigma}\big) > 0$:
(i) a positive return in the previous period implies more optimistic beliefs of the long-term traders in the current period, during $\mathcal{T}^*$, i.e.,

$$\Delta l_{u,t} > \Delta \bar{l} \text{ if } p_{t-1} > p_{t-2} \text{ for all } t \in \mathcal{T}^*,$$

(ii) more optimistic beliefs of the long-term traders in the current period imply a positive return in the current period, during $\mathcal{T}^*$, i.e.,

$$p_t > p_{t-1} \text{ if } \Delta l_{u,t} > \Delta \bar{l} \text{ for all } t \in \mathcal{T}^*.$$

4.5 Equilibrium

In this section, I solve the equilibrium of the model in two cases: (i) benchmark: the speculators' observational partitions are as coarse as those of the long-term traders, and (ii) learning bubbles: the case in which the speculators' observational partitions are finer than those of the long-term traders.

4.5.1 Benchmark Case

There always exists a fully-revealing CLEE such that the price converges to the true value at time $t = 2$, and the proof is relegated to Appendix J. Under certain conditions, there exists an equilibrium which partially reveals private information of the informed speculators at every period. I will focus on the partially-revealing equilibrium rather than other types of equilibria. I directly give the result in the following proposition by relegating the proof to Appendix J:

Proposition 4.2 (Partially-revealing Equilibrium) *There exist a partially-revealing CLEE under some parameter values of M and θ such that the following properties are true for $t = 1, 2, \dots, T-1$:*

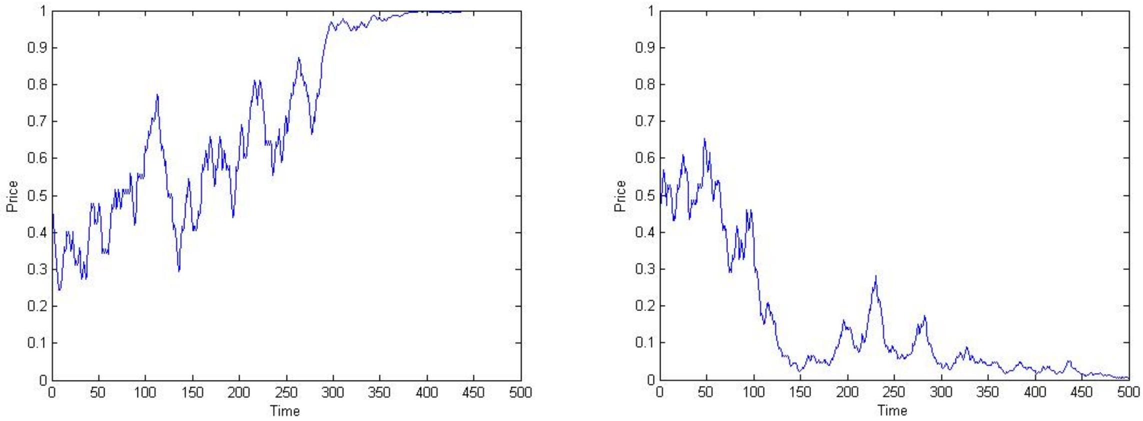

Figure 4.3: Simulated examples of partially revealing equilibrium when $\omega = G$ (left) and $\omega = B$ (right)
(Parameter values: $T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}$)

(1) ***State-dependent Return Process****: (i) When* $\omega = G$, $p_t \geq p_{t-1}$ *if* $\epsilon_t \in \{--, -, \emptyset\}$, *and* $p_t < p_{t-1}$ *if* $\epsilon_t \in \{+, ++\}$, *(ii) When* $\omega = B$, $p_t \geq p_{t-1}$ *if* $\epsilon_t \in \{--, -\}$, *and* $p_t < p_{t-1}$ *if* $\epsilon_t \in \{\emptyset, +, ++\}$,

(2) ***Momentum of Expected Return****: For all* $\omega \in \{G, B\}$, $E_t^I[p_{t+1}] \geq p_t$ *if* $p_t \geq p_{t-1}$, *and* $E_t^I[p_{t+1}] < p_t$ *if* $p_t < p_{t-1}$.

The State-dependent Return Process Property (*Property-S*) states that the return in the current period depends on the true state as well as the innovation to the supply of shares. Furthermore, the price is more likely to go up in the good state, and is more likely to go down in the bad state.[21] The Momentum of Expected Return Property (*Property-M*) states that the return at time $t+1$ is expected to continue in the same direction as the return at time t. Figure 4.3 shows simulated examples of partially-revealing CLEE which satisfy both *Property-S* and *Property-M*.

The evolution of the long-term traders' beliefs under *Property-S* is given by the following linear equation with the initial condition $l_1 = \log(\pi/(1-\pi))$:

$$l_t = l_{t-1} + \Delta l_{s,t}, \tag{4.6}$$

[21] Since the direction of the return at time t is determined only by ω and ϵ_t, it is independent of the previous returns.

where

$$\Delta l_{s,t} = \begin{cases} \log\left(\frac{1+\theta}{1-\theta}\right), & \text{if } s_{t-1} = u_{t-1}; \\ -\log\left(\frac{1+\theta}{1-\theta}\right), & \text{otherwise;} \end{cases}$$

for all $t = 2, 3, \ldots, T$. The change in the long-term traders' beliefs at time t is either positive or negative depending on the sign of s_{t-1}. Therefore, the long-term traders always engage in positive feedback trading because of their delayed learning. The aggregate demand of the long-traders is given by

$$x_t^U = \frac{1}{\gamma}\left[l_t + log\left(\frac{1-p_t}{p_t}\right)\right]. \tag{4.7}$$

Property-M shows that prices are expected to have momentum given the information set of the speculators. This is because of the long-term traders' delayed response to observed information as well as the portfolio constraints of the speculators. Therefore, the aggregate demand of the speculators for all non-bubble paths $\lambda_t \in \hat{\Lambda}_t$ is given by a 'bang-bang' solution due to their short horizon for all $t = 1, 2, \ldots, T-1$:

$$x_t^I = \begin{cases} M, & \text{if } p_t \geq p_{t-1}; \\ -M, & \text{otherwise.} \end{cases}$$

Solving the market clearing condition, the equilibrium price at time t given l_t, y_t and x_t^I is represented as[22]

$$p_t = \frac{1}{1 + \exp\Big(-l_t + \gamma(y_t - x_t^I)\Big)}. \tag{4.8}$$

The following lemma is immediately derived from the above result:

[22]The market clearing condition is given by $x_t^I + x_t^U = y_t$.

Lemma 4.3 *The return at time t has the same sign as* $\Delta l_{s,t} - \gamma\big[\epsilon_t - (x^I_t - x^I_{t-1})\big]$*, i.e.,*

$$sign(p_t - p_{t-1}) = sign\Big(\Delta l_{s,t} - \gamma\big[\epsilon_t - (x^I_t - x^I_{t-1})\big]\Big).$$

The aggregate demand of the speculators is given by a 'bang-bang' solution due to speculators' short horizon and *Property-M* for all $t = 1, 2, \ldots, T-1$:

$$x^I_t = \begin{cases} M, & \text{if } p_t \geq p_{t-1}; \\ -M, & \text{otherwise.} \end{cases}$$

Finally, the proof in the appendix shows the existence of some parameter regions which satisfy *Property-S* and *Property-M* given the optimal demand of the traders. It is worth mentioning about the parameter regions which support the partially-revealing equilibrium. If prices change too much in one period, the curvature of the price function (4.8) becomes too significant relative to main factors such as l_t, x^I_t and y_t. On the other hand, small enough price changes minimize the effect of curvature of the price function. Therefore, the price change in each period needs to be small enough for the equilibrium to exist. The proof in the appendix shows that such condition is achieved when M is small enough. That is, the volume of informed trading needs to be smaller than some threshold not to fully reveal private information of the informed traders.[23]

4.5.2 Learning Bubbles

The Possibility of Learning Bubbles

In this section, I show that learning bubbles can arise if the uninformed traders perform coarser observational learning than the informed traders. I assume that the speculators have fine enough return partitions which enable them to infer the realizations of ϵ_t precisely from price history.[24] In Section 4.5.2, I show how the equilibrium becomes different if the speculators are

[23] In case informed trading volume is large enough, there only exists fully-revealing equilibrium.

[24] Note that ϵ_t is the only source of randomness in this model. That is, there is only a finite number of feasible price paths because ϵ_t has only a finite number of realizations. Therefore, there always exists a set of fine enough history partitions $\mathcal{J}^I_t$ which contains each price path in separate history partition. Since real numbers

also constrained by the same binary partitions of the long-term traders.

I briefly describe the general idea of the proof here: Suppose that a certain path of price history conveys a technical signal that predicts a series of upward price movements in the subsequent future. Also suppose that the speculators' observational partitions are finer than those of the long-term traders, and the technical signal is precisely understood only by the speculators. The speculators can simultaneously shift their trading strategies depending on the arrival of the technical signal while the uninformed traders only recognize the possibility of the arrival of the technical signal. Unable to know the exact shift in the informed traders' strategies, the uninformed traders update their beliefs by assigning an appropriate weight to the arrival of the technical signal given their observation. Because the long-term traders understand only the average behaviors of prices over their observational partitions, they always become more optimistic given positive returns as long as the technical signal arrives with a small probability. Consequently, the long-term traders excessively extrapolate past returns in case the technical signal indeed arrives without their knowledge. Consequently, a feedback loop of price changes emerges because the long-term traders' positive feedback trading gets amplified over time by its own effects.

I call the collection of trading times in which a feedback loop of upward price changes arises a 'bubble period', and the collection of all other trading times a 'non-bubble period'. I denote the bubble period to be $\mathcal{T}^* \equiv \{\tau, \tau+1, \ldots, \tau+b-1\}$ where $b \geq 1$ is the length of the bubble period. I call the price path at time $t = \tau - 1$ which triggers the bubble period in the subsequent future a 'trigger path', which is denoted by $\lambda^b_{\tau-1}$.[25] I call any price path during the bubble period a 'bubble path', and denote Λ^*_t to be the set of all possible bubble paths at time t. I also denote $\hat{\Lambda}_t \equiv \Lambda_t \setminus \Lambda^*_t$ to be the set of all 'non-bubble paths', which is a set of price history void of any bubble path.[26] The timing of a learning bubble is summarized in Figure 4.4.

I give the following proposition by relegating the proof to Appendix K.

are dense, there exists such sequence of real numbers $-\infty < r^I_1 < r^I_2 < \ldots < r^I_N < \infty$ which separates all possible realizations of returns for each period, and this constructs a return partition given in Definition 4.4.1 for each period.

[25]Note that the return at $\tau - 1$ needs to be greater than or equal to zero (i.e., $s_{\tau-1} \geq 0$) due to Definition 4.4.4.

[26]The relative complement of a set $\mathcal{A}$ in a set $\mathcal{B}$ is defined to be $\mathcal{B} \setminus \mathcal{A} = \{z \in \mathcal{B} \,|\, z \notin \mathcal{A}\}$.

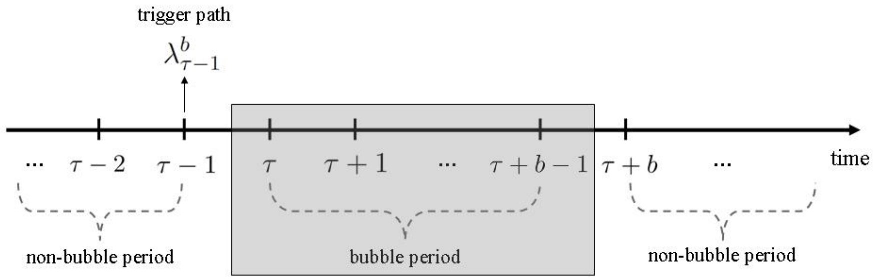

Figure 4.4: Timeline of a learning bubble

Proposition 4.4 (Learning Bubble Equilibrium) *When η is small enough, there exists a learning bubble equilibrium under some parameter values such that the following properties are true for the set of bubble paths $\{\Lambda^*_t\}_{t=\tau}^{\tau+b-1}$ and a trigger path $\lambda^b_{\tau-1} \in \Lambda_{\tau-1}$ with $Pr(\lambda^b_{\tau-1}|B) > 0$ and $\frac{\partial Pr(\lambda^b_{\tau-1}|B)}{\partial \eta} > 0$.*

***I.** For all non-bubble paths $\lambda_t \in \hat{\Lambda}_t$:*

> *(1) **State-dependent Return Process**: (i) When $\omega = G$, $p_t \geq p_{t-1}$ if $\epsilon_t \in \{--, -, \emptyset\}$ and $p_t < p_{t-1}$ if $\epsilon_t \in \{+, ++\}$, (ii) When $\omega = B$, $p_t \geq p_{t-1}$ if $\epsilon_t \in \{--, -\}$ and $p_t < p_{t-1}$ if $\epsilon_t \in \{\emptyset, +, ++\}$,*
>
> *(2) **Momentum of Expected Return**: for all $\omega \in \{G, B\}$ (i) $E^I_t[p_{t+1}] \geq p_t$ if $p_t \geq p_{t-1}$, (ii) $E^I_t[p_{t+1}] < p_t$ if $p_t < p_{t-1}$,*

***II.** For all bubble paths $\lambda_t \in \Lambda^*_t$:*

> *(1) **State-independent Return Process**: $p_t > p_{t-1}$ for all $\epsilon_t \in \{--, -, \emptyset, +, ++\}$,*
>
> *(2) **Positive Momentum of Expected Return**: $E^I_t[p_{t+1}] > p_t$.*

The two properties in the case of the non-bubble paths characterize the price behaviors when prices follow regular paths that do not involve feedback patterns: the State-dependent Return Process Property (*Property-S-I*) and the Momentum of Expected Return Property (*Property-M-I*). On the other hand, the two properties in the case of the bubble paths characterize the price behaviors when prices follow irregular paths that involve feedback patterns: the State-independent Return Process Property (*Property-S-II*) and the Positive Momentum of Expected Return Property (*Property-M-II*). In summary, there occurs a feedback loop of upward price

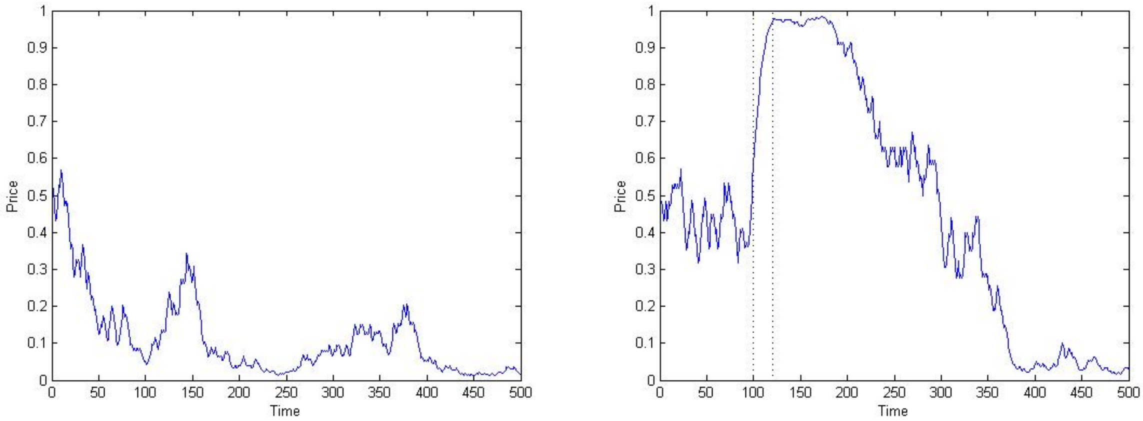

Figure 4.5: Simulated examples of learning bubble equilibrium without reaching the trigger path (left) and with reaching the trigger path (right)
(The trigger path for a bubble is three consecutive periods of $\epsilon_t \in \{--\}$ at each time $t = 98, 99, 100$ ($\kappa(\lambda^b_{\tau-1}) = \eta^3$), Parameter values: $\omega = B, T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}, \eta = \rho = 0.001, b = 20$)

changes when $\lambda_t \in \Lambda^*_t$. Figure 4.5 shows simulated examples of a learning bubble equilibrium which follows the four properties.

I first solve for the optimal choices of the traders assuming given the four properties. Then, I show that the equilibrium which emerges out of those optimal choices indeed has such properties. For notational convenience, I denote $\kappa(\lambda^b_{\tau-1}) \equiv Pr(\lambda_{\tau-1} = \lambda^b_{\tau-1}|B)$ to be the probability of the arrival of the trigger path given $\omega = B$. Using *Property-S-I* and *Property-S-II*, I solve for the evolution of long-term traders' beliefs in each period.

Lemma 4.5 *The evolution of the long-term traders' beliefs is given by the following linear equation with the initial condition* $l_1 = \log(\pi/(1-\pi))$*:*

$$l_t = l_{t-1} + \Delta l_{s,t}$$

where (i) the change in belief:

$$\Delta l_{s,t} = \begin{cases} \log\left(\frac{1+\theta}{(1-\theta)(1-\Phi_{t-1}(\lambda_t))+2\Phi_{t-1}(\lambda_t)}\right), & \text{if } s_{t-1} = u_{t-1}; \\ \log\left(\frac{1-\theta}{1+\theta}\right), & \text{otherwise}; \end{cases}$$

and (ii) the probability of a bubble:

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\kappa(\lambda^b_{\tau-1})}{\kappa(\lambda^b_{\tau-1}) + \left(\frac{1-\theta}{2}\right)^{t+1-\tau}(1-\kappa(\lambda^b_{\tau-1}))}, & \text{if } \Lambda^*_{t-1} \subset J_{t-1}(\lambda_t); \\ 0, & \text{otherwise}; \end{cases}$$

for all $t = 1, 2, \ldots, T$

Proof See Appendix K.

The aggregate demand of the long-term traders is given by the same represenation as in the benchmark case regardless of bubbles. On the other hand, the aggregate demand of the speculators for all bubble paths $\lambda_t \in \Lambda^*_t$ is given by the maximum long position, i.e., $x^I_t = M$. That is, the demand becomes a price-inelastic function because the risk-neutral speculators expect the return will be strictly positive due to *Property-M-II*.[27]

When the long-term traders can clearly perceive that there is no possibility of a learning bubble given λ_t (i.e., $\Lambda^*_t \not\subset J_{t-1}(\lambda_t)$), we find that $\Phi_{t-1}(\lambda_t) = 0$. Then, $\Delta l_{u,t} = -\Delta l_{d,t} = \log\left(\frac{1+\theta}{1-\theta}\right)$ because $\Phi_{t-1}(\lambda_t) = 0$. When the long-term traders perceive the possibility of a learning bubble (i.e., $\Lambda^*_t \subset J_{t-1}(\lambda_t)$), we find that $\Phi_{t-1}(\lambda_t) > 0$. Therefore, Lemma 4.5 shows two things: (i) the long-term traders' belief updates are symmetric on both si exactly equivalent to the benchmark case if there exists no possibility of a learning bubble given the past price history, (ii) the long-term traders' belief updates given positive returns are less optimistic than the benchmark case if there exists a positive probability of a learning bubble.

Since Φ_{t-1} is monotone increasing in $\kappa(\lambda^b_{\tau-1})$, the evolution of the long-term traders' beliefs becomes similar to the benchmark case as $\kappa(\lambda^b_{\tau-1})$ becomes smaller. I confine the analysis to the case where $\kappa(\lambda^b_{\tau-1})$ is a function of η such that $\frac{\partial\kappa}{\partial\eta} > 0$. Then, the evolution of the long-term traders' beliefs becomes sufficiently close to the benchmark case for small enough η.[28] For example, there exists such function $\kappa(\lambda^b_{\tau-1}) = \eta^{\tau-1}$ if a learning bubble is triggered by $\tau - 1$ consecutive upward price movements driven by $\epsilon_t \in \{--\}$ at each time $t \in \{1, 2, \ldots, \tau-1\}$.[29]

[27] The speculators in an aggregate level take the maximum long position at $\tau - 1$ and unwind it sometime when adverse liquidity shock arrives after $\tau + b - 1$.

[28] $\Phi_{t-1}(\lambda_t)$ approaches to zero as $\eta \downarrow 0$.

[29] Note that ϵ_t is the only source of randomness to the speculators. Therefore, each history path λ_t is unique to each realization of ϵ_t given ω.

Suppose that the trigger path has been reached at time $t = \tau - 1$, i.e., $\lambda_{\tau-1} = \lambda^b_{\tau-1}$. Then, the following is immediate from Lemma 4.3:

$$p_t > p_{t-1} \text{ if } \Delta l_{u,t} > \gamma \bar{y} \text{ for all } t \in \mathcal{T}^*.$$

Furthermore, the long-term traders' belief updates are close to the level in the benchmark case if $\Phi_{t-1}(\lambda_t)$ is small. When η is small enough, it is possible to find some region of θ which satisfies the following:

$$\Delta l_{u,t} > \gamma \bar{y} \text{ if } p_{t-1} > p_{t-2} \text{ for all } t \in \mathcal{T}^*.$$

Since the two conditions in Definition 4.4.4 are satisfied, the initial upward price movement at time $t = \tau - 1$ creates a feedback loop of upward price changes during the bubble period. In the benchmark case, the long-term traders' positive feedback trading is frequently corrected by the speculators depending on the realizations of ϵ_t. On the other hand, the long-term trader' positive feedback trading is uninterrupted during the bubble period, thereby creating self-reinforcing upward price movements. It shows why positive feedback trading may or may not get amplified depending on the situation.

The Impossibility of Learning Bubbles

In this section, I show that there exists no learning bubble equilibrium if the speculators' observational partitions are as coarse as those of the long-term traders. The result in the previous section shows that the long-term traders always engage in positive feedback trading because of their delayed response to price changes. However, the long-term traders would never engage in another positive feedback trading in the subsequent future if price changes are solely created by the feedback effects of their own trading volume. As long as the uninformed traders can figure out the shift of the informed traders' trading strategies, a learning bubble, which is a feedback loop of price changes driven by excessive extrapolation of past returns, cannot occur. The following lemma shows that the two conditions in Definition 4.4.4 cannot be true together when the speculators' observational partitions are as coarse as those of the long-term traders.

Proposition 4.6 (The Impossibility of Learning Bubble) *There exists no CLEE which is a learning bubble equilibrium if the informed speculators' observational partitions are as coarse as those of the uninformed long-term traders.*

Proof See Appendix I.

Therefore, there would not be any feedback effect of price changes driven by observational learning as long as the uninformed traders can precisely detect potential triggers of feedback loops. Proposition 4.6 implies that there cannot be any feedback effects of price changes in canonical NREE models because agents in those models have the same ability of observational learning. That is, there is no extra signal which coordinates a simultaneous shift of informed agents' trading strategies without the knowledge of uninformed agents in those models.[30]

4.6 Extension

4.6.1 Crashes

In a learning bubble equilibrium in the previous section, the divergence of prices from the fundamental value is always slowly adjusted over time once the bubble period is over. In real life, more acute corrections such as crashes may occur together. It is well known that competitive traders are often unable to burst bubbles on their own because of the lack of coordination. (See, for example, De Long, Shleifer, Summers, and Waldmann (1990a), Shleifer and Vishny (1997) and Abreu and Brunnermeier (2003)) Similarly, the speculators in this model never engage in the correction of mispricing unless all other speculators coordinate to do so. The coordination of attacking mispricing can be initiated by a technical signal, thereby creating a crash following a learning bubbles.

I assume that a crash may occur only when a learning bubble already has occurred. I denote $\lambda^c_{\tau'-1}$ to be the trigger path which triggers the crash at the subsequent period $t = \tau'$. I denote

[30] Prices may depart from the fundamental value due to a series of adverse liquidity shocks in canonical NREE models, but the liquidity shocks by themselves do not create any feedback effect.

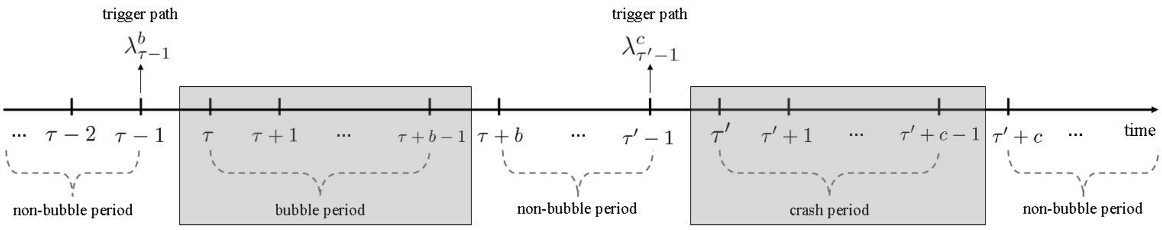

Figure 4.6: Timeline of a learning bubble with a crash

the crash period to be $\mathcal{T}^{**} \equiv \{\tau', \tau'+1, \ldots, \tau'+c-1\}$ where $c \geq 1$ is the length of the crash period. I denote $\lambda^c_{\tau'-1}$ to be the trigger path which triggers the crash in the subsequent trading time $t = \tau'$. I call each path of the price history during the crash period a 'crash path' and denote $\Lambda^*_t, \Lambda^{**}_t$ to be the set of all bubble paths and crash paths at time t, respectively. I also denote $\hat{\Lambda}_t \equiv \Lambda_t \setminus (\Lambda^*_t \cup \Lambda^{**}_t)$ to be the set of all 'non-bubble paths', which is a set of the price history void of any crash or bubble path. The timing of a learning bubble with a crash is summarized in Figure 4.6.

The learning bubble equilibrium with a crash is exactly same as the one described in Section 4.5.2 except that there additionally exists the set of crash paths $\{\Lambda^{**}_t\}_{t=\tau'}^{\tau'+c-1}$ and a trigger path $\lambda^c_{\tau'-1} \in \Lambda_{\tau'-1}$ with $Pr(\lambda^c_{\tau'-1}|B) > 0$ and $\frac{\partial Pr(\lambda^c_{\tau'-1}|B)}{\partial \eta} > 0$. Furthermore, there is an additional property for the crash paths as follows:

III. For all crash paths $\lambda_t \in \Lambda^{**}_t$:

(1) **State-independent Negative Return Process**: $p_t < p_{t-1}$,

(2) **Negative Momentum of Expected Return**: $E^I_t[p_{t+1}] < p_t$.

The State-independent Negative Return Process Property (*Property-S-III*) states that the return is always negative during the crash period and the Negative Momentum of Expected Return Property (*Property-M-III*) states that the speculators expect strict negative returns during the period correctly. Note that a crash may or may not happen depending on the sample path of the price history, i.e., a crash occurs only when $\lambda^c_{\tau'-1}$ is reached. Figure 4.7 shows a simulated example of a learning bubble equilibrium with a crash.

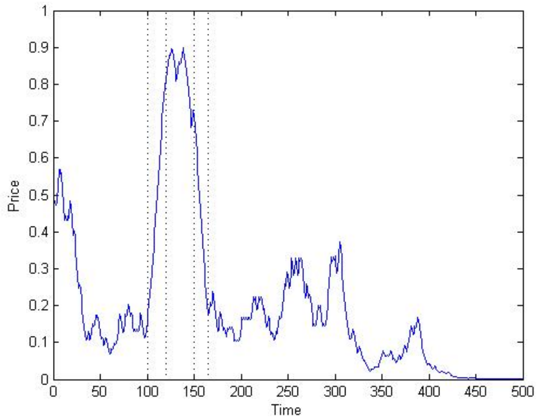

Figure 4.7: A simulated example of a learning bubble with a crash
(The trigger path for a bubble is three consecutive periods of $\epsilon_t \in \{--\}$ at each time $t = 98, 99, 100$ ($\kappa(\lambda^b_{\tau-1}) = \eta^3$), and the trigger path for a crash given the bubble period is $\epsilon_t \in \{++\}$ at time $t = 150$ ($\kappa(\lambda^c_{\tau'-1}) = \eta$), Parameter values: $\omega = B, T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}, \eta = \rho = 0.001, b = 20, c = 15$)

Using *Property-S-I*, *Property-S-II* and *Property-S-III*, I obtain the evolution of the long-term traders' beliefs in each period.

Lemma 4.7 *The evolution of the long-term traders' beliefs is given by the following linear equation:*

$$l_t = l_{t-1} + \Delta l_{s,t}$$

where (i) the change in belief:

$$\Delta l_{s,t} = \begin{cases} \log\left(\frac{1+\theta}{(1-\theta)(1-\Phi_{t-1}(\lambda_t))+2\Phi_{t-1}(\lambda_t)}\right), & \textit{if } s_{t-1} = u_{t-1}; \\ \log\left(\frac{1-\theta}{(1+\theta)(1-\Phi_{t-1}(\lambda_t))+2\Phi_{t-1}(\lambda_t)}\right), & \textit{otherwise}; \end{cases}$$

and (ii) the probability of a bubble and a crash:

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\kappa(\lambda^b_{\tau-1})}{\kappa(\lambda^b_{\tau-1})+\left(\frac{1-\theta}{2}\right)^{t+1-\tau}(1-\kappa(\lambda^b_{\tau-1}))} & \textit{if } \Lambda^*_{t-1} \subset J_{t-1}(\lambda_t); \\ \left(\frac{\kappa(\lambda^c_{\tau'-1})}{\kappa(\lambda^c_{\tau'-1})+\left(\frac{1+\theta}{2}\right)^{t+1-\tau'}(1-\kappa(\lambda^c_{\tau'-1}))}\right)\left(\frac{\kappa(\lambda^b_{\tau-1})}{\kappa(\lambda^b_{\tau-1})+\left(\frac{1+\theta}{2}\right)^{t+1-\tau'}\left(\frac{1-\theta}{2}\right)^{b}(1-\kappa(\lambda^b_{\tau-1}))}\right), & \textit{if } \Lambda^{**}_{t-1} \subset J_{t-1}(\lambda_t); \\ 0, & \textit{otherwise}; \end{cases}$$

for all $t = 1, 2, \ldots, T$ *and* $\kappa(\lambda^c_{\tau'-1}) \equiv Pr(\lambda^c_{\tau'-1} | \Lambda^*_{\tau+b-1})$.

Proof See Appendix L.

Due to *Property-M-III*, the aggregate demand of the speculators is given by $x^I_t = -M$ for all $t \in \mathcal{T}^{**}$. From Lemma 4.3, we find that

$$p_t < p_{t-1} \text{ if } \Delta l_{d,t} < -\gamma \bar{y} \text{ for all } t \in \mathcal{T}^{**}.$$

Furthermore, the belief updates of the long-term traders are close to the level in the benchmark case if $\Phi_{t-1}(\lambda_t)$ is small. Therefore, it is possible to have the following for some parameter values as long as the trigger path arrives with a small probability:

$$\Delta l_{d,t} < -\gamma \bar{y} \text{ if } p_{t-1} < p_{t-2} \text{ for all } t \in \mathcal{T}^{**}.$$

A feedback loop of downward price changes emerge through a similar mechanism of creating a feedback loop of upward price changes. I relegate the rest of the proof to Appendix L, which establishes the existence of parameter regions which satisfy the additional conditions.

4.6.2 Slow-growing Learning Bubble

In a learning bubble, prices move in one direction due to feedback loops. Therefore, they are acute price changes, which do not allow any downward fluctuations. In this section, I explore a more realistic type of self-feeding price movements where both upward and downward price movements occur during the bubble period. Even though there are some price fluctuations, the price process tends to show a strong upward trend during the bubble period. A slow-growing learning bubble equilibrium is exactly same as the one in Section 4.5.2 except for the second property as follows:

Proposition 4.8 (Slow-growing Learning Bubble Equilibrium) *When η is small enough, there exists a slow-growing learning bubble equilibrium under some parameter values such that the* *following properties are true for the set of bubble paths* $\{\Lambda_t^*\}_{t=\tau}^{\tau+b-1}$ *and a trigger path* $\lambda_{\tau-1}^b \in \Lambda_{\tau-1}$ *with* $Pr(\lambda_{\tau-1}^b|B) > 0$ *and* $\frac{\partial Pr(\lambda_{\tau-1}^b|B)}{\partial \eta} > 0$.

- *For all paths* $\lambda \in \Lambda_t$*:*

  Momentum of Expected Return*: for all* $\omega \in \{G, B\}$ *(i)* $E_t^I[p_{t+1}] \geq p_t$ *if* $p_t \geq p_{t-1}$*, (ii)* $E_t^I[p_{t+1}] < p_t$ *if* $p_t < p_{t-1}$*,*

I. *For all non-bubble paths* $\lambda_t \in \hat{\Lambda}_t$*:*

State-dependent Return Process*: (i) When* $\omega = G$*,* $p_t \geq p_{t-1}$ *if* $\epsilon_t \in \{--, -, \emptyset\}$ *and* $p_t < p_{t-1}$ *if* $\epsilon_t \in \{+, ++\}$*, (ii) When* $\omega = B$*,* $p_t \geq p_{t-1}$ *if* $\epsilon_t \in \{--, -\}$ *and* $p_t < p_{t-1}$ *if* $\epsilon_t \in \{\emptyset, +, ++\}$*,*

II. *For all bubble paths* $\lambda_t \in \Lambda_t^*$*:*

Quasi State-dependent Return Process: $p_t \geq p_{t-1}$ *if* $\epsilon_t \in \{--, -, \emptyset\}$ *and* $p_t < p_{t-1}$ *if* $\epsilon_t \in \{+, ++\}$.

The State-dependent Return Process Property (*Property-S*) is exactly the same as in learning bubbles while the Quasi State-dependent Return Process (*Property-QS*) states that the return process in the bad state behaves in the same way in the good state once the price history gets into the bubble paths. Figure 4.8 shows a simulated example of a slow-growing learning bubble equilibrium.

Using *Property-S* and *Property-QS*, I obtain the evolution of the long-term traders' beliefs in each period.

Lemma 4.9 *The evolution of the long-term traders' beliefs is given by the following linear equation:*

$$l_t = l_{t-1} + \Delta l_{s,t}$$

where (i) the change in belief:

$$\Delta l_{s,t} = \begin{cases} \log\left(\frac{1+\theta}{(1-\theta)(1-\Phi_{t-1}(\lambda_t))+(1+\theta)\Phi_{t-1}(\lambda_t)}\right), & \text{if } s_{t-1} = u_{t-1}; \\ \log\left(\frac{1-\theta}{(1+\theta)(1-\Phi_{t-1}(\lambda_t))+(1-\theta)\Phi_{t-1}(\lambda_t)}\right), & \text{otherwise}; \end{cases}$$

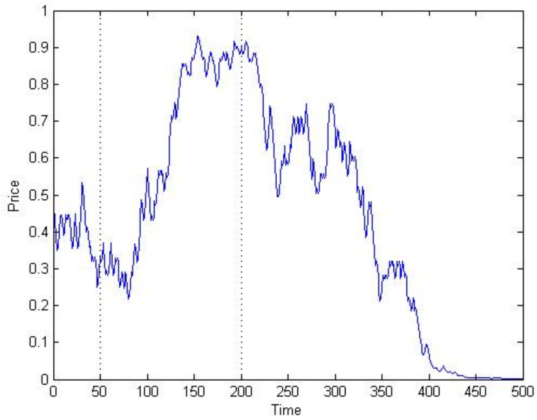

Figure 4.8: A simulated example of a slow-growing learning bubble
(The trigger path for a bubble is three consecutive periods of $\epsilon_t \in \{--\}$ at each time $t = 48, 49, 50$ ($\kappa(\lambda^b_{\tau-1}) = \eta^3$), Parameter values: $\omega = B, T = 500, \theta = 0.08, M = 0.1, \bar{y} = 0.01, \gamma = 1, \pi = \frac{1}{2}, \eta = \rho = 0.001, b = 150$)

for all $t = 1, 2, \ldots, T$ *and (ii) the probability of a bubble:*

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{e^*_t(k)\kappa(\lambda^b_{\tau-1})}{e^*_t(k)\kappa(\lambda^b_{\tau-1}) + e_t(k)(1-\kappa(\lambda^b_{\tau-1}))}, & \textit{for } t \in \mathcal{T}^*; \\ 0, & \textit{otherwise}; \end{cases}$$

where $e^*_t(k) = {}_{t-\tau}C_k\left(\frac{1+\theta}{2}\right)^k\left(\frac{1-\theta}{2}\right)^{t-\tau-k}$ *and* $e_t(k) = {}_{t-\tau}C_k\left(\frac{1-\theta}{2}\right)^k\left(\frac{1+\theta}{2}\right)^{t-\tau-k}$ *given* k *upward price movements during the period* $\{\tau, \tau+1, \ldots, t-1\} \in \mathcal{T}^*$*.*

Proof See Appendix M.

The aggregate demand of the speculators is given by the same bang-bang solution as in the benchmark case. Note that the probability of having an upward price movement given the bad state is $\frac{1-\theta}{2}$ in non-bubble paths while it is $\frac{1+\theta}{2}$ in bubble path. From Lemma 4.3, we find that

$$Pr(p_t > p_{t-1} | \lambda_t \in \Lambda^*_t, B) > Pr(p_t > p_{t-1} | \lambda_t \in \hat{\Lambda}_t, B) \text{ if } \Delta l_{u,t} > 0 \text{ for all } t \in \mathcal{T}^*.$$

Furthermore, the belief updates of the long-term traders are close to the level in the benchmark case if $\Phi_{t-1}(\lambda_t)$ is small. Therefore, it is possible to have the following for some parameter values as long as the trigger path arrives with a small probability:

$$\Delta l_{u,t} > 0 \text{ if } p_{t-1} > p_{t-2} \text{ for all } t \in \mathcal{T}^*.$$

Unlike in a learning bubble, feedback loops are defined as a probabilistic shift of the price process which allows more upward price movements during the bubble period. I relegate the rest of the proof to Appendix M, which establishes the existence of parameter regions which satisfies the equilibrium properties.

4.7 Empirical Implications

4.7.1 Bubble and Crashes

There have been well-known incidents of bubbles and crashes through the history of financial markets (e.g., Dutch tulip mania (1634-1637), South Sea Bubble (1719-1720), Great Crash (1929), Internet Bubble (1992-2000) and Credit Crunch (2007)). While the characteristics of bubbles may vary a lot depending on the economic situations during the incidents, there has been some evidence that at least some bubble incidents feature (i) informed traders who ride the bubbles, and (ii) uninformed traders who keep increasing their shares over time, and lose a significant amount of wealth after the collapses of the bubbles. For example, Brunnermeier and Nagel (2004) and Temin and Voth (2004) find that informed traders rode bubbles due to predictable investor sentiment during Internet Bubble and South Sea Bubble periods, respectively. While such phenomena could be well explained by the combination of limited arbitrage and behavioral biases such as optimistic beliefs or investor sentiment (e.g., De Long, Shleifer, Summers, and Waldmann (1990a) and Shiller (2000)), the question still remains where the behavioral biases come from and how the behavioral biases could have survived without being adjusted despite the relatively long history of financial markets. Since a biased belief is a system of subjective beliefs which is inconsistent with the equilibrium, it should be critically assumed that the bias is not corrected even after many repetitions of the same situation to maintain the stability of the equilibrium. For example, Bray (1982) shows that an equilibrium with subjective beliefs driven by misspecified models is unstable if the misspecification can be adjusted over time.[31] Another interesting observation is that the bubbles and crashes have been repeated through the history of financial markets, but they seem to occur at a very low frequency or at least not too often.[32] Therefore, it raises another question why the behavioral biases may affect the financial market intermittently.[33]

The result of this chapter shows that bubble-like price patterns arise intermittently due to the feedback effects of price changes driven by excessive extrapolation of past returns. Since gradual amplification of optimism is caused by natural responses to cognitive limits, the equilibrium outcome is stable even in the long run unless there is any significant change in cognitive resources. Furthermore, I demonstrate that the bubble-like price patterns should arise at a low frequency in the equilibrium. Proposition 4.4 shows that the technical signal which triggers financial fads needs to arrive at a low frequency.[34] I summarize the finding in the following corollary.

Corollary 4.10 *The frequency of financial fads $Pr(\lambda^b_{\tau-1}|B)$ needs to be low in a learning bubble equilibrium.*

Bubbles and crashes in real life are much complicated economic events which involve many other factors such as consumption and investment of private sectors as well as public sectors. As is shown in many historical examples of financial fads or manias, the feedback mechanism which is studied in this chapter would contribute to the formation of bubbles along with other feedback mechanisms such as wealth effects, consumption habits and credit expansions.

[31]Bray (1982) finds that the temporary equilibrium with subjective beliefs eventually converges to the rational expectations equilibrium with objective beliefs in the long run if the adjustment of misspecification is allowed.

[32]See Greenwood and Nagel (2009).

[33]One of the possible explanations mentioned by Greenwood and Nagel (2009) is related to the consequence of adaptive learning such that young generations who have not directly experienced stock market downturns are more prone to the optimism that fuels bubbles. While it could be true for some big bubbles which occur not more than once in each generation, the story cannot explain a series of smaller bubbles which occur multiple times in one generation.

[34]In a learning equilibrium, we need to have small enough η. Since $\frac{\partial Pr(\lambda^b_{\tau-1}|B)}{\partial \eta} > 0$, $Pr(\lambda^b_{\tau-1}|B)$ becomes smaller as η gets smaller.

4.7.2 Price Behaviors

Momentum and Reversal driven by Technical Indicators

There exists a large volume of empirical evidence regarding short-term momentum effects (e.g., Jegadeesh and Titman (1993), Lakonishok, Shleifer, and Vishny (1994) and Jegadeesh and Titman (2001)) and long-term reversal effects (e.g., De Bondt and Thaler (1985), Lehmann (1990), Lee and Swaminathan (2000) and Jegadeesh and Titman (2001)). Theoretical explanations in general show that delayed response to information can create momentum effects. For example, Daniel, Hirshleifer, and Subramanyam (1998) show that momentum occurs due to delayed overreaction driven by self-attribution bias. Hong and Stein (1999) show that underreaction to information occurs due to slow information diffusion across the population of traders. Vayanos and Woolley (2008) show that momentum occurs because migrations among delegated portfolios create delayed response when investors gradually learn fund managers' ability. This chapter confirms the findings of the previous literature because the momentum in this chapter is also mainly driven by positive feedback trading which is caused by the long-term traders' delayed learning.[35]

Empirical evidence of Nofsinger and Sias (1999) and Hvidkjaer (2006) finds that investors engage in positive feedback trading, which may contribute to momentum effects. While momentum effects can be potentially explained by positive feedback trading described in De Long, Shleifer, Summers, and Waldmann (1990b), the inconsistency of reversal following momentum poses challenges to this attempt (e.g., Chan, Jegadeesh, and Lakonishok (1996)). The difference of this chapter from the previous theoretical literature is that there are two types of momentum effects: (i) momentum effects without feedback loops, and (ii) momentum effects with feedback loops. Momentum generally arises due to positive feedback trading regardless of the occurrence of feedback loops, and momentum is on average not followed by reversal because momentum is on average heading toward the right direction. On the other hand, momentum which is caused with feedbacks loops always lead to reversal.[36] Therefore, this finding fills the gap between

[35] There are two important frictions which create price momentum in this model: First, the uninformed traders' response to price history is delayed. Second, the informed traders have portfolio constraints which makes them unable to push the price immediately to the expected level.

[36] Depending on the occurrence of crashes, the speed of reversal would vary.

the existing theoretical explanations of momentum effects based on positive feedback trading and empirical observations, by showing that momentum may or may not accompany reversal afterwards depending on the existence of feedback loops.

Positive Feedback Trading and Excessive Extrapolation of Past Returns

In this chapter, the long-term traders who have delayed learning optimally engage in positive feedback trading because prices are on average moving in the right direction. In any partially-revealing equilibrium, Monotone Likelihood Ratio Properties (MLRP) of $\Delta l_{s,t}$ is a necessary condition whether it is a learning bubble equilibrium or not,[37] i.e.,

$$\Delta l_{d,t} < 0 < \Delta l_{u,t} \quad \text{for all } t \in \mathcal{T}.$$

MLRP states that the price is more likely to go up in the good state, and also opposite is true in the bad state.[38] Since the long-term traders cannot learn private information of the speculators immediately due to their coarse observational learning, prices slowly converge to the true fundamental value over time.[39]

This chapter also sheds light on excessive extrapolation of past returns, which may contribute to various empirical phenomena. For example, Lakonishok, Shleifer, and Vishny (1994) argue that momentum and reversal may have been caused by the suboptimal trading behavior of some investors with naive expectation which puts excessive weight on the recent history of prices.[40] This chapter argues that the seemingly 'excessive' extrapolation which creates positive feedback trading may not be naive after all, and is rather a consequence of natural responses to the scarcity of cognitive resources. The long-term traders understand that they may excessively extrapolate past returns in case the speculators shift their trading strategy simultaneously. Since such event occurs with a low frequency, the long-term traders also recognize that the probability of making excessive extrapolation is small. Therefore, they optimally engage in extrapolation

[37] It is a direct consequence from Lemma J.2, Lemma K.5 and Lemma K.6.

[38] Empirical evidence such as Beaver, Lambert, and Morse (1980) finds that price information indeed has significant forecasting power regarding fundamentals.

[39] This is related to Norman (1972) who shows a distance diminishing learning operator eventually converges to true value.

[40] See, for example, Benartzi (2001) for further evidence of excessive extrapolation of past returns.

of past returns, which may become excessive with a small probability. Therefore, the tendency of excessive extrapolation of past returns would survive even in the long run without being adjusted unless there is significant improvement in cognitive resources.

Excess Price Volatility and Price Jumps

Another empirical implication of this chapter is that extra price volatility can be generated endogenously by feedback loops without any exogenous shocks. It has been well known that price volatility is too high to be justified by fundamental volatility (e.g., Shiller (1981) and LeRoy and Porter (1981)). Shiller (1990) suggests that feedback trading may be responsible for excess volatility. Frankel and Froot (1990) suggest that trading volumes based on technical analysis may be responsible for large fluctuations in exchange rates which are not justified by fundamentals. In line with the argument of Shiller (1990) and Frankel and Froot (1990), this chapter finds that excess volatility can arise without any fundamental reasons or external shocks. This is because feedback loops of price changes are triggered by internally emerging signals in price history. There are well-documented empirical evidence on discontinuous jumps in the asset price (e.g., Eraker (2001)). It has been widely used in many asset pricing models as jump diffusion process (e.g., Merton (1976) and Bates (1996)). While price jumps are often interpreted as abnormal shocks due to exogenous arrivals of new information, this chapter suggests an additional mechanism of generating price jumps internally.[41]

4.7.3 Technical Analysis

Academics have been skeptical about technical analysis. Nevertheless, technical analysis has been widely used by practitioners for a long time. Recent academic research such as Brock, Lakonishok, and LeBaron (1992) and Lo, Mamaysky, and Wang (2000) finds empirical evidence significant forecasting power of technical analysis. The result of this chapter implies that technical analysis may possess significant power for forecasting future prices because of its self-fulfilling nature. A series of specific amount of returns is defined as technical patterns which signal buying or selling to the informed traders. In real life, however, there are numerous

[41]See, for example, Merton (1976).

types of technical signals such as chart patterns (e.g., head-and-shoulder) or time-series patterns (e.g., moving average of past prices).[42] My findings imply that technical signals deliver non-fundamental information regarding the timing of the shift of other traders' strategies rather than information regarding the fundamental value.

Practitioners who perform technical analysis often use terms such as 'primary trend' and 'secondary trend'. Primary trend means price movements in a long-term period (e.g., a year or more), and secondary trend means price movements towards the opposite direction of the primary trend in a short-term period. Some secondary trends are popular enough to have a unique name such as 'bear market rally', which often means a short-term rebound of prices after a large drop. That is, a bear market rally is an upward price trend which follows a large slide of prices, but does not last too long before it is reversed to long-term downward trends. The result of the previous section gives an intriguing prediction regarding the span of rallies given different kinds of technical indicators. If a rally is triggered by rising prices, it can last longer. On the other hand, if a rally is triggered by falling prices, it can not sustain itself too long. Bull market rallies are sustained more easily than bear market rallies if those rallies are uninformative price movements simply driven by the feedback loop of price changes during learning bubbles. That is, technical patterns hidden in the past price history with an upward trend can support longer rallies than technical patterns hidden in the past price history with a downward trend. I give the following corollary by relegating the proof to Appendix N:

Corollary 4.11 *A financial fad triggered by an upward trend is more likely to be sustainable than the one triggered by a downward trend, i.e.,*

$$\kappa(\lambda^d_{\tau-1}) > \kappa(\lambda^u_{\tau-1}).$$

for any trigger paths $\lambda^d_{\tau-1}$ and $\lambda^u_{\tau-1}$ such that $\lambda^u_{\tau-1}$ is identical to $\lambda^d_{\tau-1}$ except that it has more upward movements than $\lambda^d_{\tau-1}$ does.

[42] For example, Zhu and Zhou (2009)) analyze the usefulness of moving average trading rule in portfolio allocation problems.

4.8 Conclusion

This chapter develops a model of financial fads in which traders' observational learning is constrained by discrete partitions. The traders are boundedly rational in the sense that they are fully aware of their limitations, and the learning is rational under such limits. In particular, the traders only understand the average behavior of prices over the discrete partitions of past price history. Nevertheless, the result of observational learning is correct on average with any given partition.

I call a feedback loop of price changes that is driven by excessive extrapolation of past returns a 'learning bubble', which is a manifestation of self-reinforcing financial fads. Learning bubbles cannot occur if both the informed and uninformed traders share the same observational partitions. Suppose that the informed traders' observational partitions are finer than those of the uninformed traders. Then, the informed traders may simultaneously shift their trading strategies depending on the arrival of certain price paths while the uninformed traders only recognize the possibility of such events. Unable to know the exact shift in the informed traders' strategies, the uninformed traders excessively extrapolate past returns. Consequently, a learning bubble emerge because the uninformed traders' positive feedback trading gets amplified over time by its own feedback effects. I further find that crashes may arise through the same mechanism as that creating learning bubbles.

In this chapter, prices are on average moving in the right direction. Therefore, the uninformed traders who have delayed learning optimally engage in positive feedback trading. Momentum arises as a result of positive feedback trading, while reversal does not necessarily follow momentum. I find that momentum is always followed by reversal when momentum is driven by feedback loops of price changes. I also argue that seemingly 'excessive' extrapolation may not be naive after all, and is rather a consequence of natural responses to scarce cognitive resources. Therefore, the equilibrium outcome driven by excessive extrapolation is stable even in the long run. I also find that technical signals convey information about non-fundamentals such as the timing of rallies rather than information about fundamentals. The technical signals could possess significant power for forecasting future prices because of its self-fulfilling nature.

Many financial phenomena may arise due to traders' heuristic thinking. If heuristic simplifications are natural responses to limited cognitive resources, the traders would choose optimal heuristics given the equilibrium price process. If so, the equilibrium price process would be also altered by the optimally chosen heuristics. The endogenous relationship between the equilibrium price process and the optimal choice of heuristics under cognitive constraints is at the heart of this chapter. The optimally chosen heuristics increase individual-level efficiency of trading at the cost of occasional individual-level mistakes. This chapter shows that such individual-level mistakes caused by the heuristics generate the amplification effects of mispricing in the aggregate level. This is because the equilibrium prices suffer aggregate-level instability with some probabilities which are small enough to be overlooked by each individual trader. My findings shed light on intermittently-arising financial instability caused by naive trend-chasing behaviors in various types of markets. For example, future work could explore trend-chasing behaviors in bond or currency markets by incorporating the underlying economics factors of each market within this bounded rationality framework. My findings may give even broader implications beyond financial markets regarding the relationship between heuristic judgements and system instability caused by trend-chasing behaviors of economic agents.

Chapter 5

Information Acquisition Chains

5.1 Motivation

It has been often assumed that information is exogenously given to investors in financial markets to study financial equilibrium. Under such an assumption, the flow of information to investors is inelastic regardless of the intensity of shocks to fundamentals. In real life, however, any change in the intensity of the shocks to fundamentals may result in significant changes of the incentive of acquiring information. This chapter attempts to incorporate such an impact of fundamental shocks on information acquisition into the analysis of financial equilibrium.

The main focus of this chapter is on the intertemporal interaction of information acquisition in the presence of fundamental shocks in each period. The result of this chapter shows information acquisition of agents in each period is interconnected like chains over different periods. The "arbitrage chains" of Dow and Gorton (1994) show that agents with limited horizon shun from trading on their private information on a risky asset unless the subsequent generation of agents trade on that information because they cannot carry their holdings of the risky asset until the liquidation. This chapter extends the argument on the intertemporal arbitrage under limited horizon to the intertemporal information acquisition under limited horizon: Short-lived agents in the current generation would refrain from acquiring information if the subsequent generation of agents does not acquire that information. That is, the incentive of agents in the subsequent generation could be significantly altered by the behavior of agents in the current generation. I will call such intertemporal dependence of information acquisition as "information acquisition chains". This chapter studies how such information acquisition chains alter the behaviors of equilibrium prices in the presence of fundamental shocks.

The model provides the following findings. First, information acquisition in the past and the current period is strategic substitutes to information acquisition in the current period. On the other hand, information acquisition in the future is strategic complements to information acquisition in the current period. Second, the prices in the later periods cannot become uninformative without making the prices in the earlier periods uninformative. It is because the payoff of short-lived agents in the current period is purely determined by the valuation of the subsequent generation of agents and the noise in the supply in the subsequent period. Therefore, uninformative prices in the future results in the failure of information acquisition in the current period. Third, a big enough shock in the future could eliminate the incentive of acquiring information in any early periods. Since it breaks information acquisition chains, the price in any earlier periods before the arrival of the shock becomes uninformative. Fourth, the price delay to information could arise because agents do not have enough incentive to acquire information due to the volatile future prospect of a traded asset.

The recent empirical evidence implies that the degree of price reaction to information is mainly related to informational factors rather than other frictions. For example, Hou and Moskowitz (2005) finds that investor recognition is more responsible for price delay than size, liquidity, or microstructure effects. Furthermore, fundamental shocks affect the flow of information into the prices. Zhang (2006) finds that price underreaction to information is greater when fundamental volatility is higher. Especially, their evidence shows that post-news price drift increases with more uncertainty.

This chapter is organized as follows. Section 5.3 describes the basic setup of the model, which is a multi-period pure-exchange economy with overlapping generations of agents in a continuum. Section 5.4 solves an equilibrium of information acquisition in all periods. Section 5.5 analyzes how the intertemporal interactions of information acquisition occur.

5.2 Literature Review

There is no doubt that agents have limited ability of acquiring information. One of the most obvious reason would be scarce cognitive resource as was pointed out by Kahneman (1973). There are also physical constraints which limits information acquisition such as time and distance. Moreover, there are non-physical constraints such as knowledge, memory, relation(or social networks). More frictions to acquiring information will require economic agents to put more efforts to collect desired signals. Facing a finite limit of resource for acquiring information, agents would have to make optimal choice in information acquisition problems. There exists a vast amount of literature which explores the optimal information acquisition problem under costly information acquisition settings. That is, agents can purchase signals with monetary costs. This chapter also uses a costly information acquisition framework close to the ones in Grossman and Stiglitz (1980), Verrecchia (1982), Kim and Verrecchia (1991), Barlevy and Veronesi (2000), Holden and Subrahmanyam (2002), Peress (2004), Mendelson and Tunca (2004), Cespa (2008), Cespa and Foucault (2008).

Although information acquisition in dynamic setting is a realistic one, there have not been many studies on this issue due to its technical complexity. A few exceptions are Holden and Subrahmanyam (2002), Peng (2005), Peng and Xiong (2006), Mendelson and Tunca (2004), Cespa (2008). Holden and Subrahmanyam (2002) analyze this problem numerically with costly information acquisition setting. Peng (2005), Peng and Xiong (2006) studies it in continuous time setting using entropy approach using a representative agent. Mendelson and Tunca (2004), Cespa (2008) uses Kyle model extension using costly information acquisition approach. This chapter differs from previous literature of dynamic information acquisition in two things: (i) This chapter studies dynamic information acquisition problem in multi-period competitive market with differential information, (ii) This chapter features a finite horizon overlapping generations (OLG) model to focus on the behavior of agents with limited horizon, and (iii) This chapter analyzes the intertemporal interaction between different generations of agents in terms of information acquisition.

Overlapping Generations model has been widely used in financial economics especially with

CARA-normal setting involving information asymmetries, e.g., De Long, Shleifer, Summers, and Waldmann (1990a), Dow and Gorton (1994), Allen, Morris, and Shin (2006), Biais, Bossaerts, and Spatt (2006), Watanabe (2008).[1] Both Biais, Bossaerts, and Spatt (2006), Watanabe (2008) features overlapping generations extension of Admati (1985) in infinite horizon with dividends. This chapter features finite horizon OLG model without any dividend. The most crucial difference from the previous literature is that this chapter features information acquisition in a finite horizon. As one will see in the later section, the mathematical structure of Admati (1985) in a static version is preserved in finite horizon OLG version without dividend while aggregate parameters are transformed due to the nature of multiple periods.

Other line of literature close to this chapter could be the ones related to the beauty contest in financial market or forecasting forecasts of others, e.g., Townsend (1983), Sargent (1991), Biais and Bossaerts (1998), Allen, Morris, and Shin (2006), Gao (2008), Makarov and Rytchkov (2008). These models typically find that agents bias toward public information because the payoff depends on what others think. This chapter also finds that agents collect information what others will know in the future. However, this chapter differs from this strand of literature because the aggregation of information is rather complete in this model due to the assumption of continuum of agents.[2] Furthermore, this chapter focuses on information acquisition problem. Higher order belief is normally hard to model because of its explosive increase of state variables. Similarly as in He and Wang (1995), higher order belief is well controlled in this model because agents only care about the aggregate level belief. One of the major findings of this chapter is that the aggregate belief is formed by information acquisition activities which aims to know forecasts of others.

As is specified earlier, the risky asset is in a fixed supply over all periods. Such nonexistence of any noise in the supply would fully reveal all the private information of informed agents in

[1] As is shown in Watanabe (2008), Geanakoplos (2008), an infinite horizon OLG model creates multiple equilibria. Furthermore, Loewenstein and Willard (2006) finds such "bubble" equilibrium with any deviation from the fundamental value in the infinite horizon model of De Long, Shleifer, Summers, and Waldmann (1990a) results in the violation of no Ponzi game condition. The infinitely elastic supply of risk free asset in infinite horizon turns out to be too extreme setting. By confining the horizon of the model to a finite horizon, this chapter does not suffer from the sunspots equilibria which creates explosive amount of borrowing from unknown source.

[2] For example, the prices in Allen, Morris, and Shin (2006) is the average expectation of agents in the economy.

the absence of other source of noise. In such case, there will be no incentive to acquire any information. The literature with NREE featuring information acquisition such as Grossman and Stiglitz (1980) typically assumes the existence of exogenous noise in the supply of the risky asset in order to give enough incentive of information acquisition. That is, the intensity of noise in the price is completely inelastic to any process in the economy. In stead of assuming that there exists exogenous noise in the supply of the risky assets, I endogenize the noise in the price by assuming the existence of a positive measure of noise traders besides agents present in a unit measure. Such endogenous noise trading is not only more realistic but also greatly simplifies the analysis. By having noise trading of which intensity is comoving with confidence of overall information by informed agents, we are able to shut down the channel of informativeness change due to risk aversion while keeping the channel of information acquisition intact. Therefore, the result with endogenous noise trading will be still robust even with exogenous noise trading.

5.3 The Basic Setup

Consider a multiperiod pure exchange economy in which trading occurs over T periods. There are one riskless asset and one risky asset. I set the price of the riskless asset equal to unity. The riskless asset is in a perfectly elastic supply and pays R units at the end of each period. The risky asset in the economy is in a positive fixed supply $\bar{x}$. The risky asset is liquidated at time $T+1$, and the payoff which is equal to the fundamental value of the risky asset is given to the agents who hold the shares in the liquidation time. The fundamental value of the risky asset at time t is denoted by V_t, and the risky asset pays V_{T+1} at time $T+1$. I assume that the fundamental value of the risky asset at time t evolves according to the following stochastic process:

$$V_{t+1} = V_t + \epsilon_{V,t+1} \tag{5.1}$$

where $\epsilon_{V,t+1}$ is a new shock to the fundamental value at time $t+1$. Each $\epsilon_{V,t}$ follows an independent normal distribution with mean zero and variance $\sigma^2_{V,t}$. I further assume that the initial fundamental value is zero, i.e, $V_0 = 0$. Note that the initial fundamental value, V_0, and the intensity of fundamental shocks in the future, $\Sigma_V \equiv (\sigma^2_{V,1}, \sigma^2_{V,2}, \ldots, \sigma^2_{V,T+1})$, are common knowledge to all agents.

There are two type of agents in the economy: (i) rational agents, and (ii) noise traders. All agents in this economy are infinitesimal, therefore they are price-takers. Agents live for two periods, and the population of agents in each period is fixed. That is, old agents who die at the end of second period of life are replaced by an equal number of young agents at the start of the next period. Agents invest in assets at the first period of their life, and consume a single good at the second period. There is a continuum of economic agents born in period $t = 1, 2, \ldots, T$, and I normalize the density of rational agents to one while I assume that noise traders are present in measure $\mu > 0$. I denote the set of rational agents to be $\mathcal{A}_t$ and the set of noise traders to be $\mathcal{N}_t$.

All agents born in period t (i.e., $a \in \mathcal{A}_t \cup \mathcal{N}_t$) observe the past history of prices as well as the current price. I denote the set of prices up to period t by $\mathcal{P}_t = \{P_1, P_2, \ldots, P_t\}$. Furthermore, a rational agent (i.e., $a \in \mathcal{A}_t$) can acquire a private signal such that

$$S_a = V_t + \epsilon_{S,a}, \tag{5.2}$$

where $\epsilon_{S,a} \sim N(0, \phi^{-1}_{t,a})$. The inverse of variance is defined to be the precision of signal. The precision is a linear function of monetary investment y such that

$$\phi_{t,a}(y) = \alpha_t y_a \tag{5.3}$$

where α is an efficiency parameter of investment in information. Higher α means the the signal delivers more precise information given the same amount of investment in information. Appendix O provides rationale for using such a linear precision function. For notational convenience, I will denote $S_a(y_a)$ to be agent a's private signal given his investment in information y_a. For simplicity, I especially consider a discrete set of choices for investing in information such that agents can choose either to invest a fixed level of cost $\bar{y}$ in information or not to invest at all, i.e., the set of choices is given by $Y = \{0, \bar{y}\}$. In case agent i invest $\bar{y}$ in information, he would acquire a private signal, $S_a(\bar{y})$, with precision, $\phi_a(\bar{y}) = \alpha_t \bar{y}$. Otherwise, he acquires a private signal, $S_a(0)$, with precision, $\phi_a(0) = 0$, i.e., agent a's private signal is valueless if no investment is made.

I will call the rational agents who engage in costly information acquisition "informed"(with a superscript i), and call the other portion of the rational agents "uninformed" (with a superscript u) for the convenience of discussion. The set of the informed and the uninformed rational agents in period t is denoted by $\mathcal{A}_t^i$ and $\mathcal{A}_t^u$, respectively. I will use a superscript n for noise traders. I denote λ_t to be the informed portion of agents in $\mathcal{A}_t$, thus $1 - \lambda_t$ would be the uninformed portion of agents in period t. Therefore, λ_t is positive as long as there exists a positive measure of agents who choose to be informed, and is zero if almost all the rational agents in period t choose not to be informed.[3] I also define the vector of the informed portion of agents over all periods to be $\Lambda \equiv (\lambda_1, \lambda_2, \ldots, \lambda_T)$.

In a similar fashion as De Long, Shleifer, Summers, and Waldmann (1990a), the noise traders misperceive the fundamental value of the risky asset by a random noise variable.[4]In this model, the noise traders' misperception arises due to their private signals which are irrelevant to fundamentals. I assume that every noise trader in all the periods always invests in information individually, therefore receives his own private signal S^n. Each noise trader perceives that he belongs to the set of the rational agents $\mathcal{A}_t$ rather than the set of the noise traders $\mathcal{N}_t$.[5] Therefore, he perceives that he receives a signal S_a given in (5.2). Unlike the private signal of

[3]"almost all" means that there could be a measure-zero group of agents who actually engage in costly information acquisition activities.

[4]There are two channels by which fundamental shocks affect informativeness of prices. First channel is information acquisition channel, and second channel is risk aversion channel. When a fundamentals shock arrives, it alters agents' portfolio because agents may change their information acquisition decision and reduce their exposure due to risk aversion. When liquidity trading is inelastic toward such fundamental shocks, the volatility of liquidity shock is more pronounced because the intensity of informed trading weakens relative to that of liquidity trading. It may be interesting to study such phenomena by itself, however, it does not serve our purpose in this chapter. By endogenizing noise trading so that its intensity is leveled with that of informed trading regardless of the magnitude of shocks, we can shut down the channel of variation in informativeness through risk aversion. Therefore, we can study the effect of information acquisition on the price process in the presence of varying fundamental shocks in isolation by employing endogenous noise trading. The endogenous noise trading is not particularly essential feature of the model which drives the result, however, it allows us to obtain closed form solutions of equilibrium as well as analyze the first channel in isolation from the second channel.

[5]While each noise trader realizes there are noise traders present in the economy, he does not perceive himself as a noise trader.

a rational agent, S^n reflects a consensus learning from the previous price history and a random component orthogonal to fundamentals. Since a noise trader a misperceives the nature of the private signal to be equivalent to that of the rational agents (5.2), the perceived private signal of each noise trader $\hat{S}^n$ is given by

$$\hat{S}^n_t = V_t + \epsilon_{S,a}. \tag{5.4}$$

On the other hand, the true of nature of the private signal is in fact equivalent to the following:

$$S^n_t = E[V_t|\mathcal{P}_{t-1}] + \eta_t, \tag{5.5}$$

where $\eta_t \sim N(0, \sigma^2_{\eta,t})$ and $\sigma^2_{\eta,t} \equiv \frac{\theta}{\alpha_t \bar{y}}$. That is, the precision function of noise traders' signal is given by $\phi^n_t = \alpha^n_t \bar{y}$ with an efficiency parameter $\alpha^n_t \equiv \frac{\alpha_t}{\theta}$. Furthermore, note the distribution of η_t is identical across all the noise traders $a \in \mathcal{N}_t$. Therefore, η_t is common misperception which applies to every noise trader in the same period while it is unobserved by rational agents. One of the potential interpretation of η_t could be investor sentiment which is common and prevalent among noise traders in period t. θ could be interpreted as the degree of volatility in noise traders' misperception or intrinsic uncertainty of the noise traders.[6] Therefore, the assumption implies that overall higher quality of private signals in a period is linked to less susceptibility of noise traders to large degree of misperception.

Each young agent $a \in \mathcal{A}_t \cup \mathcal{N}_t$ at period t has an identical initial wealth W_0 at the first period of his life and no shares in riskless asset or risky assets. I denote $X_{a,t}$ to be the holdings of the risky assets at the first period of agent i's life. Then, the wealth of agent i in the second period of his life will be

$$W_{a,t+1} = (W_0 - y_a)R + X^\top_{a,t}(P_{t+1} - RP_t)$$

where y_a is agent i's investment in information. Note that y_a is always zero in case of noise

[6] As in rational agents' case, Appendix O gives the justification of having such a linear form of precision functions.

traders, i.e., $i \in \mathcal{N}_t$. Each agent $i \in \mathcal{A}_t \cup \mathcal{N}_t$ has a identical constant absolute risk aversion (CARA) utility function with a risk aversion parameter ρ, i.e.,

$$\mathcal{U}(W_{a,t+1}) = -\exp\Big(-\rho W_{a,t+1}\Big).$$

Therefore, the market clearing condition in each period t is given by

$$\int_{a \in \mathcal{A}_t \cup \mathcal{N}_t} X_{a,t} da = \bar{x}, \tag{5.6}$$

for all $t = 1, 2, \ldots, T$.

5.4 Equilibrium

The overall equilibrium is solved in two steps. In the first step, I solve for the equilibrium prices in the asset market given the vector of informed populations over time. In the second step, I obtain the overall equilibrium by solving each agent's ex-ante utility maximization regarding the choice of information acquisition in each period given the asset market equilibrium derived in the first step.

5.4.1 Asset Market Equilibrium

In this chapter, I focus only on the class of linear equilibrium in the asset market equilibrium. All agents in this economy are price-takers since they are infinitesimal. Since each informed agent receive differential information, individual errors in private signals would cancel out by the law of large numbers when they are aggregated.[7] Therefore, the equilibrium price could reflect the true fundamental value by aggregating information among informed agents. This model could be considered as a multiperiod extension of Hellwig (1980) and Admati (1985) model.

[7] There is a well-known problem raised by Judd (1985) regarding the failure of the law of large numbers in case of a continuum of agents. Using the exact law of large number with Fubini extension suggested by Sun (2006), the measurability problem could be resolved. See also Feldman and Gilles (1985), Uhlig (1996) and Al-Najjar (2004) for more discussions.

First, I take the equilibrium population of informed agents Λ as given. That is, I suppose that there exist λ_t portion of informed rational agents and $1-\lambda_t$ portion of uninformed rational agents in each period t. As in typical noisy rational expectation equilibrium models, I conjecture that the equilibrium price function is a linear function of state variables $E[V_t|\mathcal{P}_t], V_t, \eta_t, \bar{x}$: (i) if $\lambda_t \in (0,1]$,

$$P_t = a_t E[V_t|\mathcal{P}_t] + b_t V_t + c_t \eta_t - d_t \bar{x}. \tag{5.7}$$

where a_t, b_t, c_t, d_t are constants, (ii) if $\lambda_t = 0$,

$$P_t = a_t E[V_t|\mathcal{P}_t] + c_t \eta_t - d_t \bar{x}. \tag{5.8}$$

where a_t, c_t, d_t are constants,

Learning

Since the equilibrium price function as well as $E[V_t|\mathcal{P}_t], \bar{x}$ are common knowledge in equilibrium, observing P_t is equivalent to observing a sufficient statistic $\zeta_t \equiv V_t + \frac{c_t}{b_t}\eta_t$. Extracting ζ_t by observing P_t, each agent's updating belief on V_t at time t using ζ_t could be solved by the following linear filtering:

Lemma 5.1 *Given the equilibrium price and private information, the conditional expectation and conditional variance of each rational agent a is determined by the following linear filter for each period $t = 1, 2, \ldots, T+1$.*
(i) conditional expectation:

$$E[V_t|\mathcal{P}_t, S(\bar{y})] = E[V_t|\mathcal{P}_{t-1}] + \beta^i_{S,t}(S_{a,t} - E[V_t|\mathcal{P}_{t-1}]) + \beta^i_{\zeta,t}(\zeta_t - E[V_t|\mathcal{P}_{t-1}]); \tag{5.9}$$

$$E[V_t|\mathcal{P}_t, S(0)] = E[V_t|\mathcal{P}_{t-1}] + \beta^u_{\zeta,t}(\zeta_t - E[V_t|\mathcal{P}_{t-1}]); \tag{5.10}$$

(ii) conditional variance:

$$Var[V_t|\mathcal{P}_t, S(\bar{y})] = \left(Var[V_t|\mathcal{P}_{t-1}]^{-1} + \frac{b_t^2}{c_t^2\sigma_{\eta,t}^2} + \alpha_t\bar{y}\right)^{-1}, \tag{5.11}$$

$$Var[V_t|\mathcal{P}_t, S(0)] = \left(Var[V_t|\mathcal{P}_{t-1}]^{-1} + \frac{b_t^2}{c_t^2\sigma_{\eta,t}^2}\right)^{-1}, \tag{5.12}$$

where $\beta^i_{S,t}, \beta^i_{\zeta,t}, \beta^u_{\zeta,t}$ *are constants.*

Proof See Appendix P for the proof. ∎

On the other hand, the conditional expectation and variance of noise traders is given by the following:

Lemma 5.2 *The perceived expectation and variance of each noise trader* $a \in \mathcal{N}_t$ *given* $\mathcal{P}_t$ *and* S^n_t *is given by the following:*

$$\begin{aligned}
\widehat{E}[V_t|\mathcal{P}_t] &= E[V_t|\mathcal{P}_t]; \\
\widehat{E}[V_t|\mathcal{P}_t, S^n_t] &= (1+\beta^i_{S,t})E[V_t|\mathcal{P}_{t-1}] + \beta^i_{S,t}\eta_t + \beta^i_{\zeta,t}(\zeta_t - E[V_t|\mathcal{P}_{t-1}]); \\
\widehat{Var}[V_t|\mathcal{P}_t, S^n_t] &= Var[V_t|\mathcal{P}_t, S(\bar{y})].
\end{aligned}$$

for all $t = 1, 2, \ldots, T$.

Optimal Choice of Portfolio

Each agent a in period t solves the following optimization problem:

$$\max_{X_{a,t}} \quad E\left[-e^{-\rho W_{a,t+1}}\middle|\mathcal{P}_t, S_a(y_a)\right] \tag{5.13}$$

$$\text{subject to} \quad W_{t+1} = W_0 + (P_{t+1} - RP_t)X_{a,t}$$

By the well-known property of CARA-normal setup, the optimal portfolio choice of agent a in each group of agents (i.e., for each $\mathcal{A}_t^i, \mathcal{A}_t^u, \mathcal{N}_t$) is determined as the following, respectively:

$$X_{a,t}^i = \frac{E[P_{t+1}|\mathcal{P}_t, S_a(\bar{y})] - RP_t}{\rho Var[P_{t+1}|\mathcal{P}_t, S_a(\bar{y})]}, \tag{5.14}$$

$$X_{a,t}^u = \frac{E[P_{t+1}|\mathcal{P}_t, S_a(0)] - RP_t}{\rho Var[P_{t+1}|\mathcal{P}_t, S_a(0)]}, \tag{5.15}$$

$$X_{a,t}^n = \frac{\widehat{E}[P_{t+1}|\mathcal{P}_t, S^n(\bar{y})] - RP_t}{\rho \widehat{Var}[P_{t+1}|\mathcal{P}_t, S^n(\bar{y})]}, \tag{5.16}$$

for all $t = 1, 2, \ldots, T$. For notational convenience, I will denote $\Sigma_t^i \equiv Var[P_{t+1}|\mathcal{P}_t, S_a(\bar{y})]$ and $\Sigma_t^u \equiv Var[P_{t+1}|\mathcal{P}_t, S_a(\bar{0})]$.

Note that each individual demand $X_{a,t}^i$ contains an idiosyncratic error $\epsilon_{S,a}$ from the private signal. In aggregate level, however, idiosyncratic noises in private signals cancel out in the equilibrium price by the law of large numbers. In the following lemma, I show that the aggregate demand by each type of agents could be represented as a linear function of state variables:

Lemma 5.3 *The aggregate demand of each group of agents in period t for the risky asset is given by a linear function of state variables:*

$$\int_{a\in\mathcal{A}_t^i} X_{a,t}^i da = \frac{\lambda_t}{\rho\Sigma_t^i}\Big[(k_{1,t}^i - Ra_t)E[V_t|\mathcal{P}_t] + (k_{2,t}^i - Rb_t)V_t + (k_{3,t}^i - Rc_t)\eta_t - (d_{t+1} - Rd_t)\bar{x}\Big];$$

$$\int_{a\in\mathcal{A}_t^i} X_{a,t}^u da = \frac{1-\lambda_t}{\rho\Sigma_t^u}\Big[(a_{t+1} + b_{t+1} - Ra_t)E[V_t|\mathcal{P}_t] - Rb_tV_t - Rc_t\eta_t - (d_{t+1} - Rd_t)\bar{x}\Big];$$

$$\int_{a\in\mathcal{N}_t} X_{a,t}^n da = \frac{\mu}{\rho\Sigma_t^i}\Big[(k_{1,t}^n - Ra_t)E[V_t|\mathcal{P}_t] + (k_{2,t}^n - Rb_t)V_t + (k_{3,t}^n - Rc_t)\eta_t - (d_{t+1} - Rd_t)\bar{x}\Big],$$

where

$$\Sigma_t^i = (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})^2\Big(Var[V_t|\mathcal{P}_t, S_a(\bar{y})] + \sigma_{V,t+1}^2 + \frac{c_{t+1}^2}{b_{t+1}^2}\sigma_{\eta,t+1}^2\Big);$$

$$\Sigma_t^u = (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})^2\Big(Var[V_t|\mathcal{P}_t] + \sigma_{V,t+1}^2 + \frac{c_{t+1}^2}{b_{t+1}^2}\sigma_{\eta,t+1}^2\Big).$$

and $k_{1,t}^i, k_{2,t}^i, k_{3,t}^i, k_{1,t}^u, k_{2,t}^u, k_{3,t}^u$ are constants.

Proof See Appendix Q for the proof. ∎

Solving Asset Market Equilibrium

Since the perceived precision of noise traders is identical to that of informed traders, the intensity of informed trading and noise trading is in a linear relationship as is observed in (5.14) and (5.16). Using the portfolio choice of each type of agents, the market clearing condition (5.6) is shown to be equivalent to the following:

$$\int_{a\in\mathcal{A}_t^i} X_{a,t}da + \int_{a\in\mathcal{A}_t^u} X_{a,t}da + \int_{a\in\mathcal{N}_t} X_{a,t}da = \bar{x}, \tag{5.17}$$

for all $t = 1, 2, \ldots, T$. Using Lemma 5.3, it is easily demonstrated that the equilibrium price P_t which solves the equation (5.17) is indeed a linear function conjectured in (5.7):

Theorem 5.4 *(i) Given Λ, a linear equilibrium price which clears the asset market in period t is given by*

$$P_t = a_t E[V_t|\mathcal{P}_t] + b_t V_t + c_t \eta_t - d_t \bar{x};$$

where a_t, b_t, c_t, d_t are non-negative constants. (ii) The precision of revealed information through the equilibrium price P_t in period t is equal to the signal-to-noise ratio Π_t such that

$$\Pi_t \equiv \frac{b_t^2}{c_t^2 \sigma_\eta^2} = \frac{\lambda_t^2 \phi_t}{\mu^2 \theta}.$$

Proof See Appendix Q for the proof. ∎

Note that a_t, c_t, d_t are positive constants, and b_t is zero if $\lambda_t = 0$, and a positive constant if $\lambda_t > 0$ for $t = 1, 2, \ldots, T-1$. In case almost no agent in the current period engages in costly information acquisition, the equilibrium price function only depends on prior information, noise trading and the supply of the risky asset rather than the current fundamental value. Only when a positive measure of agents observe the current fundamental value by engaging in costly information acquisition (i.e., $\lambda_t > 0$), the equilibrium price function becomes a linear function in the fundamental value. The dependence of price functions on the true fundamental value through the density of informed agents plays a very important role in the following analysis.

The quality of revealed information through the equilibrium price improves as there are more informed agents, i.e., the signal-to-noise ratio goes up as λ_t goes up. On the other hand, the quality of revealed information deteriorates as the density of noise traders increases or the intensity of noise trading gets higher, i.e., the signal-to-noise ratio goes down as μ or σ_η goes up. The following lemma summarizes the comparative statics:

Lemma 5.5 *The price informativeness in period t increases in (i) the portion of the informed agents, (ii) precision of private information, and decreases in (i) the portion of the noise traders, (ii) the intensity of noise.*

5.4.2 Overall Equilibrium

In the second step, I solve for the overall equilibrium by endogenizing the equilibrium price functions in agents' ex-ante utility maximization problems with respect to the choice of investment in information at each period. Before we move on to solving the overall equilibrium, I analyze the price informativeness in each period.

As is shown in previous literature such as Grossman and Stiglitz (1980), the ex-ante utility maximization problem in case of CARA-normal setup boils down to the minimization problem of conditional variance given the cost of information acquisition.

Lemma 5.6 *A rational agent $a \in \mathcal{A}_t$'s ex-ante utility conditional on the past history of prices is represented by the following:*

$$E[\mathcal{J}(W_{a,t+1}; \mathcal{P}_t, S(y_a))|\mathcal{P}_{t-1}] = -\sqrt{Var[P_{t+1}|\mathcal{P}_t, S(y_a)]}\exp\left(\rho R y_a\right)\Upsilon_{a,t},$$

where $\Upsilon_{a,t}$ is a constant.

Proof See Appendix R for proof. ∎

The equilibrium of costly information acquisition is given by a strategy profile $\{y_a^*\}_{i\in\mathcal{A}_t}$ such that

$$\text{for all } a \in \mathcal{A}_t,\ y_a^* \in \underset{y_a\in Y}{\text{argmax}} E[\mathcal{J}(W_{a,t+1}; \mathcal{P}_t, S_a(y_a))|\mathcal{P}_{t-1}] \tag{5.18}$$

for all $t = 1, 2, \ldots, T-1$. By using the result of Lemma 5.6, the next lemma shows that agent i in period t would invest in information if and only if the cost of information acquisition is less than the benefit of that.

Lemma 5.7 *(i) Given $\lambda_{t+1} \in (0,1]$, agent $a \in \mathcal{A}_t$ prefers acquiring private information at cost $\bar{y}$ to not acquiring any private information at no cost if and only if*

$$\underbrace{\bar{y}}_{cost} \leq \underbrace{\frac{1}{2\rho R}\left[\log\left(\frac{Var[V_t|\mathcal{P}_t, S_i(0)] + \sigma^2_{V,t+1} + \left(\frac{\mu^2\theta}{\lambda^2_{t+1}\phi_{t+1}}\right)\mathbb{1}_{t<T+1}}{Var[V_t|\mathcal{P}_t, S_i(\bar{y})] + \sigma^2_{V,t+1} + \left(\frac{\mu^2\theta}{\lambda^2_{t+1}\phi_{t+1}}\right)\mathbb{1}_{t<T+1}}\right)\right]}_{benefit}, \tag{5.19}$$

where

$$\begin{aligned} Var[V_t|\mathcal{P}_t, S_a(0)] &= \left(\frac{1}{Var[V_{t-1}|\mathcal{P}_{t-1}] + \sigma^2_{V,t}} + \frac{\lambda_t^2\phi_t}{\mu^2\theta}\right)^{-1}; \\ Var[V_t|\mathcal{P}_t, S_a(\bar{y})] &= \left(\frac{1}{Var[V_{t-1}|\mathcal{P}_{t-1}] + \sigma^2_{V,t}} + \frac{\lambda_t^2\phi_t}{\mu^2\theta} + \alpha_t\bar{y}\right)^{-1}; \end{aligned}$$

(ii) Given $\lambda_{t+1} = 0$, agent $a \in \mathcal{A}_t$ never prefers acquiring private information as long as $\bar{y} > 0$.

Proof See Appendix R for the proof. ∎

Suppose λ_t^o is the solution of (5.19) with an equality for each time t. Then, the following gives the overall equilibrium:

$$\lambda_t^* = \begin{cases} 0, & \text{if } \lambda_t^o < 0; \\ \lambda_t^o, & \text{if } 0 < \lambda_t^o < 1; \\ 1, & \text{if } 1 \leq \lambda_t^o; \end{cases}$$

Note that λ_t is directly linked to only λ_{t-1} and λ_{t+1}. That is, the decision of information acquisition in the current period is only affected by the immediately prior or posterior periods. However, the impact in farther past or future could still be transferred via adjacent periods. Therefore, information acquisition decisions could be considered as a chain of collective decisions over all the periods. Each agent needs to consider other agents in every period although they do not exist in the same period. I call this intertemporal dependence of information acquisition among different generations of agents as “information acquisition chains”. Due to this endogenous dependency across adjacent periods, any impact on information acquisition in the past or future would affect every period through information acquisition chains. I shall study how it forms the equilibrium in financial market in the next section.

5.5 Comparative Statics

This section explores the various aspects of information acquisition chains. First, I analyze the intertemporal dependence of information acquisition. Second, I study the relationship between information acquisition chains and fundamental shocks.

5.5.1 Intertemporal Dependence

Note that the condition of agent a's information acquisition endogenously depends on the decision of other agents in other periods as well as other agents in the same period through the signal-to-noise ratio. An agent's individual decision of information acquisition does not affect the overall information contents of the price since each agent is infinitesimal. It is analogous to the fact that each agent is a price-taker because his individual portfolio choice alone does not affect the equilibrium price.

In equilibrium, agents can learn information through two channels: (i) observing private information at a cost $\bar{y}$ and (ii) learning aggregated private information through the equilibrium prices. Therefore, it is not very hard to find that information acquisition by other agents in the current generation is strategic substitute. When the second channel conveys enough information, agents do not find enough incentive to use the first channel. That is, agents do not find it profitable to acquire private information if past or current prices offer too much information. The current generation (i.e., agents born in period t) can only learn from the prices up to period t while the later generations (i.e., agents born in period $t+1$ and afterwards) can learn using the price in period t. Therefore, private information of agents in the current period is a strategic substitute to information acquisition of the current generation as well as the later generations.

Indeed, one could observe that the condition (5.19) becomes less tight as the signal-to-noise ratio of the current generation Π_t goes up. On the other hand, information acquisition of later generations tends to be strategic complement because it increases the incentive of current generation. Agents in the current period would not acquire any information if no agent in the subsequent generation is investing in information (i.e., $\lambda_{t+1} = 0$). If $\lambda_t > 0$, it is clear that the condition becomes less tight as the signal-to-noise ratio of the later generation Π_{t+1} gets larger. That is, agents in the current period would have more incentive of acquiring information as the signal-to-noise ratio in the subsequent period goes up.

In summary, the incentive of investing in information by a rational agent in period t decreases as more agents are investing in the current period while the incentive increases while more agents are investing in the subsequent period. The following proposition formalize the discussion so far regarding strategic substitutability and complementarity.

Proposition 5.8 *Information in the future is a strategic complement substitute to information acquisition of agents in the current or past period while information in the present is a strategic substitute.*

Proof From Lemma 5.6, it can be shown that the first order derivative of the ex-ante utility $\frac{\partial(E[\mathcal{J}(W_{a,t+1};\mathcal{P}_t,S(\bar{y}))|\mathcal{P}_{t-1}]-E[\mathcal{J}(W_{a,t+1};\mathcal{P}_t,S(0))|\mathcal{P}_{t-1}]]}{\partial\lambda_\tau}$ is positive if $\tau > t$, and negative otherwise. ■

5.5.2 Impact of Anticipated Fundamental Shocks

The increase of fundamental volatility in the future does not affect the current fundamentals immediately. However, it could affect the behavior of asset prices in the current period by altering the incentive of information acquisition in the current period. The following lemma shows that the increase in uncertainty in the past increases the incentive of information acquisition while the increase of uncertainty in the future reduces the incentive of information acquisition.

Lemma 5.9 *The increase in the unresolved uncertainty of the fundamental value increases the incentive of acquiring information while the increase in the volatility of a fundamental shock in the future reduces the incentive of acquiring information in the current period.*

Proof Using Lemma 5.6 and Lemma 5.7, the following could be easily shown

$$\frac{\partial(E[\mathcal{J}(W_{a,t+1};\mathcal{P}_t,S(\bar{y}))|\mathcal{P}_{t-1}]-E[\mathcal{J}(W_{a,t+1};\mathcal{P}_t,S(0))|\mathcal{P}_{t-1}]]}{\partial Var[V_{t-1}|\mathcal{P}_{t-1}]} > 0,$$

and

$$\frac{\partial(E[\mathcal{J}(W_{a,t+1};\mathcal{P}_t,S(\bar{y}))|\mathcal{P}_{t-1}]-E[\mathcal{J}(W_{a,t+1};\mathcal{P}_t,S(0))|\mathcal{P}_{t-1}]]}{\partial\sigma_{t+1}} < 0.$$

for all $t = 1, 2, \ldots, T$. ∎

Therefore, a high volatility in the future period would deter the agents in the current period from having more private information. In an extreme case, a surge of fundamental volatility in future periods could form a structural break for price processes by breaking information acquisition chains. In the following proposition, I show that a shock with high enough volatility could break the information acquisition chains.

Proposition 5.10 *If information acquisition is costly, high enough volatility completely removes the incentive of acquiring information. That is, if $y_t > 0$ there exists some $\underline{\sigma}_{t+1}$ such that $\sigma_{t+1} > \underline{\sigma}_{t+1}$ implies $\lambda_t = 0$.*

Proof I will denote the right hand side of (5.19) as $g(\sigma_{t+1})$, which is a monotone increasing function of σ_{t+1}. Suppose $\lambda_t = 0$ when $\sigma_{t+1} = 0$. Then, we can trivially find that $\underline{\sigma}_{t+1} = 0$ due to the monotonicity of $g(\cdot)$. Now suppose $\lambda_t \in (0, 1]$ when $\sigma_{t+1} = 0$. It is easily shown that $g(\cdot)$ approaches to infinity as σ_{t+1} approaches to infinity. Since $g(\cdot)$ is continuous, there must exist $\underline{\sigma}_{t+1}$ such that $g(\bar{\sigma}_{t+1}) = \bar{y}$. Due to the monotonicity of $g(\cdot)$, the claim is proven. ∎

A highly volatile shock, which is predicted to arrive in the future, could adversely affect the incentive of information acquisition thereby creating further impacts on the price process beyond its fundamental impacts. Lemma 5.7 *(ii)* shows that the agents in the current period would not invest in information if the agents in any future generation do not invest in information because the subsequent generation also would not invest in information unless the subsequent generation of subsequent generation invest in information and so on. Therefore, this result is analogous to

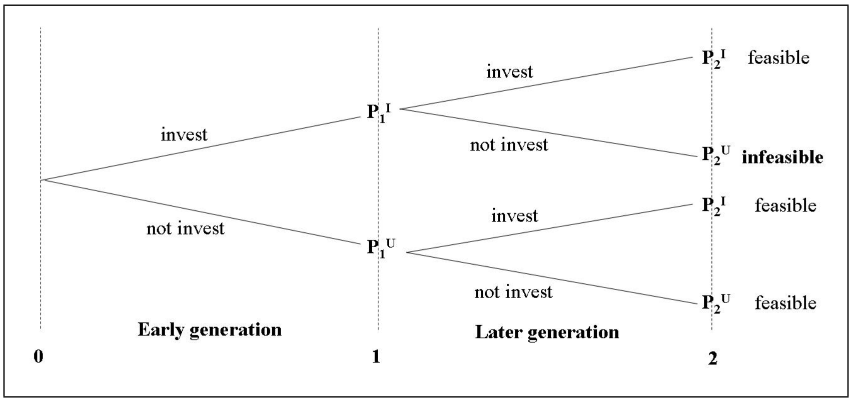

Figure 5.1: Illustration of information acquisition chains in case of three periods model P_t^I and P_t^U stands for the price in period t with and without extra information acquisition at time $t = 1, 2$, respectively.

the "arbitrage chain" argument of Dow and Gorton (1994) which finds that informed arbitragers may not trade on their private information on a distant event unless subsequent generations trade on that information. Their findings go one step further by finding that the agents do not want to acquire information unless future generations are going to acquire that information. I call this connected sequence of information acquisition as information acquisition chains:

Proposition 5.11 (Information Acquisition Chains) *An early generation of the agents never engages in costly information acquisition unless the later generations do afterwards until the liquidation.*

Figure 5.1 illustrates examples of equilibria in a three period case. It is impossible to have an equilibrium such that the generation in period 1 engages in costly information acquisition while the generation in period 2 does not.

This shed a light on the cause of price delays to information. The prediction of the model shows that price delays may arise due to increased fundamental shocks in the presence of endogenous information acquisition. The result of this chapter shows that underreaction to information will be more pronounced when there are more uncertainty about the fundamental process in the future. Furthermore, the degree of underreaction also depends on the difficulty of acquiring information.

The recent empirical evidence are consistent with the predictions: For example, Hou and Moskowitz (2005) finds that investor recognition is more responsible for price delay than size, liquidity, or microstructure effects. That is, Price delay to information is mainly related to informational factors rather than other frictions. Zhang (2006) also finds that price underreaction to information is greater when fundamental volatility is higher. Especially, their evidence shows that post-news price drift increases with more uncertainty.

5.6 Conclusion

This chapter studies a finite-horizon overlapping generations model where agents endogenously acquire information on a risky asset of which fundamental value fluctuates due to new fundamental shocks. Since short-lived agents can not carry the asset until the liquidation of the long-lived risky asset, an early generation of agents does not engage in costly information acquisition unless later generations engage in information acquisition consecutively until the liquidation like a chain. The result shows that fundamental shocks of high magnitude in the future may eliminate the incentive of acquiring information in earlier periods, thereby breaking "information acquisition chain" from the earliest generation. The result gives a prediction regarding price delay to information as well as extra impact of fundamental shocks such that an extra fundamental shock could result in amplified fundamental uncertainty in the present without having an immediate effect on the current fundamentals.

Appendices

Appendix A

Linear filtering problem could be solved by a standard algorithm called Kalman filter. (For example, see Hamilton (1994) or Wang (1994))

Lemma A.1 *Let* ξ_t *denote a* n*-vector of state variables,* y_t *denote a* m*-vector of observed signals. Suppose the dynamics of* y_t *is given by the following system of equations:*

$$\begin{aligned} \xi_t &= A_t \xi_{t-1} + B_t \epsilon_{\xi,t} \\ y_t &= H_t \xi_t + \epsilon_{y,t} \end{aligned}$$

where A_t, B_t *and* H_t *are matrices of parameters of dimension* $n \times n, n \times k, m \times n$*, respectively.* $\epsilon_{\xi,t}$ *and* $\epsilon_{y,t}$ *are* k*-vector and* m*-vector of innovations, respectively.* $\epsilon_{\xi,t}$ *and* $\epsilon_{y,t}$ *are independent, and their distributions are given by* $\epsilon_{\xi,t} \sim \mathcal{N}(0, \mathcal{Q}_t)$ *and* $\epsilon_{y,t} \sim \mathcal{N}(0, R_t)$*. Then, the conditional expectation and variance of* ξ_t *is given by the following recursive filters:*

$$\hat{\xi}_t = A_t \hat{\xi}_{t-1} + K_t (y_t - H_t A_t \hat{\xi}_{t-1}) \tag{A.1}$$

$$O_t = (I_n - K_t H_t)(A_t O_{t-1} A_t^\top + B_t \mathcal{Q}_t B_t^\top) \tag{A.2}$$

where $K_t = (A_t O_{t-1} A_t^\top + B_t \mathcal{Q}_t B_t^\top) H_t^\top [H_t (A_t O_{t-1} A_t^\top + B_t \mathcal{Q}_t B_t^\top) H_t^\top + R_t]^{-1}$*, and* I_n *is a* $(n \times n)$ *identity matrix.*

Proof of Theorem 3.2: In case of market makers' filtering problem, the state variables are $\xi_t \equiv (V, S_t)^\top$ and the observed variables are $y_t \equiv (\zeta_t, Y_t)^\top$. Also, innovations are given by

$\epsilon_{\xi,t} \equiv \epsilon_{S,t+1}$, $\epsilon_{y,t} \equiv [\epsilon_{U,t+1}, \epsilon_{Y,t}]^\top$, and coefficients are given by

$$
\begin{aligned}
A_t &= \begin{pmatrix} 1 & 0 \\ 0 & a_S \end{pmatrix}, \\
B_t &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
H_t &= \begin{pmatrix} a_{X,t} & a_U + b_{X,t} \\ 1 & a_Y \end{pmatrix}, \\
\mathcal{Q}_t &= \sigma^2_{S,t+1}, \\
R_t &= \begin{pmatrix} \sigma^2_{U,t+1} & 0 \\ 0 & \sigma^2_{Y,t} \end{pmatrix}.
\end{aligned}
$$

Using Lemma A.1, we could derive the following Kalman filter of market makers:

$$
\begin{pmatrix} \hat{V}^M_t \\ \hat{S}^M_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a_S \end{pmatrix} \begin{pmatrix} \hat{V}^M_{t-1} \\ \hat{S}^M_{t-1} \end{pmatrix} + K_t \begin{pmatrix} \zeta_t - E[\zeta_t|\mathcal{F}^M_{t-1}] \\ Y_t - E[Y_t|\mathcal{F}^M_{t-1}] \end{pmatrix}
$$

where $K_t = (A_t O_{t-1} A_t^\top + B_t \mathcal{Q}_t B_t^\top) H_t^\top [H_t(A_t O_{t-1} A_t^\top + B_t \mathcal{Q}_t B_t^\top) H_t^\top + R_t]^{-1}$. Also, the mean-square error of forecasting is given by

$$
O_t = (I_n - K_t H_t)(A_t O_{t-1} A_t^\top + B_t \mathcal{Q}_t B_t^\top).
$$

∎

Appendix B

Proof of Lemma 3.3:

Since $E[\zeta_t|\mathcal{F}_{t-1}^M] = b_{X,t}V_t$ and $E[Y_t|\mathcal{F}_{t-1}^M] = V_t$, it could be shown that

$$\begin{aligned}
\hat{V}_t^M &= \hat{V}_{t-1}^M + k_{V,t}^{\zeta}[\zeta_t - (a_{X,t}\hat{V}_{t-1}^M + (b_{X,t} + a_U)a_S\hat{S}_{t-1}^M)] + k_{V,t}^{Y}[Y_t - (\hat{V}_{t-1}^M + a_Y a_S\hat{S}_{t-1}^M)] \\
&= \hat{V}_{t-1}^M + k_{V,t}^{\zeta}[\Delta X_t + \Delta U_t + a_{X,t}P_{t-1} + b_{X,t}a_S\hat{S}_{t-1}^M - c_{X,t}(Y_t - P_{t-1} - a_Y a_S\hat{S}_{t-1}^M) \\
&\quad -(a_{X,t}\hat{V}_{t-1}^M + (b_{X,t} + a_U)a_S\hat{S}_{t-1}^M)] + k_{V,t}^{Y}(V - P_{t-1} - a_Y a_S\hat{S}_{t-1}^M) \\
&= P_{t-1} + k_{V,t}^{\zeta}\Delta X_t + a_U k_{V,t}^{\zeta}(S_t - a_S\hat{S}_{t-1}^M) + (-c_{X,t}k_{V,t}^{\zeta} + k_{V,t}^{Y})(Y_t - P_{t-1} - a_Y a_S\hat{S}_{t-1}^M) + k_{V,t}^{\zeta}\epsilon_{U,t+1} \; .
\end{aligned}$$

Since $Q_{t+1} \equiv V - P_t$, the excess return is given by

$$\begin{aligned}
Q_{t+1} &= -k_{V,t}^{\zeta}\Delta X_t + (V - P_{t-1}) - a_U k_{V,t}^{\zeta}(S_t - a_S\hat{S}_{t-1}^M) + (c_{X,t}k_{V,t}^{\zeta} - k_{V,t}^{Y})(Y_t - P_{t-1} - a_Y a_S\hat{S}_{t-1}^M) \\
&\quad -k_{V,t}^{\zeta}\epsilon_{U,t+1} \; .
\end{aligned}$$

Equivalently, the equilibrium excess return at date t could be represented as

$$Q_{t+1} = a_{Q,t+1}\Delta X_t + b_{Q,t+1}\Psi_t + c_{Q,t+1}\varepsilon_{t+1}$$

where

$$\begin{aligned}
a_{Q,t+1} &\equiv -k_{V,t}^{\zeta}, \\
b_{Q,t+1} &\equiv \left(1, -a_U k_{V,t}^{\zeta}, c_{X,t}k_{V,t}^{\zeta} - k_{V,t}^{Y}\right), \\
c_{Q,t+1} &\equiv \left(0, \; -k_{V,t}^{\zeta}, \; 0\right).
\end{aligned}$$

∎

Appendix C

Proof of Lemma 3.4:

Note that

$$\begin{aligned}
V-P_t &= -k^{\zeta}_{V,t}\Delta X_t+(V-P_{t-1})-a_U k^{\zeta}_{V,t}(S_t-a_S\hat{S}^M_{t-1})\\
&\quad +(c_{X,t}k^{\zeta}_{V,t}-k^{Y}_{V,t})(Y_t-P_{t-1}-a_Y a_S\hat{S}^M_t)-k^{\zeta}_{V,t}\epsilon_{U,t+1},\\
S_{t+1}-a_S\hat{S}^M_t &= -a_S k^{\zeta}_{S,t}\Delta X_t+a_S(1-a_U k^{\zeta}_{S,t})(S_t-a_S\hat{S}^M_{t-1})\\
&\quad +a_S(c_{X,t}k^{\zeta}_{S,t}-k^{Y}_{S,t})(Y_t-P_{t-1}-a_Y a_S\hat{S}^M_t)-a_S k^{\zeta}_{S,t}\epsilon_{U,t+1}+\epsilon_{S,t+1},\\
Y_{t+1}-P_t-a_Y a_S\hat{S}^M_t &= (V-P_t)+a_Y(S_{t+1}-a_S\hat{S}^M_t)+\epsilon_{Y,t+1}.
\end{aligned}$$

Therefore, it is straight forward to show that

$$\Psi_{t+1}=a_{\Psi,t+1}\Delta X_t+b_{\Psi,t+1}\Psi_t+c_{P,t+1}\varepsilon_{t+1}$$

where

$$\begin{aligned}
a_{\Psi,t+1} &= (-k^{\zeta}_{V,t},-a_S k^{\zeta}_{S,t},-k^{\zeta}_{V,t}-a_Y a_S k^{\zeta}_{S,t})^{\top},\\
b_{\Psi,t+1} &= \begin{pmatrix} 1 & -a_U k^{\zeta}_{V,t} & c_{X,t}k^{\zeta}_{V,t}-k^{Y}_{V,t}\\ 0 & a_S(1-a_U k^{\zeta}_{S,t}) & a_S(c_{X,t}k^{\zeta}_{S,t}-k^{Y}_{S,t})\\ 1 & -a_U k^{\zeta}_{V,t}+a_Y a_S(1-a_U k^{\zeta}_{S,t}) & c_{X,t}k^{\zeta}_{V,t}-k^{Y}_{V,t}+a_Y a_S(c_{X,t}k^{\zeta}_{S,t}-k^{Y}_{S,t}) \end{pmatrix},\\
c_{\Psi,t+1} &= \begin{pmatrix} 0 & -k^{\zeta}_{V,t} & 0\\ 1 & -a_S k^{\zeta}_{S,t} & 0\\ a_Y & -k^{\zeta}_{V,t}-a_Y a_S k^{\zeta}_{S,t} & 1 \end{pmatrix}.
\end{aligned}$$

∎

Appendix D

There is a standard formula which computes the certainty equivalence of expected utilities in case of CARA utilities. (For example, see Dow and Rahi (2003))

Lemma D.1 *Suppose A is a symmetric $m \times m$ matrix, b is an m-vector, d is a scalar, and w*

is an m-dimensional normal variate: $w \sim N(0,\Sigma)$, Σ *positive definite. Then, we can find the following certainty equivalence of expected utilities if* $(I - 2\Sigma A)$ *is positive definite*

$$E\Big[\exp(w^\top Aw + b^\top w + d)\Big] = |I - 2\Sigma A|^{-\frac{1}{2}} \exp\Big[\frac{1}{2} b^\top (I - 2\Sigma A)^{-1}\Sigma b + d\Big]. \tag{D.1}$$

Proof of Lemma 3.5:

Conjecture that the value function has the form as the following:

$$J(W_t;\Psi_t;t) = -\exp\Big(-\gamma_t W_t - \frac{1}{2}\Psi_t{}^\top \Omega_t \Psi_t + \kappa_t\Big) \tag{D.2}$$

Then, it leads to

$$\begin{aligned}
&E[J(W_{t+1};\Psi_{t+1};t+1)|\mathcal{F}_t^I] \\
&= E\Big[-\exp\Big(-\gamma_{t+1}W_{t+1} - \frac{1}{2}(\Psi_{t+1}{}^\top \Omega_{t+1}\Psi_{t+1}) + \kappa_{t+1}\Big)\Big|\mathcal{F}_t^I\Big] \\
&= E\Big[-\exp\Big(-\gamma_{t+1}\Big\{W_t + \Delta X_t(a_{Q,t+1}\Delta X_t + b_{Q,t+1}\Psi_t + c_{Q,t+1}\varepsilon_{t+1})\Big\} \\
&\quad - \frac{1}{2}(a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1})^\top \Omega_{t+1}(a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1}) + \kappa_{t+1}\Big)\Big|\mathcal{F}_t^I\Big] \\
&= E\Big[-\exp\Big(-\gamma_{t+1}\Big\{W_t + \Delta X_t(a_{Q,t+1}\Delta X_t + b_{Q,t+1}\Psi_t)\Big\} \\
&\quad - \frac{1}{2}(a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t)^\top \Omega_{t+1}(a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t) \\
&\quad - \Big\{\gamma_{t+1}c_{Q,t+1}^\top \Delta X_t + c_{\Psi,t+1}^\top \Omega_{t+1}(a_{\Psi,t+1}\Delta X_t + b_{\Psi,t+1}\Psi_t)\Big\}^\top \varepsilon_{t+1} \\
&\quad - \frac{1}{2}\varepsilon_{t+1}{}^\top c_{\Psi,t+1}^\top \Omega_{t+1} c_{\Psi,t+1}\varepsilon_{t+1} + \kappa_{t+1}\Big)\Big|\mathcal{F}_t^I\Big]
\end{aligned}$$

Using Lemma R.8, it can be shown that

$$\begin{aligned}
&E[J(W_{t+1};\Psi_{t+1};t+1)|\mathcal{F}_t^I] \\
=\;& -\rho_{t+1}\exp\Bigg(-\gamma_{t+1}\Big\{W_t+\Delta X_t(a_{Q,t+1}\Delta X_t+b_{Q,t+1}\Psi_t)\Big\} \\
&-\frac{1}{2}(a_{\Psi,t+1}\Delta X_t+b_{\Psi,t+1}\Psi_t)^\top\Omega_{t+1}(a_{\Psi,t+1}\Delta X_t+b_{\Psi,t+1}\Psi_t) \\
&+\frac{1}{2}\Big\{\gamma_{t+1}c_{Q,t+1}^\top\Delta X_t+c_{\Psi,t+1}^\top\Omega_{t+1}(a_{\Psi,t+1}\Delta X_t+b_{\Psi,t+1}\Psi_t)\Big\}^\top\Xi_{t+1} \\
&\times\Big\{\gamma_{t+1}c_{Q,t+1}^\top\Delta X_t+c_{\Psi,t+1}^\top\Omega_{t+1}(a_{\Psi,t+1}\Delta X_t+b_{\Psi,t+1}\Psi_t)\Big\}+\kappa_{t+1}\Bigg)
\end{aligned}$$

where $\Xi_{t+1}\equiv(\Sigma_{t+1}^{-1}+c_{\Psi,t+1}^\top\Omega_{t+1}c_{\Psi,t+1})^{-1}$ and $\rho_{t+1}=\sqrt{|\Xi_{t+1}|/|\Sigma_{t+1}|}$.

The first-order condition yields

$$(\delta_1+\delta_2)\Delta X_t=(\beta_1+\beta_2)\Psi_t$$

where

$$\begin{aligned}
\delta_1 &= \gamma_{t+1}^2 c_{Q,t+1}\Xi_{t+1}c_{Q,t+1}^\top \\
\delta_2 &= -2\gamma_{t+1}a_{Q,t+1}-a_{\Psi,t+1}^\top\Omega_{t+1}a_{\Psi,t+1}+(c_{\Psi,t+1}^\top\Omega_{t+1}a_{\Psi,t+1})^\top\Xi_{t+1}c_{\Psi,t+1}^\top\Omega_{t+1}a_{\Psi,t+1} \\
&\quad+2\gamma_{t+1}c_{Q,t+1}\Xi_{t+1}c_{\Psi,t+1}^\top\Omega_{t+1}a_{\Psi,t+1} \\
\beta_1 &= \gamma_{t+1}(b_{Q,t+1}-c_{Q,t+1}\Xi_{t+1}c_{\Psi,t+1}^\top\Omega_{t+1}b_{\Psi,t+1}) \\
\beta_2 &= a_{\Psi,t+1}^\top\Omega_{t+1}b_{\Psi,t+1}-(c_{\Psi,t+1}^\top\Omega_{t+1}^\top a_{\Psi,t+1})^\top\Xi_{t+1}c_{\Psi,t+1}^\top\Omega_{t+1}b_{\Psi,t+1}
\end{aligned}$$

Therefore, the solution for the optimization problem is given by

$$\Delta X_t=F_t\Psi_t,\ \ \text{for all } 1\le t<T \tag{D.3}$$

where $F_t\equiv(\delta_1+\delta_2)^{-1}(\beta_1+\beta_2)$.

The second-order condition yields the condition for optimality:

$$\delta_1+\delta_2>0 \tag{D.4}$$

Define

$$
\begin{aligned}
M_t \quad \equiv \quad & F_t^\top(\delta_1+\delta_2)F_t \\
& - (c_{\Psi,t+1}^\top \Omega_{t+1} b_{\Psi,t+1})^\top \Xi_{t+1} (c_{\Psi,t+1}^\top \Omega_{t+1} b_{\Psi,t+1}) \\
& + b_{\Psi,t+1}^\top \Omega_{t+1} b_{\Psi,t+1}
\end{aligned}
$$

Then, we derive

$$
E[J(W_{t+1};\Psi_{t+1};t+1)|\mathcal{F}_t^I] = -\rho_{t+1}\exp\left(-\gamma_{t+1}W_t - \frac{1}{2}\Psi_t{}^\top M_t \Psi_t + \kappa_{t+1}\right) \tag{D.5}
$$

From (D.3)and(D.5), we obtain the following for $t < T$:

$$
\gamma_t = \gamma_{t+1},\ \Omega_t = M_t,\ \text{and}\ \kappa_t = \kappa_{t+1} - 2\log\rho_{t+1}, \tag{D.6}
$$

and for $t = T$:

$$
\gamma_T = \gamma,\ \Omega_T = \Theta_{3,3},\ \text{and}\ \kappa_T = 0. \tag{D.7}
$$

where $\Theta_{3,3}$ is a 3×3 matrix of zeros. Then, we can recursively solve for λ_t, Ω_t and κ_t, which proves the conjecture on the value function is indeed correct. ∎

Appendix E

Proof of Lemma 3.7: Assume the same order flow of each informed trader's order flow in Theorem 3.1. Then, the sufficient statistic at date t which market makers observe in case of N informed traders is defined as

$$
\begin{aligned}
\zeta_t \quad &\equiv \quad \Delta\mathbb{X}_t + \Delta U_t + Na_{X,t}P_{t-1} + Nb_{X,t}a_S\hat{S}_{t-1}^M - Nc_{X,t}(Y_t - P_{t-1} - a_Y a_S \hat{S}_{t-1}^M) \quad (\text{E.1}) \\
&= \quad Na_{X,t}V + (Nb_{X,t} + a_U)S_t + \epsilon_{U,t+1}. \quad (\text{E.2})
\end{aligned}
$$

Define $a'_{X,t} = Na_{X,t}, b'_{X,t} = Nb_{X,t}$. By simply substituting $a_{X,t}, b_{X,t}$ and ΔX_t with $a'_{X,t}, b'_{X,t}$ and $\Delta \mathbb{X}_t$ in the proof, we could obtain the same result for Theorem 3.2, Lemma 3.3, Lemma 3.4. Now, I replicate the proof of Lemma 3.5. Trader i's expected utility given the same conjecture as in (D.2) is given by

$$
\begin{aligned}
&E[J(W^i_{t+1};\Psi_{t+1};t+1)|\mathcal{F}^I_t] \\
&= E\Big[-\exp\Big(-\gamma_{t+1}\Big\{W^i_t + \Delta X^i_t(a_{Q,t+1}\Delta\mathbb{X}_t + b_{Q,t+1}\Psi_t + c_{Q,t+1}\varepsilon_{t+1})\Big\} \\
&\quad - \frac{1}{2}(a_{\Psi,t+1}\Delta\mathbb{X}_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1})^\top\Omega_{t+1}(a_{\Psi,t+1}\Delta\mathbb{X}_t + b_{\Psi,t+1}\Psi_t + c_{\Psi,t+1}\varepsilon_{t+1}) + \kappa_{t+1}\Big)\Big|\mathcal{F}^I_t\Big].
\end{aligned}
$$

Define $\Delta\bar{X}_t$ to be the average of the other informed traders' order flows. Therefore, the aggregate informed order flow is given by $\Delta\mathbb{X}_t = \Delta X^i_t + (N-1)\Delta\bar{X}_t$. Substituting this into ith trader's expected utility and evaluating the first-order condition yields[8]

$$
(\delta_1+\delta_2)\Delta X^i_t + (N-1)(\delta_2 - \gamma_{t+1}c_{Q,t+1}\Xi_{t+1}c^\top_{\Psi,t+1}\Omega_{t+1}a_{\Psi,t+1})\Delta\bar{X}_t = (\beta_1+\beta_2)\Psi_t
$$

where $\delta_1, \delta_2, \beta_1, \beta_2$ are identical to the solution in Appendix D. As in Holden and Subrahmanyam (1992), $\Delta X^i_t = \Delta\bar{X}_t$ in equilibrium. Substituting this yields the equilibrium order flow of ith informed trader

$$
\Delta X^i_t = (\delta_1 + \delta'_2)^{-1}(\beta_1+\beta_2)\Psi_t
$$

where

$$
\delta'_2 = N(\delta_2 - \gamma_{t+1}c_{Q,t+1}\Xi_{t+1}c^\top_{\Psi,t+1}\Omega_{t+1}a_{\Psi,t+1}).
$$

The rest of the proof stays the same, and it proves the initial conjecture of equilibrium order flow is indeed correct. Thus, it finishes the proof. ∎

[8]One is advised to follow the same step in Appendix D.

Appendix F

Numerical Procedure for Section 3.3.4: I follow a similar scheme as in Holden and Subrahmanyam (1994) to obtain numerical solutions for the equation system (3.14). First, I assume an arbitrary value for market makers' mean-squared error of forecasting at the final date T, which is denoted by $\tilde{O}_T$. By the definition of market makers' mean-square of forecasting at trade date t, the following should be true:

$$\tilde{O}_{t+1} = \begin{pmatrix} o^V_{t+1} & o^{VS}_{t+1} \\ o^{VS}_{t+1} & o^S_{t+1} \end{pmatrix}$$

where

$$\begin{aligned} o^V_{t+1} &= E[(V_t - \hat{V}^M_t)(V_t - \hat{V}^M_t)|\mathcal{F}^M_t], \\ o^{VS}_{t+1} &= E[(V_t - \hat{V}^M_t)(S_t - \hat{S}^M_t)|\mathcal{F}^M_t], \\ o^S_{t+1} &= E[(S_t - \hat{S}^M_t)(S_t - \hat{S}^M_t)|\mathcal{F}^M_t]. \end{aligned}$$

Note that

$$\zeta_t = a_{X,t}V + (b_{X,t} + a_U)S_t + \epsilon_{U,t+1}, \tag{F.1}$$

$$Y_t = V + a_Y S_t + \epsilon_{Y,t}. \tag{F.2}$$

Using (F.1) and (F.2), Kalman gain matrix K_t can be easily derived using $\tilde{O}_{t+1}$ like the following:

$$k^\zeta_{V,t} = \frac{1}{\sigma^2_{U,t+1}}[a_{X,t}o^V_{t+1} + (b_{X,t} + a_U)o^{VS}_{t+1}], \tag{F.3}$$

$$k^Y_{V,t} = \frac{1}{\sigma^2_{Y,t}}[o^V_{t+1} + a_Y o^{VS}_{t+1}] \tag{F.4}$$

$$k^\zeta_{S,t} = \frac{1}{\sigma^2_{U,t+1}}[a_{X,t}o^{VS}_{t+1} + (b_{X,t} + a_U)o^S_{t+1}], \tag{F.5}$$

$$k^Y_{S,t} = \frac{1}{\sigma^2_{Y,t}}[o^{VS}_{t+1} + a_Y o^S_{t+1}]. \tag{F.6}$$

Therefore, it allows us to solve (3.14) for η_{T-1} at date $T-1$. Then, $\tilde{O}_{T-1}$ could be solved like the following:

$$\begin{aligned}
o_t^V &= \frac{1}{D_t}\Big[(1-(b_{X,t}+a_U)k_{S,t}^{\zeta}-a_Y k_{S,t}^{Y})o_{t+1}^V+[(b_{X,t}+a_U)k_{V,t}^{\zeta}+a_Y k_{V,t}^{Y}]o_{t+1}^{VS}\Big],\\
o_t^{VS} &= \frac{1}{a_S D_t}\Big[(1-(b_{X,t}+a_U)k_{S,t}^{\zeta}-a_Y k_{S,t}^{Y})o_{t+1}^{VS}+[(b_{X,t}+a_U)k_{V,t}^{\zeta}+a_Y k_{V,t}^{Y}]o_{t+1}^{S}\Big],\\
o_t^S &= \frac{1}{a_S^2 D_t}\Big[(a_{X,t}k_{S,t}^{\zeta}+k_{S,t}^{Y})o_{t+1}^{VS}+[1-a_{X,t}k_{V,t}^{\zeta}-k_{V,t}^{Y}]o_{t+1}^{S}\Big]-\frac{\sigma_{S,t}^2}{a_S^2}.
\end{aligned}$$

where

$$D_t \equiv \Big|1+[a_{X,t}a_Y-(b_{X,t}+a_U)](k_{V,t}^{\zeta}k_{S,t}^{Y}-k_{S,t}^{\zeta}k_{V,t}^{Y})-a_{X,t}k_{V,t}^{\zeta}-k_{V,t}^{Y}-(b_{X,t}+a_U)k_{S,t}^{\zeta}-a_Y k_{S,t}^{Y}\Big|.$$

I continue to solve for η_{T-2}, and so on until I solve for $\tilde{O}_0$. Using trial and error method, find $\tilde{O}_T$ which leads to $\tilde{O}_0 = O_0$, which completes solving the equation system numerically.

Appendix G

First, I claim that the posterior belief of the long-term traders conditional on λ_t is equivalent to

$$\mu_t^U(\omega|\lambda_t) = \frac{Pr(J_t^U(\lambda_t)|\omega)Pr(\omega)}{\sum_{\omega'\in\Omega} Pr(J_t^U(\lambda_t)|\omega')Pr(\omega')} \equiv Pr(\omega|J_t^U(\lambda_t)).$$

From Definition 4.4.1, the consistent observational learning of the long-term traders with the history partition λ_t is equivalent to[9]

$$\mu_t^U(\omega|\lambda_t) = \sum_{\lambda_t'\in J_t^U(\lambda_t)} \frac{Pr(\lambda_t')Pr(\omega|\lambda_t')}{\sum_{\lambda_t'\in J_t^U(\lambda_t)}\sum_{\omega'\in\Omega} Pr(\lambda_t'|\omega')Pr(\omega')}.$$

[9]Note that $Pr(\lambda_t') = \sum_{\omega'\in\Omega} Pr(\lambda_t'|\omega')Pr(\omega')$.

Since λ_t's are disjoint events in Λ_t, it is immediate that $Pr(J_t^U(\lambda_t)|\omega) = \sum_{\lambda'_t \in J_t^U(\lambda_t)} Pr(\lambda'_t|\omega)$. Thus,

$$\begin{aligned}
\mu_t^U(\omega|\lambda_t) &= \sum_{\lambda'_t \in J_t^U(\lambda_t)} \frac{Pr(\lambda'_t)Pr(\omega|\lambda'_t)}{\sum_{\omega' \in \Omega} Pr(J_t^U(\lambda_t)|\omega')Pr(\omega')}, \\
&= \frac{\sum_{\lambda'_t \in J_t^U(\lambda_t)} Pr(\lambda'_t|\omega)Pr(\omega)}{\sum_{\omega' \in \Omega} Pr(J_t^U(\lambda_t)|\omega')Pr(\omega')}, \\
&= \frac{Pr(J_t^U(\lambda_t)|\omega)Pr(\omega)}{\sum_{\omega' \in \Omega} Pr(J_t^U(\lambda_t)|\omega')Pr(\omega')}, \\
&\equiv Pr(\omega|J_t^U(\lambda_t)).
\end{aligned}$$

Therefore, the claim is proved. Using the claim and Bayes' theorem, it is easily shown that

$$\mu_t^U(\omega|\lambda_t) = \frac{Pr(J_t^U(\lambda_t)|\omega)\pi}{Pr(J_t^U(\lambda_t)|G)\pi + Pr(J_t^U(\lambda_t)|B)(1-\pi)}.$$

Appendix H

Since the long-term traders solve a static optimization problem in each period, the optimization problem at time t is given by

$$\begin{aligned}
\max_{x_t^U} \quad & E_t^U\Big[-\exp\Big(-\gamma W_{T+1}^U\Big)\Big] \\
\text{subject to} \quad & W_{T+1}^U = W_0 + (V - p_t)x_t^U
\end{aligned}$$

The first order condition yields

$$-\mu_t^U(G|\lambda_t)(1-p_t)\exp\Big(-\gamma(W_0 + (1-p_t)x_t^U)\Big) + (1-\mu_t^U(G|\lambda_t))p_t\exp\Big(-\gamma(W_0 - p_t x_t^U)\Big) = 0.$$

The second order condition is satisfied as long as $0 < \mu^U(G|\lambda_t) < 1$. Since the long-term traders are identical and present in a unit mass, solving for x_t^U gives the aggregate demand of the long-term traders at time t given p_t as follows:

$$x_t^U = \frac{1}{\gamma}\Big[l_t + log\Big(\frac{1-p_t}{p_t}\Big)\Big].$$

The optimal portfolio choice of the speculators could be obtained directly from the fact that they are short-lived, competitive and risk-neutral. Since the speculators are identical and present in a unit mass, the aggregate demand of the speculators at time t given p_t could be represented as follows:

$$x_t^I \in \begin{cases} M, & \text{if } E_t^I[p_{t+1}] > p_t; \\ [-M, M], & \text{if } E_t^I[p_{t+1}] = p_t; \\ -M, & \text{if } E_t^I[p_{t+1}] < p_t. \end{cases}$$

Given the demand of both the speculators and the long-term traders, the market clearing condition is given by

$$x_t^I + x_t^U = y_t.$$

Therefore, the equilibrium price at time t given the speculators' portfolio x_t^I is represented as

$$p_t = \frac{1}{1 + \exp\Big(-l_t + \gamma(y_t - x_t^I)\Big)}.$$

In particular, the demand of the speculators at time $t = 0$ and $t = T$ is zero; therefore, the price is given by

$$\begin{aligned} p_0 &= \frac{1}{1 + \exp\Big(-l_0\Big)} = \pi; \\ p_T &= \frac{1}{1 + \exp\Big(-l_T + \gamma y_T\Big)}. \end{aligned}$$

Note that the price is a monotone increasing function in $L_t \equiv l_t - \gamma(y_t - x_t^I)$ such that $F(L_t) \equiv \frac{1}{1+\exp(-L_t)}$ and $F'(\cdot) > 0$. L_t reflects the degree of the long-term traders' beliefs, the change in the speculators' aggregate demand and the net supply of the risky asset. Since L_t is an increasing function in l_t, one can also find that the price infinitely approaches to one as l_t approaches to infinity and approaches to zero as l_t approaches to negative infinity. That is, the

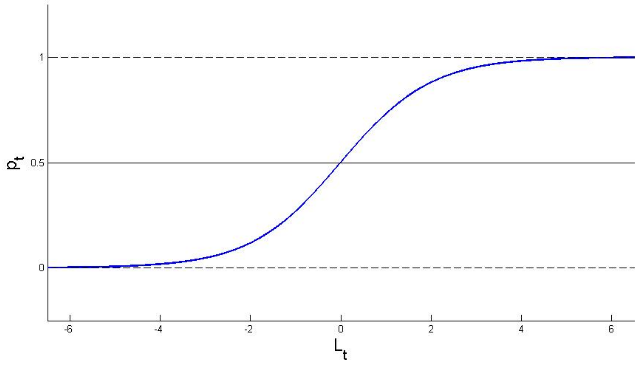

Figure H.1. Price function in L_t

price converges to either one or zero as the long-term traders' beliefs converge to either $\omega = G$ or $\omega = B$. The price function in L_t is illustrated in Figure H.1.

The return at time t has the same sign as $L_t - L_{t-1} = \Delta l_{s,t} - \gamma\big[\epsilon_t - (x_t^I - x_{t-1}^I)\big]$, i.e.,

$$\text{sign}(p_t - p_{t-1}) = \text{sign}\Big(\Delta l_{s,t} - \gamma\big[\epsilon_t - (x_t^I - x_{t-1}^I)\big]\Big).$$

Therefore, the return in the current period is likely to be higher as the evolution of the long-term traders' beliefs becomes more positive and the innovation to the supply of shares becomes more negative. If the speculators have taken the maximum long position in the previous period, the return is likely to be lower in the current period if other things are equal because the speculators from the previous period have to unwind their positions. That is, the net demand of the speculators would be at best the same or smaller if they have already taken the maximum long position in the previous period.

Appendix I

In a CLEE, the optimal demand of the speculators is a function from the current price to the set of portfolio choices given the price history and the true state of the world, i.e.,

$$x_t^I(\cdot|\lambda_t \in \Lambda_t, \omega \in \Omega) : P_t \to [-M, M]. \tag{I.1}$$

Since the speculators maximize their expected utility conditional on their information set, the optimal demand function (I.1) is equivalent to

$$x_t^I(\cdot|J \in J_t^I(\lambda_t), \omega \in \Omega) : P_t \to [-M, M].$$

That is, the set of the optimal aggregate demand of the speculators given history λ_t is identical to the optimal demand given all other possible histories $\lambda_t' \in J_t^I(\lambda_t)$ as long as the true state is the same. In the symmetric equilibrium, the equilibrium strategy is therefore unique to the pair of the given observational partition and state. The following theorem summarizes the findings:

Lemma I.1 *The aggregate demand of the speculators is uniquely determined given $J_t^I(\lambda_t)$ and $\omega \in \Omega$.*

Proof of Proposition 4.6:

Suppose that there exists a learning bubble equilibrium which satisfy (i) and (ii) in Definition 4.4.4. Suppose that the realized state of the world is the bad state, and the learning bubble has indeed occurred in the current sample path of prices, i.e. (i) is true. The feedback loop of price changes in a learning bubble drives the price to go up for sure during the period $\mathcal{T}^* \equiv \{\tau, \tau+1, \dots, \tau+b-1\}$, and the speculators know it because it is common knowledge in equilibrium. Therefore, the speculators will take the maximum long position due to their short horizon as long as the learning bubble persists. Since $x_t^I = M$ for all $t \in \mathcal{T}^*$, Lemma 4.3 shows that the return during the bubble period is always positive as long as $\Delta l_{s,t} > \gamma \bar{y}$ for all $t \in \mathcal{T}^*$. That is, the long-term traders' belief updates should be optimistic enough so that the learning bubble may persist.

Let λ_t be the price history at time t during the learning bubble, i.e., $t \in \mathcal{T}^*$. Then, the optimal strategy of the speculators at time t is unique given the observational partition of the history $\{s_1, s_2, \dots, s_t\}$ and the state ω by the result of Lemma I.1. Since the long-term traders in the subsequent period would also learn from the same observational partition $J_t(\lambda_{t+1}) \equiv \{s_1, s_2, \dots, s_t\} = J_t^I(\lambda_t)$, their assessment of the posterior probability of an upward price movement at time t conditional on the bad state and the observed price history is given by

$$Pr(u_t|J_t(\lambda_{t+1}), B) = Pr(\Delta l_{u,t} \geq \gamma\epsilon_t) = 1 \geq Pr(u_t|J_t(\lambda_{t+1}), G),$$

for any feasible equilibrium strategy of the speculators in the good state. This leads to

$$\Delta l_{u,t+1} \equiv \log\left(\frac{Pr(u_t|J_t(\lambda_{t+1}), G)}{Pr(u_t|J_t(\lambda_{t+1}), B)}\right) \leq 0.$$

Hence, the condition (i) in Definition 4.4.4 is violated. It contradicts the supposition. ∎

Appendix J

Proof of the Existence of a Fully-Revealing Equilibrium:

Suppose that there exists a fully-revealing equilibrium. Then, the change in the log-likelihood ratio at time $t = 2$ is given by $\Delta l_{u,2} = \infty, \Delta l_{d,2} = -\infty$. Therefore, $E_t^I[p_2] = 1$ if $\omega = G$, and $E_t^I[p_2] = 0$ otherwise. The speculators' optimal portfolio is therefore $x_t^I = M$ if $\omega = G$, $x_t^I = -M$ otherwise. Since the long-term traders at time $t = 1$ do not perform any observation inference due to their delayed learning, we have $l_1 = l_0$. By Lemma 4.3, it could be easily shown that $p_1 \geq p_0$ for all ϵ_t if and only if $\bar{y} < M$ and $p_1 < p_0$ for all ϵ_t if and only if $-\bar{y} > -M$. ∎

Proof of Proposition 4.2:

I will solve for the optimal choices of the traders assuming both *Property-S* and *Property-M*, and show that the equilibrium obtained out of those optimal choices indeed has both properties. First, I derive the evolution of the long-term traders' beliefs under *Property-S*. From Lemma 4.1, the log-likelihood ratio of the long-term traders at time t is given by the following linear equation with the initial condition $l_1 = \log(\pi/(1-\pi))$:

$$l_t = l_{t-1} + \Delta l_{s,t}, \tag{J.1}$$

where

$$\Delta l_{s,t} = \begin{cases} \log\left(\frac{1+\theta}{1-\theta}\right), & \text{if } s_{t-1} = u_{t-1}; \\ -\log\left(\frac{1+\theta}{1-\theta}\right), & \text{otherwise;} \end{cases}$$

for all $t = 2, 3, \ldots, T$. Since the sizes of the change in the log-likelihood ratio in the case of the upward and downward price movements are symmetric except for their directions for all $t = 2, 3, \ldots, T$, I denote $\Delta l \equiv \Delta l_{u,t} = -\Delta l_{d,t}$ for notational convenience. Given the log-likelihood ratio l_t, the long-term traders at time t demand the aggregate quantity defined in (4.7). On the other hand, the aggregate demand of the speculators is given by a 'bang-bang' solution due to their short horizon and *Property-M* for all $t = 1, 2, \ldots, T-1$:

$$x_t^I = \begin{cases} M, & \text{if } p_t \geq p_{t-1}; \\ -M, & \text{otherwise.} \end{cases}$$

Note that the speculators' strategy is invariant to the past price history λ_{t-1}, and only depends on the sign of the current return r_t (equivalently, the sign of the current price change $p_t - p_{t-1}$). Given the aggregate demand of the speculators as well as the long-term traders, the Walrasian auctioneer sets the price which clears the market according to the clearing condition given by $x_t^I + x_t^U = y_t$. Given the past price history λ_t, there exists a unique market clearing price in each state of the world ω and each realization of the new innovation to the supply of shares ϵ_t:[10]

	$\epsilon_t = -$	$\epsilon_t = \emptyset$	$\epsilon_t = +$
$\omega = G$	$\frac{1}{1+\exp(-l_t+\gamma(y_{t-1}-\bar{y}-M))}$	$\frac{1}{1+\exp(-l_t+\gamma(y_{t-1}-M))}$	$\frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+\bar{y}+M))}$
$\omega = B$	$\frac{1}{1+\exp(-l_t+\gamma(y_{t-1}-\bar{y}-M))}$	$\frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+M))}$	$\frac{1}{1+\exp(-l_t+\gamma(y_{t-1}+\bar{y}+M))}$

Table J.1. Feasible Equilibrium Prices given ω and ϵ_t for $t = 1, 2, \ldots, T-1$

[10] There could potentially exist other market clearing prices given the speculators' bang-bang trading strategy. However, those prices are in the out-of-equilibrium path. Suppose that the market is cleared at one of those prices in the out-of-equilibrium path. Because the price in the out-of-equilibrium path realizes the current return in the opposite sign to the one in the equilibrium path, *Property-S* is not true any more. Since the equilibrium price properties are common knowledge, the long-term traders' beliefs are not updated by (J.1). Therefore, the price in the out-of-equilibrium path would not be sustainable because of the changes in the long-term traders' portfolio choices.

Since $\Delta l_{s,t}$ is only dependent on the previous return $s_{t-1} \in \{u_{t-1}, d_{t-1}\}$, the market clearing price at time t is uniquely determined by ω, ϵ_t and s_{t-1}. Given such equilibrium price movements, the speculators, who are also able to learn from the price movement either up or down, can infer whether $\epsilon_t \in \{-, \emptyset\}$ or $\epsilon_t \in \{+\}$ if $\omega = G$. On the other hand, they infer $\epsilon_t \in \{-\}$ or $\epsilon_t \in \{\emptyset, +\}$ if $\omega = B$.

To verify *Property-S*, it is sufficient to show that the return at the current period indeed follows such property according to the equilibrium prices found in Table J.1.

Lemma J.2 *Property-S is true if and only if*

$$0 < \Delta l < 2\gamma M. \tag{J.2}$$

Proof Since p_t is uniquely determined by ω, ϵ_t and $\Delta l_{s,t}$, I denote the price at each state (ω, ϵ_t) at time t to be $p_t^{\omega,\epsilon_t}(\lambda_t)$. Since $l_0 = l_1$, it is easily shown that *Property-S* is trivially true at time $t = 1$. Therefore, *Property-S* is true if and only if the followings are true for all $t = 2, 3, \ldots, T$:
(i) $p_t^{G,\epsilon_t}(s_{t-1}) \geq p_{t-1}$ for all $\epsilon_t \in \{\emptyset, +\}$ for all $s_{t-1} \in S_{t-1}$;
(ii) $p_t^{G,\epsilon_t}(s_{t-1}) < p_{t-1}$ for all $\epsilon_t \in \{-\}$ for all $s_{t-1} \in S_{t-1}$;
(iii) $p_t^{B,\epsilon_t}(s_{t-1}) \geq p_{t-1}$ for all $\epsilon_t \in \{+\}$ for all $s_{t-1} \in S_{t-1}$;
(iv) $p_t^{B,\epsilon_t}(s_{t-1}) < p_{t-1}$ for all $\epsilon_t \in \{\emptyset, +\}$ for all $s_{t-1} \in S_{t-1}$;
Using Lemma 4.3, it could be easily shown that the followings are equivalent to each of the above conditions:
(i) $\Delta l_{s,t} - \gamma[\epsilon_t - (M - x_{t-1}^I(s_{t-1}))] \geq 0$ for all $\epsilon_t \in \{\emptyset, +\}, s_{t-1} \in S_{t-1}$;
(ii) $\Delta l_{s,t} - \gamma[\epsilon_t - (-M - x_{t-1}^I(s_{t-1}))] < 0$ for all $\epsilon_t \in \{-\}, s_{t-1} \in S_{t-1}$;
(iii) $\Delta l_{s,t} - \gamma[\epsilon_t - (M - x_{t-1}^I(s_{t-1}))] \geq 0$ for all $\epsilon_t \in \{+\}, s_{t-1} \in S_{t-1}$;
(iv) $\Delta l_{s,t} - \gamma[\epsilon_t - (-M - x_{t-1}^I(s_{t-1}))] < 0$ for all $\epsilon_t \in \{-, \emptyset\}, s_{t-1} \in S_{t-1}$;
Equivalently,
(i) $0 \leq \Delta l \leq 2\gamma M$;
(ii) $-\gamma\bar{y} < \Delta l < \gamma(\bar{y} + 2M)$;
(iii) $-\gamma\bar{y} \leq \Delta l \leq \gamma(\bar{y} + 2M)$;
(iv) $0 < \Delta l < 2\gamma M$;

Therefore, the necessary and sufficient condition for satisfying all (i),(ii),(iii),(iv) is

$$0 < \Delta l < 2\gamma M,$$

for all $t = 2, 3, \ldots, T$. ∎

To verify *Property-M* is true, it is sufficient to show that the expected return in the subsequent period indeed follows such property according to the equilibrium prices found in Table J.1.

Lemma J.3 *There exist M such that Property-M is true for all $M < \bar{M}$ if*

$$\gamma\theta M + \delta < \Delta l < 2\gamma M. \tag{J.3}$$

for some constant $\delta \in \big(0, \gamma(1-\theta)M\big)$.

Proof It is easily shown that *Property-M* is trivially true at time $t = T$ if is true for all $t = 1, 2, 3, \ldots, T-1$. Therefore, *Property-M* is always true if and only if the followings are true for all $t = 1, 2, 3, \ldots, T-1$:
(i) $p_t^{G,\epsilon_t}(s_{t-1}) \leq \frac{1-\theta}{2} p_{t+1}^{G,+}(u_t) + \theta p_{t+1}^{G,\emptyset}(u_t) + \frac{1-\theta}{2} p_{t+1}^{G,-}(u_t)$ for all $\epsilon_t \in \{\emptyset, +\}, s_{t-1} \in S_{t-1}$;
(ii) $p_t^{G,\epsilon_t}(s_{t-1}) > \frac{1-\theta}{2} p_{t+1}^{G,+}(d_t) + \theta p_{t+1}^{G,\emptyset}(d_t) + \frac{1-\theta}{2} p_{t+1}^{G,-}(d_t)$ for all $\epsilon_t \in \{-\}, s_{t-1} \in S_{t-1}$;
(iii) $p_t^{B,\epsilon_t}(s_{t-1}) \leq \frac{1-\theta}{2} p_{t+1}^{B,+}(u_t) + \theta p_{t+1}^{B,\emptyset}(u_t) + \frac{1-\theta}{2} p_{t+1}^{B,-}(u_t)$ for all $\epsilon_t \in \{+\}, s_{t-1} \in S_{t-1}$;
(iv) $p_t^{B,\epsilon_t}(s_{t-1}) > \frac{1-\theta}{2} p_{t+1}^{B,+}(d_t) + \theta p_{t+1}^{B,\emptyset}(d_t) + \frac{1-\theta}{2} p_{t+1}^{B,-}(d_t)$ for all $\epsilon_t \in \{-, \emptyset\}, s_{t-1} \in S_{t-1}$.
Both (i) and (iii) are true if

$$p_t^{B,+}(s_{t-1}) \leq \frac{1-\theta}{2} p_{t+1}^{B,+}(u_t) + \theta p_{t+1}^{B,\emptyset}(u_t) + \frac{1-\theta}{2} p_{t+1}^{B,-}(u_t), \tag{J.4}$$

for all $s_{t-1} \in S_{t-1}$. Likewise, both (ii) and (iv) are true if

$$p_t^{G,-}(s_{t-1}) > \frac{1-\theta}{2} p_{t+1}^{G,+}(d_t) + \theta p_{t+1}^{G,\emptyset}(d_t) + \frac{1-\theta}{2} p_{t+1}^{G,-}(d_t), \tag{J.5}$$

for all $s_{t-1} \in S_{t-1}$.

From (4.8), the price at time t is represented as a function of $L_t \equiv l_t - \gamma(y_t - x_t^I)$ such that $F(L_t) \equiv \frac{1}{1+\exp(-L_t)}$. Let $\Delta L_{t+1} \equiv \Delta l_{t+1} - \gamma(\epsilon_t - \Delta x_t^I)$. Then, we have

$$F(L_{t+1}) = F(L_t) + F'(L_t)\Delta L_{t+1} + \Re_2(L_{t+1})$$

where $\Re_2(L_{t+1}) \equiv F(L_{t+1}) - F(L_t) - F'(L_t)\Delta L_{t+1}$, which is the remainder. By Taylor theorem, there exists some positive constant $\mathcal{M}$ given L_{t+1} which satisfies

$$|\Re_2(L_{t+1})| \leq \mathcal{M}\frac{\Delta L_{t+1}^2}{2},$$

for small enough ΔL_{t+1}. Furthermore, there are only finite number of possible L_{t+1}'s because the trading horizon is finite and the number of realizations of random variable y_t is also finite. Let $\mathcal{L}_T$ be a set which includes all possible realizations of L_t given the finite trading horizon T in equilibrium, i.e.,

$$\mathcal{L}_T \equiv \left\{ l_t - \gamma(y_t - x_t^I) \middle| \forall l_t \in \{-\tau\Delta l, 0, \tau\Delta l\}_{\tau=1}^T, \forall y_t \in \{-\tau\bar{y}, 0, \tau\bar{y}\}_{\tau=1}^T, \forall x_t \in \{-M, M\} \right\}.$$

By taking the maximum of $\mathcal{M}$'s for each possible $L_{t+1} \in \mathcal{L}_T$, there exists $\mathcal{M}'$ for all $L_{t+1} \in \mathcal{L}_T$ which satisfies

$$|\Re_2(L_{t+1})| \leq \mathcal{M}'\frac{\Delta L_{t+1}^2}{2},$$

for small enough ΔL_{t+1}. From Lemma J.2 and the assumption such that $\bar{y} < M$, we have $\Delta L_{t+1} \leq (2 + 3\gamma)M$ if *Property-S* is true. Since $F'(L_{t+1})$ is strictly positive and bounded for all $L_{t+1} \in \mathcal{L}_T$, it could be shown that (J.4) and (J.5) are equivalent to $\Delta l \geq \gamma\theta M + O(M^2)$ if *Property-S* is true.[11] Therefore, there exists $\bar{M}$ given $\delta > 0$ which satisfies *Property-M* for all $M < \bar{M}$ if

$$\gamma\theta M + \delta < \Delta l < 2\gamma M. \quad \blacksquare$$

[11] $f(M) = O(M^2)$ as $M \to 0$ if and only if there exist positive numbers ξ and m such that $f(M) \leq m|M^2|$ for all $|M| < \xi$.

Finally, the following lemma shows that there exists some parameter regions which satisfy both conditions (J.2) and (J.3), and it finishes the proof of the existence of partially-revealing equilibrium:

Lemma J.4 *There exist some positive numbers* $0 < \theta_L < \theta_U < 1$ *such that a partially-revealing equilibrium exists for all* $\theta \in (\theta_L, \theta_U)$ *and for any given* $M < \bar{M}$*.*

Proof Pick a $M^* \in (0, \bar{M})$. Let $\Delta l(\theta) \equiv \log\left(\frac{1+\theta}{1-\theta}\right)$. Then, Δl is continuous and monotone increasing in $\theta \in (0, 1)$, and $\Delta l(0) = 0, \Delta l(1) = \infty$. Also, note that $\Delta l' > 0$ and $\Delta l'' > 0$. By the intermediate value theorem, there exist a unique $0 < \theta_L < \theta_H < 1$ such that $\Delta l(\theta_L) = \gamma\theta M^* + \delta$ and $\Delta l(\theta_H) = 2\gamma M^*$. Consequently, $\gamma\theta M^* + \delta < \Delta l(\theta) < 2\gamma M^*$ for all $\theta \in (\theta_L, \theta_H)$. Therefore, we conclude that given any $M^* < \bar{M}$ there always exists a pair (θ_L, θ_H) such that $\gamma\theta M^* + \delta < \Delta l(\theta) < 2\gamma M^*$ for all $\theta \in (\theta_L, \theta_H)$. ∎

Appendix K

Proof of Lemma 4.5:

The probability assessment of the long-term traders consistent with *Property-S-I* and *Property-S-II* is given by

$$Pr(u_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1+\theta}{2}, \quad Pr(d_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1-\theta}{2},$$

and

$$\begin{aligned} Pr(u_{t-1}|J_{t-1}(\lambda_t), B) &= Pr(u_{t-1}|\hat{\Lambda}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\hat{\Lambda}_{t-1}|J_{t-1}(\lambda_t), B) \\ &\quad + Pr(u_{t-1}|\Lambda^*_{t-1}, J_{t-1}(\lambda_t), B)Pr(\Lambda^*_{t-1}|J_{t-1}(\lambda_t), B) \\ &= \frac{1-\theta}{2}(1 - \Phi_{t-1}(\lambda_t)) + \Phi_{t-1}(\lambda_t), \\ Pr(d_{t-1}|J_{t-1}(\lambda_t), B) &= \frac{1+\theta}{2}(1 - \Phi_{t-1}(\lambda_t)), \end{aligned}$$

where $\Phi_{t-1}(\lambda_t) \equiv Pr(\Lambda^*_{t-1}|J_{t-1}(\lambda_t), B)$ is the long-term traders' assessment of the probability of being in a bubble path at time $t-1$ given the history λ_t. Using Bayes' Theorem, it is shown to be

$$\Phi_{t-1}(\lambda_t) = \frac{Pr(J_{t-1}(\lambda_t)|\Lambda^*_{t-1}, B)Pr(\Lambda^*_{t-1}|B)}{Pr(J_{t-1}(\lambda_t)|\Lambda^*_{t-1}, B)Pr(\Lambda^*_{t-1}|B) + Pr(J_{t-1}(\lambda_t)|\hat{\Lambda}_{t-1}, B)Pr(\hat{\Lambda}_{t-1}|B)}.$$

The probability that $\lambda_t \in \Lambda^*_t$ given $\omega = B$ is equal to $\kappa(\lambda^b_{\tau-1})$ since all bubble paths are initiated by $\lambda^b_{\tau-1}$. That is, we have

$$\kappa(\lambda^b_{\tau-1}) \equiv Pr(\lambda_{\tau-1} = \lambda^b_{\tau-1}|B) = Pr(\lambda_t \in \Lambda^*_t|B), \text{ for all } t \in \mathcal{T}^*.$$

Note that $\Lambda^*_{t-1} \subset J_{t-1}(\lambda_t)$ implies that the price movements after $\tau - 1$ have been upward movements consecutively until $t-1$. Since $Pr(J_{t-1}(\lambda_t)|\Lambda^*_{t-1}, B)$ is either one if $\Lambda^*_{t-1} \subset J_{t-1}(\lambda_t)$ or zero otherwise, it leads to[12]

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\kappa(\lambda^b_{\tau-1})}{\kappa(\lambda^b_{\tau-1}) + \left(\frac{1-\theta}{2}\right)^{t+1-\tau}(1-\kappa(\lambda^b_{\tau-1}))}, & \text{if } \Lambda^*_{t-1} \subset J_{t-1}(\lambda_t); \\ 0, & \text{otherwise}; \end{cases}$$

where $\kappa(\lambda^b_{\tau-1}) \equiv Pr(\lambda^b_{\tau-1}|B)$. From Lemma 4.1, the evolution of the long-term traders' beliefs is given by

$$l_t = l_{t-1} + \Delta l_{s,t}$$

where

$$\Delta l_{s,t} = \begin{cases} \log\left(\frac{1+\theta}{(1-\theta)(1-\Phi_{t-1}(\lambda_t)) + 2\Phi_{t-1}(\lambda_t)}\right), & \text{if } s_{t-1} = u_{t-1}; \\ \log\left(\frac{1-\theta}{1+\theta}\right), & \text{otherwise}; \end{cases}$$

for all $t = 1, 2, \ldots, T$.[13] ∎

Proof of Proposition 4.4:

[12] Let $J^U_{j,\tau} = (s_1, s_2, \ldots, s_{\tau-1})$ be the partition which includes $\lambda^b_{\tau-1}$. Note that the return is always positive during the bubble period in any bubble path. Then, $J^U_{j,t} = (s_1, s_2, \ldots, s_{\tau-1}, u_\tau, u_{\tau+1}, \ldots, u_{t-1})$ contains any possible bubble paths at time t. Therefore, there exists some element $J^U_{j,t} \in \mathcal{J}^U_t$ such that $\Lambda^*_{t-1} \in J^U_{j,t}$.

[13] Note that $\Phi_{t-1}(\lambda_t) = 0$ if $s_{t-1} = d_{t-1}$.

I will prove that the four properties of Proposition 4.4 are indeed true in the equilibrium given the aggregate demand of the long-term traders and the speculators. Note that $\Delta l_{s,t}$ is not independent of the past price history λ_{t-1} any more if the long-term traders suspect there may exist bubbles, i.e., $\Lambda_t^* \not\subset J_{t-1}(\lambda_t)$. Then, $\Phi_{t-1}(\lambda_t)$ gets bigger over time as long as $\lambda_t \in \Lambda_t^*$ because the long-term traders become more sure about the occurrence of a learning bubble. Therefore, the bubble period needs to finish at some point before the probability $\Phi_{t-1}(\lambda_t)$ becomes too sizable. First, I prove that *Property-S-I* and *Property-M-I* are true for all $\lambda_t \in \hat{\Lambda}_t$ with $\Lambda_t^* \not\subset J_{t-1}(\lambda_t)$. Then, I prove that *Property-S-I* and *Property-M-I* are true for all $\lambda_t \in \hat{\Lambda}_t$ with $\Lambda_t^* \subset J_{t-1}(\lambda_t)$, in which case *Property-S-II* and *Property-M-II* are also true for all $\lambda_t \in \Lambda_t^*$.

Lemma K.5 *There exist $\bar{M}^I$ such that Property-S-I and Property-M-I are true for all $M < \bar{M}^I$ when $\lambda_t \in \Lambda_t$ and $\Lambda_t^* \not\subset J_{t-1}(\lambda_t)$ if*

$$\gamma\theta M + \delta < \Delta l < 2\gamma M, \tag{K.1}$$

where $\Delta l \equiv \log\left(\frac{1+\theta}{1-\theta}\right)$ and δ is a positive constant such that $\delta \in \big(0, \gamma(1-\theta)M\big)$.

Proof Because $\Phi_{t-1}(\lambda_t) = 0$ for all $\lambda_t \in \Lambda_t$ with $\Lambda_t^* \not\subset J_{t-1}(\lambda_t)$, we have $\Delta l \equiv \Delta l_{u,t} = \Delta l_{d,t} = \log\left(\frac{1+\theta}{1-\theta}\right)$. Therefore, everything could be proven exactly in the same way as in Lemma J.2. ∎

Lemma K.6 *There exist $\bar{M}^{II} \le \bar{M}^I$ such that Property-S-I, Property-S-II, Property-M-I and Property-M-II are true for all $M < \bar{M}^{II}$ when $\lambda_t \in \Lambda_t$ and $\Lambda_t^* \subset J_{t-1}(\lambda_t)$ if*

$$\gamma\theta M + \delta < \Delta l_{u,t} < 2\gamma M \quad \text{for all } t \in \mathcal{T}^*. \tag{K.2}$$

Proof First, *Property-S-II* is true if *Property-S-I* is true for all $\lambda_t \in \hat{\Lambda}_t$ with $\Lambda_t^* \subset J_{t-1}(\lambda_t)$. Because the aggregate demand of the speculators is $x_t^I = M$ during $t \in \mathcal{T}^*$, Lemma 4.3 implies that the return at time $t \in \mathcal{T}^*$ is always positive as long as *Property-S-I* is true. Likewise, *Property-M-II* is true if *Property-M-I* is true for all $\lambda_t \in \hat{\Lambda}_t$ with $\Lambda_t^* \subset J_{t-1}(\lambda_t)$. It is because the price always goes up when $\lambda_t \in \Lambda_t^*$ as long as *Property-M-I* is true even when $\Lambda_t^* \subset J_{t-1}(\lambda_t)$. Therefore, it is sufficient to show that *Property-S-I* and *Property-M-I* are both true for all

$\lambda_t \in \hat{\Lambda}_t$ with $\Lambda_t^* \subset J_{t-1}(\lambda_t)$.

In the same way as in Lemma J.2, it is easily shown that *Property-S-I* is true if

$$0 < \Delta l_{u,t} < 2\gamma M. \tag{K.3}$$

and

$$-2\gamma M < \Delta l_{d,t} < 0. \tag{K.4}$$

for all $\lambda_t \in \Lambda_t$ with $\Lambda_t^* \not\subset J_{t-1}(\lambda_t)$.

When $s_{t-1} = d_{t-1}$, we have $\Delta l_{s,t} = -\Delta l$ since $\Phi_{t-1}(\lambda_t) = 0$. Therefore, the condition given in Lemma K.5 is sufficient to satisfy *Property-M-I* in case there exists any downward price movement for $t \in \mathcal{T}^*$. Therefore, we only need to show the sufficient condition for the case $s_{t-1} = u_{t-1}$. Then, *Property-M-I* is true if

(i) $p_t^{G,\epsilon_t}(s_{t-1}) \leq \eta p_{t+1}^{G,++}(u_t) + (\frac{1-\theta}{2} - \eta) p_{t+1}^{G,+}(u_t) + \theta p_{t+1}^{G,\emptyset}(u_t) + (\frac{1-\theta}{2} - \eta) p_{t+1}^{G,-}(u_t) + \eta p_{t+1}^{G,--}(u_t)$ for all $\epsilon_t \in \{--, -, \emptyset\}, s_{t-1} \in S_{t-1}$;

(ii) $p_t^{B,\epsilon_t}(s_{t-1}) \leq \eta p_{t+1}^{B,++}(u_t) + (\frac{1-\theta}{2} - \eta) p_{t+1}^{B,+}(u_t) + \theta p_{t+1}^{B,\emptyset}(u_t) + (\frac{1-\theta}{2} - \eta) p_{t+1}^{B,-}(u_t) + \eta p_{t+1}^{B,--}(u_t)$ for all $\epsilon_t \in \{--, -\}, s_{t-1} \in S_{t-1}$;

Using the same technique as in the proof of Lemma J.2, we can show that (i) and (ii) are equivalent to

$$\Delta l_{u,t} \geq \gamma\theta M + O(\Delta L_{t+1}^2) \text{ for all } t \in \mathcal{T}^*. \tag{K.5}$$

For given δ, there exists small enough $\bar{M}'$ which satisfies

$$\gamma\theta M + \delta < 2\gamma M \text{ for all } M < \bar{M}'.$$

Define $\bar{M}^{II} = \min(\bar{M}^I, \bar{M}')$. Then, we find that all four properties of Proposition 4.4 are satisfied if (K.1) and the following are true for all $M < \bar{M}^{II}$:

$$\gamma\theta M + \delta < \Delta l_{u,t} < 2\gamma M \text{ for all } t \in \mathcal{T}^*. \quad \blacksquare$$

Finally, it suffices to show that there exists an equilibrium if there exists parameter values which satisfies the conditions (K.1) and (K.2). The following lemma reveals that there exists such parameter regions, and it finishes the proof of existence of learning bubble equilibrium:

Lemma K.7 *There exist small enough η which guarantees the existence of some positive numbers $0 < \theta_L < \theta_U < 1$ such that a learning bubble equilibrium exists for all $\theta \in (\theta_L, \theta_U)$ and for any given $M < \bar{M}^{II}$.*

Proof By Lemma J.4, there exists $0 < \theta_L < \theta_U < 1$ such that (K.1) is satisfied for all $\theta \in (\theta_L, \theta_U)$ and for any given $M < \bar{M}^{II}$. Now, I claim that there exists $0 < \theta'_L < \theta'_U < 1$ such that (K.2) is satisfied for all $\theta \in (\theta'_L, \theta'_U)$ and for any given $M < \bar{M}^{II}$ if η is small enough. First, I define $\Theta_t(\eta) \equiv (\theta_{L,t}, \theta_{U,t})$ to be the interval given η, which satisfies (K.2) for all $\theta \in \Theta_t(\eta)$ and $M < \bar{M}^{II}$.

I claim that $\theta_{L,t} \to \theta_L$ and $\theta_{U,t} \to \theta_U$ as $\eta \to 0$. Let $G_{L,t}(\theta;\eta) \equiv \gamma\theta M + \delta - \Delta l_{u,t}(\theta;\eta)$ and $G_{U,t}(\theta;\eta) \equiv \Delta l_{u,t}(\theta;\eta) - \exp(2\gamma M)$. Also, let $\theta_{L,t}(\eta) \equiv \max(\{\theta \in (0,1) | G_{L,t}(\theta;\eta) = 0\})$ and $\theta_{U,t}(\eta) \equiv \max(\{\theta \in (0,1) | G_{U,t}(\theta;\eta) = 0\})$. Note that $G_{L,t}$ and $G_{U,t}$ are continuous in θ and η. Furthermore, $\Delta l_{u,t}(\theta;\eta) \to \Delta l(\theta)$ as $\eta \downarrow 0$ and $\Delta l_{d,t}(\theta;\eta) \to -\Delta l(\theta)$ as $\eta \downarrow 0$. Therefore, there exists a sufficiently small η such that $\theta_{L,t}(\eta) \in (\theta_L - \sigma, \theta_L + \sigma)$ and $\theta_{U,t}(\eta) \in (\theta_U - \sigma, \theta_U + \sigma)$ for arbitrarily small $\sigma > 0$.

Therefore, $\Theta_t(\eta)$ converges to $\Theta \equiv (\theta_L, \theta_U)$ when η is sufficiently small. There exists $\bar{\eta}$ which makes $(\cap_{t\in\mathcal{T}^*}\Theta_t(\eta)) \cap \Theta \neq \emptyset$ for all $\eta < \bar{\eta}$. Therefore, we can find an interval such that (K.2) is satisfied for all $\theta \in (\cap_{t\in\mathcal{T}^*}\Theta_t(\eta)) \cap \Theta$ when η is small enough. $\blacksquare$

Appendix L

Proof of Lemma 4.9:

The probability assessment of the long-term traders consistent with *Property-S-I* and *Property-*

S-II is given by

$$Pr(u_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1+\theta}{2}, \quad Pr(d_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1-\theta}{2},$$

and

$$\begin{aligned} Pr(u_{t-1}|J_{t-1}(\lambda_t), B) &= Pr(u_{t-1}|\hat{\Lambda}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\hat{\Lambda}_{t-1}|J_{t-1}(\lambda_t), B) \\ &\quad + Pr(u_{t-1}|\Lambda^{**}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\Lambda^{**}_{t-1}|J_{t-1}(\lambda_t), B) \\ &= \frac{1-\theta}{2}(1-\Phi_{t-1}(\lambda_t)), \\ Pr(d_{t-1}|J_{t-1}(\lambda_t), B) &= \frac{1+\theta}{2}(1-\Phi_{t-1}(\lambda_t)) + \Phi_{t-1}(\lambda_t), \end{aligned}$$

where $\Phi_{t-1}(\lambda_t) \equiv Pr(\Lambda^{**}_{t-1}|J_{t-1}(\lambda_t), B)$ is the long-term traders' assessment of the probability of being in a bubble path or a crash path at time $t-1$ given the history λ_t. Then,

$$\begin{aligned} \Phi_{t-1}(\lambda_t) &= Pr(\lambda_t \in \Lambda^{**}_t|\lambda_{\tau+b-1} \in \Lambda^*_{\tau+b-1}, J_{t-1}(\lambda_t), B)Pr(\lambda_{\tau+b-1} \in \Lambda^*_{\tau+b-1}|J_{t-1}(\lambda_t), B) \\ &\quad + Pr(\lambda_t \in \Lambda^{**}_t|\lambda_{\tau+b-1} \notin \Lambda^*_{\tau+b-1}, J_{t-1}(\lambda_t), B)Pr(\lambda_{\tau+b-1} \notin \Lambda^*_{\tau+b-1}|J_{t-1}(\lambda_t), B) \\ &= Pr(\lambda_t \in \Lambda^{**}_t|J_{t-1}(\lambda_t), \lambda_{\tau+b-1} \in \Lambda^*_{\tau+b-1}, B)\kappa^*_{t-1}(\lambda^b_{\tau-1}), \end{aligned}$$

where

$$\kappa^*_{t-1}(\lambda^b_{\tau-1}) = \frac{\kappa(\lambda^b_{\tau-1})}{\kappa(\lambda^b_{\tau-1}) + \left(\frac{1+\theta}{2}\right)^{t+1-\tau'}\left(\frac{1-\theta}{2}\right)^b(1-\kappa(\lambda^b_{\tau-1}))}.$$

Using Bayes' Theorem leads to

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{\kappa(\lambda^c_{\tau'-1})\kappa^*_{t-1}(\lambda^b_{\tau-1})}{\kappa(\lambda^c_{\tau'-1}) + \left(\frac{1+\theta}{2}\right)^{t+1-\tau'}(1-\kappa(\lambda^c_{\tau'-1}))}, & \text{if } \Lambda^{**}_{t-1} \subset J_{t-1}(\lambda_t); \\ 0, & \text{otherwise;} \end{cases}$$

where $\kappa(\lambda^c_{\tau'-1}) \equiv Pr(\lambda^c_{\tau'-1}|\Lambda^*_{\tau+b-1}, B) = Pr(\lambda^c_{\tau'-1}|\Lambda^*_{\tau+b-1})$. The evolution of the long-term traders' beliefs can be derived similarly in Lemma 4.1. ∎

Proof of the Existence of Learning Bubble Equilibrium with a Crash:

It is easily shown that the following condition needs to be satisfied upon all the conditions for a learning bubble equilibrium with a crash:

$$-2\gamma M < \Delta l_{d,t} < -\gamma\theta M - \delta \quad \text{for all } t \in \mathcal{T}^{**},$$

where δ is a positive constant such that $\delta \in \big(0, \gamma(1-\theta)M\big)$. The existence of some parameter regions which guarantee the existence of the equilibrium can be established in the same way as in Section 4.5.2. ▮

Appendix M

Proof of Lemma 4.9:

The probability assessment of the long-term traders consistent with *Property-S* and *Property-QS* is given by

$$Pr(u_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1+\theta}{2}, \quad Pr(d_{t-1}|J_{t-1}(\lambda_t), G) = \frac{1-\theta}{2},$$

and

$$\begin{aligned} Pr(u_{t-1}|J_{t-1}(\lambda_t), B) &= Pr(u_{t-1}|\hat{\Lambda}_{t-1}, J_{t-1}(\lambda_t), B)Pr(\hat{\Lambda}_{t-1}|J_{t-1}(\lambda_t), B) \\ &\quad + Pr(u_{t-1}|\Lambda^*_{t-1}, J_{t-1}(\lambda_t), B)Pr(\Lambda^*_{t-1}|J_{t-1}(\lambda_t), B) \\ &= \frac{1-\theta}{2}(1-\Phi_{t-1}(\lambda_t)) + \frac{1+\theta}{2}\Phi_{t-1}(\lambda_t), \\ Pr(d_{t-1}|J_{t-1}(\lambda_t), B) &= \frac{1+\theta}{2}(1-\Phi_{t-1}(\lambda_t)) + \frac{1-\theta}{2}\Phi_{t-1}(\lambda_t), \end{aligned}$$

where

$$\Phi_{t-1}(\lambda_t) = \begin{cases} \frac{e^*_t(k)\kappa(\lambda^b_{\tau-1}}{e^*_t(k)\kappa(\lambda^b_{\tau-1}+e_t(k)(1-\kappa(\lambda^b_{\tau-1})}, & \text{for } t \in \mathcal{T}^*; \\ 0, & \text{otherwise;} \end{cases}$$

and $e^*_t(k) = {}_{t-\tau}C_k\big(\frac{1+\theta}{2}\big)^k\big(\frac{1-\theta}{2}\big)^{t-\tau-k}$ and $e_t(k) = {}_{t-\tau}C_k\big(\frac{1-\theta}{2}\big)^k\big(\frac{1+\theta}{2}\big)^{t-\tau-k}$ given k upward price movements. The evolution of the long-term traders' beliefs can be derived similarly in Lemma 4.1. ∎

Proof of Proposition 4.8:

It is easily shown that the following condition needs to be satisfied upon all the conditions for a slow-growing learning bubble equilibrium:

$$\gamma\theta M + \delta < \Delta l_{u,t} < 2\gamma M \quad \text{for all } t \in \mathcal{T}^*,$$

and

$$-2\gamma M < \Delta l_{d,t} < -\gamma\theta M - \delta \quad \text{for all } t \in \mathcal{T}^*,$$

where δ is a positive constant such that $\delta \in \big(0, \gamma(1-\theta)M\big)$. The existence of some parameter regions which guarantee the existence of the equilibrium can be established in the same way as in Section 4.5.2. ∎

Appendix N

Proof of Corollary 4.11:

Let $\lambda^d_{\tau-1} = (r_1, r_2, \ldots, r_{\tau-1})$ be a trigger path which triggers a learning bubble at time $t = \tau$. Pick any return in the pattern at arbitrary time $1 \leq \hat{\tau} \leq \tau-1$, and suppose that it is a downward movement, i.e., $r_{\hat{\tau}} < 0$. Also, let $\lambda^u_{\tau-1}$ be another trigger path which is exactly same but $r_{\hat{\tau}}$ is replaced by $r^u_{\hat{\tau}}$, which is an upward movement at time $t = \hat{\tau}$, i.e., $r^u_{\hat{\tau}} \geq 0$. By Proposition 4.4, returns are independent over time on non-bubble paths in a learning bubble equilibrium. Therefore, the probabilities of reaching those triggers have the following relationship:

$$\kappa(\lambda^d_{\tau-1}) \equiv Pr(\lambda^d_{\tau-1}|B) = Pr(\lambda^u_{\tau-1}|B)\frac{Pr(d_\tau|B)}{Pr(u_\tau|B)} = \kappa(\lambda^u_{\tau-1})\frac{1+\theta}{1-\theta} > \kappa(\lambda^u_{\tau-1}).$$

One could easily verify that $\Delta l_{u,t}$ is decreasing as κ gets larger.[14] Any deviation of $\Delta l_{u,t}$ from Δl makes the sufficient condition for the existence tighter as is obvious in Lemma K.5 and Lemma K.6. Therefore, a learning bubble equilibrium is hard to exist when κ is larger. ∎

Appendix O

There is a standard Bayesian update formula. (For example, see p. 87 of Greene (2000).)

Lemma O.1 *Let* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *have a joint multivariate normal distribution with mean* $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ *and variance* $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$. *Then, the conditional distribution of* x_1 *given* x_2 *is normal with mean* $E[x_1|x_2] = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$ *and variance* $Var[x_1|x_2] = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}$.

The Construction of a Linear Precision Function

Suppose a signal about V_t could be bought at the cost of $\bar{y}$. The signal is given by

$$S = V_t + \epsilon_S$$

where ϵ_S is an error term which follows an i.i.d normal distribution with mean zero and variance $\sigma_{\epsilon_S}^2/\bar{y}$.[15] Note that the variance of error is proportionally decreasing in the cost of the signal. In other words, the signal becomes more precise in a linear way as the cost of a signal gets higher. It is natural that higher investment in information leads to higher return in terms of the precision of the acquired signal. The proportional decrease in the unit cost $\bar{y}$ is rather a technical assumption which obtains smooth extrapolation of discrete choices. Moscarini and Smith (2001) provides a similar extrapolation out of number of sampling.

If an agent invest y in information acquisition, he would collect $k = \lfloor y/\bar{y} \rfloor$ signals. It is easily shown that acquiring multiple i.i.d normal signals is equivalent to acquiring one signal

[14] Note that $\frac{\partial \Delta l_{u,t}}{\partial \kappa} < 0$.

[15] The assumption of i.i.d error terms is equivalent to no searching friction such that an agent would never encounter a redundant signal in his attempt of acquiring new signals.

normally distributed with a pooled precision, which is given by the number of signals times the precision of one independent signal:

Lemma O.2 *Having k i.i.d signals with variance $\sigma^2_{\epsilon_S}$ is equivalent to having a signal with variance $\sigma^2_{\epsilon_S}/k$.*

Proof Denote S_t to be the vector of k signals, i.e. $S_t \equiv (S_1, S_2, \ldots, S_k)$. His posterior variance conditional on acquired signals is given by

$$Var[F_i|S_t] = \sigma^2_{F_i}\left[\frac{\xi_{ia}^{-1}\frac{\sigma^2_\epsilon}{\bar{y}}}{k\sigma^2_{F_i} + \xi_{ia}^{-1}\frac{\sigma^2_\epsilon}{\bar{y}}}\right] \tag{O.1}$$

Therefore, it is easily found that having k independent signals is equivalent to having a signal with k times precision. ∎

Therefore, one could obtain a signal with precision $\sigma^2_{\epsilon_S}/(\bar{y}\lfloor y/\bar{y}\rfloor)$ by investing y. As $\bar{y}$ tends to be zero, the number of signals tends to infinity while the variance of each signal tends to zero. Therefore, having $\bar{y}$ approach to zero is equivalent to having a continuum of i.i.d signals with very small noises. By the result of Theorem O.2, the precision of one combined signal given investment in information acquisition c is shown to be

$$\frac{1}{\sigma^2_{\epsilon_S}}y\left\lfloor\frac{y}{\bar{y}}\right\rfloor = \frac{1}{\sigma^2_{\epsilon_S}}y - o(1/\bar{y}), \tag{O.2}$$

where $o(1/\bar{y})$ is a function converging to zero as $1/\bar{y}$ tends to infinity. By assuming that the agent is acquiring a continuum of i.i.d signals, the asymptotic precision function is given by

$$\phi_t(y) = \alpha_t y,$$

where $\alpha_t \equiv \frac{1}{\sigma^2_{\epsilon,S}}$.

Appendix P

Proof of Lemma 5.1:

Each young agent in period t forms his prior belief by observing price history $\mathcal{P}_{t-1}$ before observing any private signal and the current price. The posterior belief after observing ζ_t could easily be solved by a standard Bayesian formula in Lemma O.1:

$$\begin{aligned} E[V_t|\mathcal{P}_t] &= E[V_t|\mathcal{P}_{t-1}] + \beta^u_{\zeta,t}(\zeta_t - E[V_t|\mathcal{P}_{t-1}]); \\ Var[V_t|\mathcal{P}_t] &= \left(Var[V_t|\mathcal{P}_{t-1}]^{-1} + \frac{b_t^2}{c_t^2\sigma^2_{\eta,t}} \right)^{-1}, \end{aligned}$$

where

$$\beta^u_{\zeta,t} \equiv \frac{Var[V_t|\mathcal{P}_{t-1}]}{Var[V_t|\mathcal{P}_{t-1}] + \frac{c_t^2}{b_t^2}\sigma^2_{\eta,t}}.$$

for all $t = 1, 2, \ldots, T$. Similarly, the posterior belief after observing ζ_t and S_a is given by

$$\begin{aligned} E[V_t|\mathcal{P}_t, S(\bar{y})] &= E[V_t|\mathcal{P}_{t-1}] + \beta^i_{S,t}(S_{a,t} - E[V_t|\mathcal{P}_{t-1}]) + \beta^i_{\zeta,t}(\zeta_t - E[V_t|\mathcal{P}_{t-1}]); \\ Var[V_t|\mathcal{P}_t, S(\bar{y})] &= \left(Var[V_t|\mathcal{P}_{t-1}]^{-1} + \frac{b_t^2}{c_t^2\sigma^2_{\eta,t}} + \alpha_t\bar{y} \right)^{-1} \end{aligned}$$

where

$$\begin{aligned} \beta^i_{S,t} &\equiv \frac{Var[V_t|\mathcal{P}_{t-1}]\frac{c_t^2}{b_t^2}\sigma^2_{\eta,t}}{Var[V_t|\mathcal{P}_{t-1}](\frac{c_t^2}{b_t^2}\sigma^2_{\eta,t} + \frac{1}{\alpha_t\bar{y}}) + \frac{c_t^2\sigma^2_{\eta,t}}{b_t^2\alpha_t\bar{y}}}; \\ \beta^i_{\zeta,t} &\equiv \frac{Var[V_t|\mathcal{P}_{t-1}]\frac{1}{\alpha_t\bar{y}}}{Var[V_t|\mathcal{P}_{t-1}](\sigma^2_{\eta,t} + \frac{1}{\alpha_t\bar{y}}) + \frac{c_t^2\sigma^2_{\eta,t}}{b_t^2\alpha_t\bar{y}}}, \end{aligned}$$

for all $t = 1, 2, \ldots, T$. ∎

Appendix Q

Proof of Lemma 5.3:

From (5.7) and (5.10), it could be easily shown that

$$P_{t+1} = a_{t+1}(1 - \beta^u_{\zeta,t+1})E[V_t|\mathcal{P}_t] + (a_{t+1}\beta^u_{\zeta,t+1} + b_{t+1})\zeta_{t+1} - d_{t+1}\bar{x}. \tag{Q.1}$$

Using (Q.1) and Lemma 5.2, it is immediate that

$$\begin{aligned} E[P_{t+1}|\mathcal{P}_t, S_a(\bar{y})] &= a_{t+1}(1-\beta^u_{\zeta,t+1})E[V_t|\mathcal{P}_t] + (\beta^u_{\zeta,t+1}a_{t+1}+b_{t+1})E[V_t|\mathcal{P}_t, S_a(\bar{y})] - d_{t+1}\bar{x}; \\ E[P_{t+1}|\mathcal{P}_t, S_a(0)] &= (a_{t+1}+b_{t+1})E[V_t|\mathcal{P}_t] - d_{t+1}\bar{x}; \\ \widehat{E}[P_{t+1}|\mathcal{P}_t, S^n_t] &= a_{t+1}(1-\beta^u_{\zeta,t+1})E[V_t|\mathcal{P}_t] + (\beta^u_{\zeta,t+1}a_{t+1}+b_{t+1})\widehat{E}[V_t|\mathcal{P}_t, S^n_t] - d_{t+1}\bar{x}, \end{aligned}$$

and

$$\begin{aligned} Var[P_{t+1}|\mathcal{P}_t, S_a(\bar{y})] &= (\beta^u_{\zeta,t+1}a_{t+1}+b_{t+1})^2\Big(Var[V_t|\mathcal{P}_t, S_a(\bar{y})] + \sigma^2_{V,t+1} + \frac{c^2_{t+1}}{b^2_{t+1}}\sigma^2_{\eta,t+1}\Big); \\ Var[P_{t+1}|\mathcal{P}_t, S_a(0)] &= (\beta^u_{\zeta,t+1}a_{t+1}+b_{t+1})^2\Big(Var[V_t|\mathcal{P}_t] + \sigma^2_{V,t+1} + \frac{c^2_{t+1}}{b^2_{t+1}}\sigma^2_{\eta,t+1}\Big); \\ \widehat{Var}[P_{t+1}|\mathcal{P}_t, S^n_t] &= (\beta^u_{\zeta,t+1}a_{t+1}+b_{t+1})^2\Big(Var[V_t|\mathcal{P}_t, S_a(\bar{y})] + \sigma^2_{V,t+1} + \frac{c^2_{t+1}}{b^2_{t+1}}\sigma^2_{\eta,t+1}\Big). \end{aligned}$$

From (5.9), we derive informed agent a's conditional expectation of V_t:

$$\begin{aligned} E[V_t|\mathcal{P}_t, S_a(\bar{y})] &= (1-\beta^i_{S,t}-\beta^i_{\zeta,t})E[V_t|\mathcal{P}_{t-1}] + \beta^i_{S,t}S_a + \beta^i_{\zeta,t}\zeta_t; \\ &= (1-\beta^i_{S,t}-\beta^i_{\zeta,t})E[V_t|\mathcal{P}_t] + \beta^i_{S,t}S_a + [\beta^i_{\zeta,t} - (1-\beta^i_{S,t}-\beta^i_{\zeta,t})\beta^u_{\zeta,t}]\zeta_t; \\ &= (1-\beta^i_{S,t}-\beta^i_{\zeta,t})E[V_t|\mathcal{P}_t] + (\beta^i_{S,t}+B_t)V_t + \beta^i_{S,t}\epsilon_{S,a} + B_t\frac{c_t}{b_t}\eta_t. \end{aligned}$$

where $B_t \equiv \beta^i_{\zeta,t+1} - (1-\beta^i_{S,t+1}-\beta^i_{\zeta,t+1})\beta^u_{\zeta,t+1}$. From Lemma 5.2, we similarly derive noise trader a's conditional expectation of V_t:

$$\widehat{E}[V_t|\mathcal{P}_t, S^n_t] = (1-\beta^i_{\zeta,t})E[V_t|\mathcal{P}_t] + B_tV_t + (\beta^i_{S,t}+B_t)\frac{c_t}{b_t}\eta_t.$$

Note that $\int_{a\in\mathcal{A}^i_t}\beta^u_{\zeta,t}\epsilon_{S,a}da = 0$ by the law of large numbers.[16] Therefore, integrating (5.14),

[16] $\beta^u_{\zeta,t}$ is a constant, and $\epsilon_{S,a}$ is independent across $\mathcal{A}$. Furthermore, the distribution of $\epsilon_{S,a}$ is preserved the same in the subset $\mathcal{A}^i_t$. Therefore, the integration over $\mathcal{A}^i_t$ converges to the mean of $\epsilon_{S,a}$ by the law of large numbers. See Sun (2006) for more discussions.

(5.15 and (5.16) over $\mathcal{A}_t^i, \mathcal{A}_t^u$ and $\mathcal{N}_t$ yields the following results:

$$\begin{aligned}
\int_{a\in\mathcal{A}_t^i} X_{a,t}^i da &= \frac{\lambda_t}{\rho\Sigma_t^i}\Big[(k_{1,t}^i - Ra_t)E[V_t|\mathcal{P}_t] + (k_{2,t}^i - Rb_t)V_t + (k_{3,t}^i - Rc_t)\eta_t - (d_{t+1} - Rd_t)\bar{x}\Big];\\
\int_{a\in\mathcal{A}_t^i} X_{a,t}^u da &= \frac{1-\lambda_t}{\rho\Sigma_t^u}\Big[(a_{t+1} + b_{t+1} - Ra_t)E[V_t|\mathcal{P}_t] - Rb_tV_t - Rc_t\eta_t - (d_{t+1} - Rd_t)\bar{x}\Big];\\
\int_{a\in\mathcal{N}_t} X_{a,t}^n da &= \frac{\mu}{\rho\Sigma_t^i}\Big[(k_{1,t}^n - Ra_t)E[V_t|\mathcal{P}_t] + (k_{2,t}^n - Rb_t)V_t + (k_{3,t}^n - Rc_t)\eta_t - (d_{t+1} - Rd_t)\bar{x}\Big],
\end{aligned}$$

where

$$\begin{aligned}
k_{1,t}^i &\equiv a_{t+1} + b_{t+1} - (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})(\beta_{S,t}^i + \beta_{\zeta,t}^i);\\
k_{2,t}^i &\equiv (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})(\beta_{S,t}^i + B_t);\\
k_{3,t}^i &\equiv (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})B_t\frac{c_t}{b_t};\\
k_{1,t}^n &\equiv a_{t+1} + b_{t+1} - (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})\beta_{\zeta,t}^i;\\
k_{2,t}^n &\equiv (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})B_t;\\
k_{3,t}^n &\equiv (\beta_{\zeta,t+1}^u a_{t+1} + b_{t+1})(\beta_{S,t}^i + B_t\frac{c_t}{b_t}).
\end{aligned}$$

∎

Proof of Theorem 5.4:

First, I assume the following: $\frac{c_t}{b_t} = \frac{\mu}{\lambda_t}$ for all $t = 1, 2, \ldots, T$.

Using the result of Lemma 5.3, rearranging (5.17) yields the following affine equation of state variables:

$$f_0E[V_t|\mathcal{P}_t] + f_1V_t + f_2\eta_t - f_3\bar{x} = 0,$$

where

$$
\begin{aligned}
f_0 &\equiv \frac{\lambda_t k^i_{1,t} + \mu k^n_{1,t}}{\rho \Sigma^i_t} + \frac{1-\lambda_t}{\rho \Sigma^u_t}(a_{t+1} + b_{t+1}) - R\Xi_t a_t; \\
f_1 &\equiv \frac{\lambda_t \beta^i_{S,t} + (\lambda_t + \mu) B_t}{\rho \Sigma^i_t}(\beta^u_{\zeta,t+1} a_{t+1} + b_{t+1}) - R\Xi_t b_t; \\
f_2 &\equiv \frac{\lambda_t \beta^i_{S,t} + (\lambda_t + \mu) B_t}{\rho \Sigma^i_t}\Big(\frac{\mu}{\lambda_t}\Big)(\beta^u_{\zeta,t+1} a_{t+1} + b_{t+1}) - R\Xi_t c_t; \\
f_3 &\equiv \Xi_t d_{t+1} + 1 - R\Xi_t d_t.
\end{aligned}
$$

and

$$
\Xi_t \equiv \frac{\lambda_t + \mu}{\rho \Sigma^i_t} + \frac{1-\lambda_t}{\rho \Sigma^u_t}.
$$

It is obvious that the equation (Q.2) holds true for all realizations of the state variables if and only if f_0, f_1, f_2, f_4 are all zero. Therefore, the following is derived by solving for a_t, b_t, c_t, d_t which make f_0, f_1, f_2, f_4 zero:

$$
\begin{aligned}
a_t &= \frac{1}{R\Xi_t}\Big[\frac{\lambda_t k^i_{1,t} + \mu k^n_{1,t}}{\rho \Sigma^i_t} + \frac{1-\lambda_t}{\rho \Sigma^u_t}(a_{t+1} + b_{t+1})\Big]; \\
b_t &= \frac{1}{R\Xi_t}\Big[\frac{\lambda_t \beta^i_{S,t} + (\lambda_t + \mu) B_t}{\rho \Sigma^i_t}(\beta^u_{\zeta,t+1} a_{t+1} + b_{t+1})\Big]; \\
c_t &= \frac{1}{R\Xi_t}\frac{\mu}{\lambda_t}\Big[\frac{\lambda_t \beta^i_{S,t} + (\lambda_t + \mu) B_t}{\rho \Sigma^i_t}(\beta^u_{\zeta,t+1} a_{t+1} + b_{t+1})\Big]; \\
d_t &= \frac{1}{R\Xi_t}\Big[\Xi_t d_{t+1} + 1\Big].
\end{aligned}
$$

Indeed, we find the initial assumption is indeed true. Therefore, this finishes the proof. ∎

Appendix R

There is a standard formula which computes the certainty equivalence of expected utilities in case of CARA utilities. (For example, see Dow and Rahi (2003))

Lemma R.8 *Suppose A is a symmetric $m \times m$ matrix, b is an m-vector, d is a scalar, and w is an m-dimensional normal variate: $w \sim N(0, \Sigma)$, Σ positive definite. Then, we can find the* *following certainty equivalence of expected utilities if* $(I - 2\Sigma A)$ *is positive definite*

$$E\Big[\exp(w^\top A w + b^\top w + d)\Big] = |I - 2\Sigma A|^{-\frac{1}{2}} \exp\Big[\frac{1}{2} b^\top (I - 2\Sigma A)^{-1} \Sigma b + d\Big]. \tag{R.1}$$

Proof of Lemma 5.6:

The value function of agent $i \in \mathcal{A}_t$ given the history of prices up to period t, $\mathcal{P}_t$, and private information, $S_a(y_a)$, is given by

$$\mathcal{J}(W_{a,t+1}; \mathcal{F}_t^i) \equiv \max_{X_{a,t}} E\Big[-\exp\Big[-\rho\big(R(W_0 - y_a) + X_{a,t}(P_{t+1} - RP_t)\big)\Big]\Big|\mathcal{P}_t, S_a(y_a)\Big].$$

Using Lemma R.8, the ex-ante utility of agent i is given by

$$\begin{aligned} & E[\mathcal{J}(W_{a,1}; \mathcal{P}_t, S_a(y_a))|\mathcal{P}_t] \\ = \;& E\Big[-\exp\Big[-\rho R(W_0 - y_a) - \frac{(E[P_{t+1}|S_a(y_a), \mathcal{P}_t] - RP_t)^2}{2\Xi_{i,t}}\Big]\Big|\mathcal{P}_t\Big] \\ = \;& -\sqrt{\frac{Var[P_{t+1}|S_a(y_a), \mathcal{P}_t]}{Var[P_{t+1}|S_a(y_a), \mathcal{P}_t] + Var[E[P_{t+1}|S_a(y_a), \mathcal{P}_t]]}} \exp\Big[-\rho R(W_0 - y_a) - \frac{(E[P_{t+1}|\mathcal{P}_t] - RP_t)^2}{2Var[P_{t+1}|\mathcal{P}_t]}\Big]. \end{aligned}$$

Note that $Var[P_{t+1}|\mathcal{P}_t] = Var[P_{t+1}|S_a(y_a), \mathcal{P}_t] + Var[E[P_{t+1}|S_a(y_a), \mathcal{P}_t]]$.[17] Therefore, we obtain

$$E[\mathcal{J}(W_{a,t+1}; S(y_a), \mathcal{P}_t)|\mathcal{P}_{t-1}] = -\sqrt{Var[P_{t+1}|\mathcal{P}_t, S(y_a)]} \exp\Big(\rho R y_a\Big) \Upsilon_{a,t},$$

where

$$\Upsilon_{a,t} = \sqrt{\frac{1}{Var[P_{t+1}|\mathcal{P}_t]}} E\Big[\exp\Big[-\rho R W_0 - \frac{(E[P_{t+1}|\mathcal{P}_t] - RP_t)^2}{2Var[P_{t+1}|\mathcal{P}_t]}\Big]\Big|\mathcal{P}_{t-1}\Big].$$

∎

Proof of Lemma 5.7:

[17] This step could be easily proven by using the formula $Var(Y) = Var[E(Y|X)] + E[Var(Y|X)]$ and the fact that $E[Var(Y|X)] = Var(Y|X)$ in case X, Y are jointly normal.

By the result of Lemma 5.6, agent i prefers acquiring private information by paying $\bar{y}$ to not acquiring private information at no cost if and only if

$$-\sqrt{Var[P_{t+1}|\mathcal{P}_t, S_a(\bar{y})]}\exp\left(\rho R\bar{y}\right)\Upsilon_{a,t} \geq -\sqrt{Var[P_{t+1}|\mathcal{P}_t, S_a(0)]}\Upsilon_{a,t}.$$

Using Lemma 5.3 and Theorem 5.4, the above inequality could be shown to be equivalent to

$$\bar{y} \leq \frac{1}{2\rho R}\log\left(\frac{Var[V_t|\mathcal{P}_t, S_a(0)] + \sigma^2_{V,t+1} + \frac{\mu^2\theta}{\lambda^2_{t+1}\phi_{t+1}}}{Var[V_t|\mathcal{P}_t, S_a(\bar{y})] + \sigma^2_{V,t+1} + \frac{\mu^2\theta}{\lambda^2_{t+1}\phi_{t+1}}}\right).$$

∎

Bibliography

Abreu, D., and M. K. Brunnermeier, 2003, "Bubbles and Crashes," *Econometrica*, 71(1), 173–204.

Admati, A. R., 1985, "A Noisy Rational Expectations Equilibrium for Multi-Asset Securities Markets," *American Economic Review*, 53(3), 629–657.

Al-Najjar, N. I., 2004, "Aggregation and the Law of Large Numbers in Large Economies," *Games and Economic Behavior*, 47(1), 1–35.

Allen, F., and D. Gale, 1992, "Differential Information and Dynamic Behavior of Stock Trading Volume," *Review of Financial Studies*, 5(3), 503–529.

Allen, F., and G. Gorton, 1992, "Stock price manipulation, market microstructure and asymmetric information," *European Economic Review*, 36(2-3), 624–630.

Allen, F., S. Morris, and A. Postlewaite, 1993, "Finite Bubbles with Short Sale Constraints and Asymmetric Information," *Journal of Economic Theory*, 61(2), 206–229.

Allen, F., S. Morris, and H. S. Shin, 2006, "Beauty Contests and Iterated Expectations in Asset Markets," *Review of Financial Studies*, 19(3), 719–752.

Attari, M., A. S. Mello, and M. E. Ruckes, 2005, "Arbitraging Arbitrageurs," *Journal of Finance*, 60(5), 2471–2511.

Avery, C., and P. Zemsky, 1998, "Multidimensional Uncertainty and Herd Behavior in Financial Markets," *American Economic Review*, 88(4), 724–748.

Back, K., C. H. Cao, and G. A. Willard, 2000, "Imperfect Competition among Informed Traders," *Journal of Finance*, 55(5), 2117–2155.

Barberis, N., A. Shleifer, and R. Vishny, 1998, "A Model of Investor Sentiment," *Journal of Financial Economics*, 49(3), 307–343.

Barlevy, G., and P. Veronesi, 2000, "Information Acquisition in Financial Markets," *Review of Economic Studies*, 67(1), 79–90.

Bates, D. S., 1996, "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options," *Review of Financial Studies*, 9(1), 69–107.

Beaver, W., R. Lambert, and D. Morse, 1980, "The Information Content of Security Prices," *Journal of Accounting and Economics*, 2(1), 3–28.

Benartzi, S., 2001, "Excessive Extrapolation and the Allocation of 401(k) Accounts to Company Stock," *Journal of Finance*, 56(5), 1747 – 1764.

Bernhaedt, D., and J. Miao, 2004, "Informed Trading When Information Becomes Stale," *Journal of Finance*, 59(1), 339–390.

Biais, B., P. Bossaerts, and C. Spatt, 2006, "Equilibrium Asset Pricing And Portfolio Choice Under Asymmetric Information," Working paper, Toulouse University, California Institute of Technology and Carnegie Mellon University.

Biais, B. R., and P. L. Bossaerts, 1998, "Asset Prices and Trading Volume in a Beauty Contest," *Review of Economic Studies*, 65(2), 307–340.

Bikchandani, S., D. Hirshleifer, and I. Welch, 1992, "A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascasdes," *Journal of Political Economy*, 100(5), 992–1026.

Bray, M., 1982, "Learning, Estimation, and the Stability of Rational Expectations," *Journal of Economic Theory*, 26(2), 318–339.

Brock, W., J. Lakonishok, and B. LeBaron, 1992, "Simple Technical Trading Rules and the Stochastic Properties of Stock Returns," *Journal of Finance*, 47(5), 1731–1764.

Brunnermeier, M. K., 2001, *Asset Pricing under Asymmetric Information: Bubbles, Crashes, Technical Analysis, and Herding*. Oxford University Press.

Brunnermeier, M. K., and S. Nagel, 2004, "Hedge Funds and the Technology Bubble," *Journal of Finance*, 59(5), 2013–2040.

Brunnermeier, M. K., and L. H. Pedersen, 2009, "Market Liquidity and Funding Liquidity," *Review of Financial Studies*, 22(6), 2201–2199.

Camerer, C., 1989, "Bubbles and Fads in Asset Prices," *Journal of Economic Surveys*, 3(1), 3 – 41.

Cespa, G., 2008, "Information Sales and Insider Trading with Long-Lived Information," *Journal of Finance*, 63(2), 639–672.

Cespa, G., and T. Foucault, 2008, "Insiders-Outsiders, Transparency and the Value of the Ticker," Working Paper.

Chakraborty, A., and B. Yilmaz, 2004a, "Informed manipulation," *Journal of Economic Theory*, 114(1), 132–152.

——— , 2004b, "Manipulation in market order models," *Journal of Financial Markets*, 7(2), 187–206.

Chan, L. K. C., N. Jegadeesh, and J. Lakonishok, 1996, "Momentum Strategies," *Journal of Finance*, 51(5), 1681–1713.

Chen, J., S. Hanson, H. Hong, and J. C. Stein, 2008, "Do Hedge Funds Profit from Mutual-Fund Distress?," National Bureau of Economic Research Working Paper No.

Daniel, K., D. Hirshleifer, and A. Subramanyam, 1998, "Investor Psychology and Security Market Under- and Overreactions," *Journal of Finance*, 53(6), 1839–1885.

De Bondt, W. F. M., and R. Thaler, 1985, “Does the Stock Market Overreact?,” *Journal of Finance*, 40(3), 793–805.

De Long, J. B., A. Shleifer, L. H. Summers, and R. J. Waldmann, 1990a, “Noise Trader Risk in Financial Markets,” *Journal of Political Economy*, 98(4), 703–738.

——— , 1990b, “Positive Feedback Investment Strategies and Destabilizing Rational Speculation,” *Journal of Finance*, 45(2), 374–397.

Decamps, J.-P., and S. Lovo, 2006, “Informational cascades with endogenous prices: The role of risk aversion,” *Journal of Mathematical Economics*, 42(1), 109–120.

Donaldson, R. G., and H. Y. Kim, 1993, “Price Barriers in the Dow Jones Industrial Average,” *Journal of Financial and Quantitative Analysis*, 28(3), 313–330.

Dow, J., 1991, “Search Decisions with Limited Memory,” *Review of Economic Studies*, 58(1), 1–14.

Dow, J., and G. Gorton, 1994, “Arbitrage Chains,” *Journal of Finance*, 49(3), 819–849.

Dow, J., and R. Rahi, 2003, “Informed Trading, Investment and Welfare,” *Journal of Business*, 76(3), 439–454.

Eraker, B., 2001, “Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices,” *Journal of Finance*, 59, 1367–1403.

Feldman, M., and C. Gilles, 1985, “An Expository Note on Individual Risk without Aggregate Uncertainty,” *Journal of Economic Theory*, 35(1), 26–32.

Foster, F. D., and S. Viswanathan, 1996, “Strategic Trading When Agents Forecast the Forecasts of Others,” *Journal of Finance*, 51(4), 1437–1478.

Frankel, J. A., and K. A. Froot, 1990, “Chartists, Fundamentalists, and Trading in the Foreign Exchange Market,” *American Economic Review*, 80(2), 181–185.

Gao, P., 2008, "Keynesian Beauty Contest, Accounting Disclosure, and Market Efficiency," *Journal of Accounting Research*, 46(4), 785–807.

Geanakoplos, J., 2008, "Overlapping Generations Models of General Equilibrium," Working paper, Cowles Foundation, Yale University.

Gilovich, T., and D. Griffin, 2002, "Introduction - Heuristics and Biases: Then and Now," in *Heuristics and Biases: The Psychology of Intuitive Judgment*, ed. by T. Gilovich, D. Griffin, and D. Kahneman. Cambridge University Press.

Glosten, L., and P. Milgrom, 1985, "Bid, Ask, and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," *Journal of Financial Economics*, 14(1), 71–100.

Goldstein, I., and A. Gumbel, 2008, "Manipulation and the Allocational Role of Prices," *Review of Economic Studies*, 75(1), 133–164.

Greene, W. H., 2000, *Econometric Analysis.* Prentice Hall, 4th edn.

Greenwood, R., and S. Nagel, 2009, "Inexperienced Iinvestors and Bubbles," *Journal of Financial Economics*, 93(2), 239–258.

Gromb, D., and D. Vayanos, 2002, "Equilibrium and welfare in markets with financially constrained arbitrageurs," *Journal of Financial Economics*, 66(2-3), 361–407.

Grossman, S. J., and J. E. Stiglitz, 1980, "On the Impossiblity of Informationally Efficient Markets," *American Economic Review*, 70(3), 393–408.

Hamilton, J. D., 1994, *Time Series Analysis.* Princeton University Press.

Harris, L., 1991, "Clustering of Stock Prices," *Review of Financial Studies*, 4(3), 389–415.

He, H., and J. Wang, 1995, "Differential Information and Dynamic Behavior of Stock Trading Volume," *Review of Financial Studies*, 8(4), 919–972.

Hellwig, M. F., 1980, "On the Aggregation of Information in Competitive Markets," *Journal of Economic Theory*, 22, 477–498.

——— , 1982, "Rational Expectations Equilibrium with Conditioning on Past Prices: A Mean-Variance Example," *Journal of Economic Theory*, 26(2), 279–312.

Hirshleifer, D., 2001, "Investor Psychology and Asset Pricing," *Journal of Finance*, 56(4), 1533–1597.

Hirshleifer, D. A., A. Subrahmanyam, and S. Titman, 2006, "Feedback and the Success of Irrational Investors," *Journal of Financial Economics*, 81(2), 311–338.

Holden, C. W., and A. Subrahmanyam, 1992, "Long-lived Private Information and Imperfect Competition," *Journal of Finance*, 47(1), 247–270.

——— , 1994, "Risk Aversion, Imperfect Competition and Long-lived Information," *Economics Letters*, 44, 181–190.

——— , 2002, "News Events, Information Acquisition, and Serial Correlation," *Journal of Business*, 75(1), 1–32.

Hong, H., and J. C. Stein, 1999, "A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets," *Journal of Finance*, 54(6), 2143–2184.

Hou, K., and T. J. Moskowitz, 2005, "Market Frictions, Price Delay, and the Cross-Section of Expected Returns," *Review of Financial Studies*, 18(3), 981–1020.

Huddart, S., J. S. Hughes, and C. B. Levine, 2001, "Public Disclosure and Dissimulation of Insider Trades," *Econometrica*, 69(3), 665–681.

Hvidkjaer, S., 2006, "A Trade-Based Analysis of Momentum," *Review of Financial Studies*, 19(2), 457–491.

Jarrow, R. A., 1992, "Market manipulation, bubbles, corners, and short squeezes," *Journal of Financial and Quantitative Analysis*, 27(3), 311–336.

Jegadeesh, N., 2000, "Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation: Discussion," *Journal of Finance*, 55(4), 1765–1770.

Jegadeesh, N., and S. Titman, 1993, "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance*, 48(1), 65–91.

———, 2001, "Profitability of Momentum Strategies: An Evaluation of Alternative Explanations," *Journal of Finance*, 56(2), 699–720.

Jehiel, P., 2005, "Analogy-based Expectation Equilibrium," *Journal of Economic Theory*, 123(2), 81–104.

Judd, K. L., 1985, "The law of large numbers with a continuum of IID random variables," *Journal of Economic Theory*, 35(1), 19–25.

Kahneman, D., 1973, *Attention and Effort.* Prentice-Hall, Englewood Cliffs, NJ.

Kim, O., and R. E. Verrecchia, 1991, "Market reaction to anticipated announcements," *Journal of Financial Economics*, 30(2), 273–309.

Kindleberger, C. P., 2000, *Manias, Panics, and Crashes: A History of Financial Crises.* Wiley, Hoboken, New Jersey, fourth edn.

Kyle, A. S., 1985, "Continuous Auctions and Insider Trading," *Econometrica*, 53(6), 1315–1335.

Lakonishok, J., A. Shleifer, and R. W. Vishny, 1994, "Contrarian Investment, Extrapolation, and Risk," *Journal of Finance*, 49(5), 1541–1578.

Lee, C. M. C., A. Shleifer, and R. H. Thaler, 1991, "Investor Sentiment and the Closed-End Fund Puzzle," *Journal of Finance*, 46(1), 75–109.

Lee, C. M. C., and B. Swaminathan, 2000, "Price Momentum and Trading Volume," *Journal of Finance*, 55(5), 2017–2069.

Lee, I. H., 1998, "Market Crashes and Informational Avalanches," *Review of Economic Studies*, 65(4), 741–759.

Lehmann, B. N., 1990, "Fads, Martingales, and Market Efficiency," *Quarterly Journal of Economics*, 105(1), 1–28.

LeRoy, S. F., and R. D. Porter, 1981, "The Present-Value Relation: Tests Based on Implied Variance Bounds," *Econometrica*, 49(3), 555–574.

Lo, A. W., H. Mamaysky, and J. Wang, 2000, "Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation," *Journal of Finance*, 55(4), 1705–1765.

Loewenstein, M., and G. A. Willard, 2006, "The Limits of Investor Behavior," *Journal of Finance*, 61(1), 231–258.

Makarov, I., and O. Rytchkov, 2008, "Forecasting the Forecasts of Others: Implications for Asset Pricing," Working Paper.

Mendelson, H., and T. I. Tunca, 2004, "Strategic Trading, Liquidity, and Information Acquisition," *Review of Financial Studies*, 17(2), 295–337.

Merton, R. C., 1976, "Option Pricing When Underlying Stock Returns Are Discontinuous," *Journal of Financial Economics*, 3, 125–144.

Milgrom, P., and N. Stokey, 1982, "Information, trade and common knowledge," *Journal of Economic Theory*, 26(1), 17–27.

Moscarini, G., and L. Smith, 2001, "The Optimal Level of Experimentation," *Econometrica*, 69(6), 1629–1644.

Mullainathan, S., J. Schwartzstein, and A. Shleifer, 2008, "Coarse Thinking and Persuasion," *Quarterly Journal of Economics*, 123(2), 577–619.

Niederhoffer, V., 1965, "Clustering of Stock Prices," *Operations Research*, 13(2).

——— , 1966, "A New Look at Clustering of Stock Prices," *Journal of Business*, 39(2).

Nofsinger, J. R., and R. W. Sias, 1999, "Herding and Feedback Trading by Institutional and Individual Investors," *Journal of Finance*, 54(6), 2263–2295.

Norman, M. F., 1972, *Markov Processes and Learning Models*. Academic Press, New York.

Odean, T., 1998, "Volume, Volatility, Price, and Profit When All Traders Are Above Average," *Journal of Finance*, 53(6), 1887–1934.

Ozdenoren, E., and K. Yuan, 2007, "Feedback Effects and Asset Prices," Working Paper, University of Michigan.

Park, A., and H. Sabourian, 2008, "Herding and Contrarian Behavior in Financial Markets," Working Paper, University of Toronto and University of Cambridge.

Peng, L., 2005, "Learning with Information Capacity Constraints," *Journal of Financial and Quantitative Analysis*, 40, 307–329.

Peng, L., and W. Xiong, 2006, "Investor Attention, Overconfidence, and Category Learning," *Journal of Financial Economics*, 80, 563–602.

Peress, J., 2004, "Wealth, Information Acquisition and Portfolio Choice," *Review of Financial Studies*, 17(2), 879–914.

Sargent, T., 1991, "Equilibrium with signal extraction from endogenous variables," *Journal of Economic Dynamics and Control*, 15(2), 245–273.

Scheinkman, J. A., and W. Xiong, 2003, "Overconfidence and Speculative Bubbles," *Journal of Political Economy*, 111(6), 1183–1219.

Shiller, R. J., 1981, "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?," *American Economic Review*, 71(3), 421–436.

——— , 1990, "Market Volatility and Investor Behavior," *American Economic Review*, 80(2), 58–62.

——— , 2000, *Irrational Exuberance.* Princeton University Press, Princeton, New Jersey.

——— , 2002, "Bubbles, Human Judgment, and Expert Opinion," *Financial Analysts Journal*, 58(3), 18–26.

Shleifer, A., and R. W. Vishny, 1997, "The Limits of Arbitrage," *Journal of Finance*, 52(1), 35–55.

Sonnemans, J., 2006, "Price clustering and natural resistance points in the Dutch stock market: A natural experiment," *European Economic Review*, 50(8).

Sun, Y., 2006, "The Exact Law of Large Numbers via Fubini Extension and Characterization of Insurable Risks," *Journal of Economic Theory*, 126(1), 31–69.

Temin, P., and H.-J. Voth, 2004, "Riding the South Sea Bubble," *American Economic Review*, 94(5), 1654–1668.

Tirole, J., 1982, "On the Possibility of Speculation under Rational Expectations," *Econometrica*, 50(5), 1163–1181.

Townsend, R. M., 1983, "Forecasting the Forecasts of Others," *Journal of Political Economy*, 91(4), 546–588.

Uhlig, H., 1996, "A Law of Large Numbers for Large Economies," *Economics Theory*, 8(1), 41–50.

Vayanos, D., and P. Woolley, 2008, "An Institutional Theory of Momentum and Reversal," Working paper, London School of Economics.

Verrecchia, R. E., 1982, "Information Acquisition in a Noisy Rational Expectations Economy," *Econometrica*, 50(6), 1415–1430.

Wang, J., 1993, "A Model of Intertemporal Asset Prices Under Asymmetric Information," *Review of Financial Studies*, 60, 249–282.

——— , 1994, "A Model of Competitive Stock Trading," *Journal of Political Economy*, 102(1), 127–168.

Watanabe, M., 2008, "Price Volatility and Investor Behavior in an Overlapping Generations Model with Information Asymmetry," *Journal of Finance*, 63(1), 229–272.

Zhang, X. F., 2006, "Information Uncertainty and Stock Returns," *Journal of Finance*, 61(1), 105–137.

Zhu, Y., and G. Zhou, 2009, "Technical Analysis: An Asset Allocation Perspective on the Use of Moving Averages," *Journal of Financial Economics*, 92(3), 519–544.