Credit Risk in Derivative Products

BY

Marcel Nicolas Martin

A dissertation submitted to the University of London in partial fulfillment of the requirements for the degree

PhD

University of London
London Business School

March 27, 1997

"Credit Risk in Derivative Products," a dissertation prepared by Marcel Nicolas Martinin partial fulfillment of the requirements for the degree, Doctor of Philosophy, has been approved and accepted by the following:

Ian Cooper

William Perraudin

Antonio Roma

March 27, 1997

Committee in charge:
Prof. W. Perraudin
Prof. A. Roma

ABSTRACT

After two decades of rapid growth in terms of volume and sophistication, there is growing recognition among both academics and practitioners that the development of transfer mechanisms for market risk has reached a mature stage. Investors have available to them an unprecedented instrumentarium of financial instruments allowing them to implement complicated exposure profiles with respect to interest rate, equity, foreign exchange and commodity risk. By contrast, development of similar mechanisms for credit risk has only recently achieved significant attention. While the growing involvement of regulatory institutions in this area attests to the urgency of the task, there is no consensus as to how credit risk should best be modeled, priced and managed. The objective of this thesis is to make a contribution to the debate on how to best model, price and manage credit risk. After a brief review of the literature, I proceed to outline a framework for the pricing of credit contingent claims. Given the decision to characterize credit risk in terms of a probability of default and a set of payoffs conditional on default having occurred, I provide for both elements to be subject to uncertainty. The concept of a forward probability of default is introduced, capturing the idea of default probabilities varying randomly over time and facilitating incorporation of a correlation structure between credit risk and market risk. In contrast, conventional approaches, by subsuming uncertainty into one credit risk component, limit the scope of a given model to address comprehensively the pricing and risk management issues associated with credit risk and fail to capture interaction with other risks. Development of a pricing methodology as well as articulation of the relationship between credit risk and market risk require that the components of credit risk be embedded in a market structure in which credit risky and credit risk free instruments are being traded. I propose an arbitrage free securities market in which treasury, Libor and credit risky corporate bonds are traded. Specification of a correlation structure between credit risk and market risk requires the presence of the treasury instrument, assumed to be subject to market risk only. Inclusion of the Libor instrument is motivated in terms of the need to specify spread elements unrelated to credit risk. The corporate instrument incorporates credit risk. Restrictions on the distribution of recovery rates necessary to make use of the no arbitrage paradigm are identified. Equivalent martingale measure densities associated with the use of alternative numeraire assets are derived and securities under these measures

are characterized. Using the developed methodology, a number of pricing results for contracts on credit risk variables are provided. A second application provides analysis of quantification and management of credit risk in conventional contracts. Model specification to a level of sophistication unwarranted by the requirements of a particular application will result in a waste of resources. Whether there is merit in making allowance for uncertainty in credit risk or sacrificing tractability in return for incorporating a correlation structure between credit risk and market risk is ultimately an empirical question. Using US data on corporate bonds issued by financial institutions I examine the impact of credit risk specification on the valuation of securities subject to credit risk. Relative to the particular metric I impose, a model incorporating credit risk uncertainty and correlation with market risk dominates the same model with the correlation parameter constrained to be zero, and both models dominate a naive model which stipulates that credit risk is not subject to uncertainty. Consequently, the empirical results validate the relevance of my modelling choices.

Table of Contents

ACKNOWLEDGMENTS

My greatest debt is owed to my supervisor, Professor Ian Cooper. Much of the present analysis was inspired by ideas I would not have encountered without his intellectual guidance. Without his constant support and encouragement it is difficult to conceive how this particular thesis would have been completed.

Much appreciated were conversations with Dr. Massoud Mussavian, which greatly contributed to my understanding of some technical issues. Discussions with Dr. Mark Brittan-Jones, Dr. Andrea Buraschi, Professor Stephen Schaefer and Professor Walter Torous are gratefully acknowledged.

Financial assistance for this work was provided by Salomon Brothers International, the Finance Department and the PhD Programme at London Business School, for which I am very grateful.

Finally, I would like to express my greatest gratitude towards my family for their continued support throughout the years that led to the completion of this thesis.

Chapter 1: Overview

The subject of the present thesis is credit risk in derivative contracts. The objective of this work is to make a contribution to the current debate on how to best model, measure, price and manage credit risk. This contribution is considered to be innovative in the following sense.

On a theoretical level, the current thesis proposes a new credit risk specification. This specification is distinct from existing credit models.

On an empirical level, the current thesis employs a new data set and applies a new metric in assessing the merit of the theoretical specification introduced in the analysis below.

This overview is intended to provide a coherent summary of this thesis, emphasizing the innovative features relative to established approaches to credit risk; furthermore, it should provide a synthesis of the range of issues for which the current analysis should prove relevant.

For the purpose of this thesis, credit risk is defined to subsume two types of uncertainty. On the one hand, credit risk is used to denote the risk of loss of value due to adverse changes in the credit quality of a given issuer; this risk is present prior to the event of default. On the other hand, credit risk is also used to refer to the uncertainty relating to the loss of value in the event of default. This uncertainty relates to the realization of value at or after the event of default and potentially depends on the nature of the complex bargaining process ensuing after default has occurred.

In an environment in which market conditions were stable, credit risk was the most relevant source of uncertainty faced by financial institutions. Given the relative simplicity of financial contracts at the time, this uncertainty could be dealt with through simple techniques. However, during the 1970's, political and economic events led to an increased fluctuation in market variables, triggering a phase of innovation in financial contracts aimed at facilitating the transfer of this market risk. This phase of innovation can now be considered as completed: The demand by financial institutions to adopt specific market risk exposure profiles is matched by a comprehensive supply of market risk transfer mechanisms. Regarding credit risk, this is not the case: The lack of established and proven credit risk transfer mechanisms is in stark contrast to the rising demand for such mechanisms.. This demand has arisen due to a number of circumstances.

The financial industry is currently characterized by increased competition due to the trend towards globalization. Because of this, margins on market risk-based trading income are decreasing. In the search for new sources of income, banks are therefore eager to develop new markets, specifically markets to trade credit risk. Furthermore, banks will try to manage existing activities affected by credit risk in more sophisticated ways. Both of these objectives are currently creating a demand to develop sophisticated techniques to deal with credit risk.

Additional demand for such techniques is generated by regulators, whose primary objective is to provide for the stability of the financial system by ensuring that financial institutions have capital resources commensurate with the risks they are exposed to.

In short, there is a demand by a broad range of financial market participants for a consistent paradigm capable of addressing issues related to the modelling, measurement, management and pricing of credit risk.

This demand has stimulated development of credit models which can broadly be categorized as belonging to one of two groups.

Models in the so-called capital structure category model credit risk as an imbalance between the assets and liabilities of a given issuer. Default is assumed to occur when a value process hits a reorganization boundary. Early instances of such models assume exogenous reorganization boundaries; current approaches provide for this boundary to be determined endogenously, due to the possibility of pre or post-default bargaining between different claims holders of the credit risky issuer's securities. Modelling efforts following this approach have been credited with providing formalizations of credit risk reflecting the economic mechanisms generating credit risk. However, implementations of this approach have been less than successful because of the requirement of modelling the complete liability structure of a given issuer. At the same time, specification of the reorganization boundary is a non trivial issue in practice as the outcome of bargaining process defining this boundary is highly unpredictable. Consequently, the ability of models in this category to address problems relating to the pricing of complicated credit contingent claims remains untested. No applications of such models to risk management issues exist.

A second class of models, termed reduced form models, have sought to model credit risk in terms of a probability of default and a recovery rate. The advantage of such a specification is its parsimony, which should facilitate implementation of models belonging to this class. On the other hand, such model are subject to serious identification problems. The reduced form specification is criticized for incorporating less economic content than the capital structure approach. Furthermore, models in this category are not general enough in order to address comprehensively issues relating to credit risk. Applications of models in this category to the pricing of credit contingent claims are few and no applications to risk management issues can be found.

It must be concluded that existing credit risk models are incapable of addressing comprehensively the demand by market participants for an integrated credit risk measurement methodology.

The analysis in the current thesis is aimed at addressing this demand, proposing a new theoretical specification for credit risk, illustrating its usefulness for specific pricing and risk management applications and finally verifying empirically the significance of the modelling choices made.

On a theoretical level, the analysis in the current thesis is distinct from established approaches with respect to two sets of issues.

On the one hand, while the present modelling approach is consistent with the reduced form approach to credit risk modelling in terms of modelling credit risk by specifying a probability of default and a recovery rate, a number of innovative technical features are introduced. An explicit specification for the probability of default is provided. Since this variable is allowed to be stochastic, this modelling choice requires specification of a forward probability of default. Allowance is made for the recovery rate to be stochastic. Finally, a correlation structure between market risk and credit risk is explicitly specified.

On the other hand, the present analysis differs from established models with regard to the market structure modelled. Specifically, the model proposed assumes that treasury, interbank and corporate assets are traded in an integrated securities market. This modelling choice is motivated by the objective to integrate a number of existing models. Interbank assets trade at spread to treasury assets. In established models, this spread is motivated primarily in terms of credit risk attached to interbank assets. However, the interbank market is empirically free of default. This fact generates the requirement to formulate spread components unrelated to credit risk. At the same time, in order to avoid arbitrage between treasury and interbank assets, imposition of trading restrictions are shown to be necessary. The spread between interbank assets and corporate assets is motivated by the presence of credit risk attached to corporate bonds. The analysis of credit risk is thus embedded in a richer market structure relative to established models.

Based on the innovative technical features of the model and the elaborate market structure, a number of theoretical results are derived. It is shown that restrictions must be placed on the distributional assumptions relating to the recovery rate process in order to make use of the no-arbitrage paradigm. It is also demonstrated that certain processes are fully specified by volatility parameters under a probability measure which is relevant for pricing purposes. This is an attractive feature for implementation purposes since volatility parameters are more easily estimated than drift parameters. This part of the analysis can be interpreted as extending the by now classical Heath Jarrow Morton analysis from market risk to credit risk. Furthermore, pricing functions facilitating valuation of contracts defined in terms of arbitrary credit contingent payoffs are identified.

The generality of the theoretical model is illustrated through two applications.

First, the model is applied to the pricing of specific credit contingent claims. Pricing results relating to four different underlying assets are provided: The treasury asset, an interbank deposit, a credit risky corporate bond and a corporate bond whose credit characteristics are continuously refreshed to preserve a given level of credit quality. Examination of pricing results for forward and futures contracts on prices and yields relating to these four assets yields new insights with respect to four specific issues: The impact of the presence of the treasury interbank spread on contract valuation is examined; the impact of the credit refreshing feature is made explicit; explicit convexity adjustments due to payoffs being non-linear functions of prices arising from the presence of credit risk are provided; and finally, the impact of the continuous settlement feature of futures contracts in the presence of credit risk is illustrated. A second set of pricing applications is concerned with the model's ability to price a number of emerging structures containing credit components. It is shown that the credit risk model proposed in the present thesis is capable to handle the pricing of specific components of credit risk. This is due to the modular specification of credit risk in terms of a probability of default and a recovery rate: It seems that contracts emerging in the markets are structured in terms of these components. Given that, in contrast to the majority of other approaches, both components are subject to uncertainty, the present modelling approach seems well suited to address the pricing and hedging issues associated with these credit structures.

Subsequently, a number of risk management applications of the model are illustrated.

Practitioners have focused on exposure and loss statistics to measure credit risk. It is shown that exposure and loss statistics are easily formulated in terms of the credit risk specification proposed by the current thesis. Furthermore, calculation of these statistics is illustrated to be very similar to the corresponding concepts relating to market risk. Regulators and financial institutions are using Value at Risk (VaR) measures to estimate maximum losses arising from adverse movements in market variables. It is illustrated that the current credit risk specification facilitates calculation of a Value at Risk measure for credit risk, identifying the maximum loss, subject to a given confidence level and liquidation period, due to adverse changes in issuer/counterparty credit quality. Both the calculation of exposure and loss statistics as well as potential Value at Risk calculations for credit risk will depend on the nature of comovements of market risk and credit risk. The explicit modelling of correlation between the probability of default and risk free interest rates proposed in the present thesis permits for the interaction between market risk and credit risk in such calculations to be taken into account.

The two applications of the credit risk specification demonstrate that the model provides a very general framework in which pricing and risk management issues can be addressed in a unifying framework. For both sets of applications, it is demonstrated that the modelling innovations facilitate a greater understanding of how uncertainty in specific credit components affects the pricing of specific contracts or the magnitude of some credit risk statistic.

The final chapter of this thesis is concerned with an empirical verification of the relevance of the modelling choices introduced in the earlier chapters. From a technical point of view, the credit risk model is more general than other models, and whether explicit modelling of uncertainty in credit risk or correlation between market risk and credit risk is warranted is ultimately an empirical question. A number of problems exist with regard to empirical work on credit risk. Credit risk, comprising significant issuer specific components, is idiosyncratic by its nature; this fact means that data constraints are severe. Because there is no consensus as to how credit risk is best modelled, data problems are compounded by lack of information as to the true credit risk model. Finally, estimation techniques used in the context of estimating market risk models (for example, term structure models) still display considerable shortcomings; it is unlikely such techniques perform better in a credit risk context. It is ultimately for these reason that few empirical studies relating to credit risk exists; among those studies that do exists, the majority focuses on statistical properties of credit models as opposed to pricing properties.

Based on such considerations, for the purpose of empirical verification I formulate a metric which is based on pricing performance rather than a statistical measure of fit: I examine how well a model based on a given credit risk specification performs when confronted with the task of pricing a credit contingent claim. Implementation of this principle is based on the following algorithm. First, a nested set of three credit models is specified. The simplest model is based on a fixed credit spread, while the most sophisticated model assumes a stochastic credit spread as well as non-zero correlation between credit spreads and risk free interest rates. The three models are parameterized using data on A-rated straight corporate bonds issued by US financial institutions. The models are then implemented, using trinomial trees. Given that call schedules and price data is available for A-rated callable corporate bonds issued by US financial institutions, the model prices (corresponding to the three credit risk specifications) of the calls embedded in these securities can be calculated using the trees. Using a market-based measure of the embedded calls, pricing errors associated with each of the three credit risk specification can be calculated. The results indicate that, relative to a model in which credit spreads are constant, incorporation of uncertainty and, subsequently, correlation between credit spreads and risk free interest rates reduces the pricing errors of the embedded calls. The empirical results therefore validate the modelling choices of the present thesis to the extent that the additional technical features distinguishing the credit risk specification from other credit risk models are shown to generate significant improvements when applied to the pricing of credit contingents claims. The assertion that credit risk should be modelled as being subject to uncertainty, as well as the contention that correlation between market risk and credit risk is relevant for the pricing of credit contingent claims, have thus been established as refutable propositions.

Chapter 2: Literature

2.1. Credit Risk in Debt Instruments

Introduction

The term structure of interest rates has been a long standing research field in financial economics. A substantial part of the literature is concerned with the analysis of debt instruments free of credit risk[1], focussing on domestic currency bonds issued or effectively guaranteed by a sovereign government[2]. The equilibrium approach to modelling the risk free[3]term structure, pioneered by CIR (1981a), CIR (1985a), CIR (1985b), starts from a description of the economy, specification of the evolution of a number of state variables subsuming uncertainty and assumptions about investor preferences. Recent variants of this approach include Ahn and Thompson (1988), who analyze the term structure based on a jump-diffusion specification for the state variables, and Longstaff and Schwartz (1993), who propose a model in which the term structure is determined by the value of the short rate and its volatility, both of which are assumed to follow diffusion processes.

The equilibrium approach produces the term structure of interest rates as an output. As such, the model term structure is frequently at variance with the observed term structure. Given the objective to identify mispricings in the term structure itself, such models will provide useful information. For the purpose of pricing contracts contingent on the term structure, the fact that these models misprice the underlying instruments is problematic. While some authors have proposed introduction of time

[1] In the current thesis, the term credit risk is used to denote changes in value due to changes in credit quality before the occurence of default as well as the actual event of default. The term default risk is reserved to denote the contingency of the latter event.

[2] While in these approaches the definition of a credit risk free issuer is synonymous with a sovereign, there is a growing interest in analysis of sovereign debt issues subject to credit risk. To the extent that such instruments, taking account of the required currency translation, are equivalent to domestic credit risky corporate bonds, analysis will be identical to that pursued in the current thesis and the specific literature is not reviewed here.

[3] Unless otherwise stated, the term risk free will be used as being synonymous with credit risk free and does not preclude the presence of other risks.

varying coefficients in order to obtain consistency between model and observed term structures (Hull and White (1989), Maghsood (1996), Jamshidian (1995), Jamshidian (1996)), the problem of attributing the model's failure to replicate market data to either misspecification of variable dynamics or omission of relevant state variables remains.

A second problem relates to the fact that pricing formulae generated by these models invariably include parameters subsuming investors attitudes toward various sources of risk; estimation of these parameters has proved an elusive exercise.

Early arbitrage approaches refrained from giving a complete description of the economy and focused on the relative pricing of traded instruments. In contrast to the equilibrium approach, specification of the dynamics of a finite number of state variables (Vasicek (1977) uses the short rate, Brennan and Schwartz (1982) propose the short rate and the long rate as state variables), coupled with the assumption that bond markets are free of arbitrage, is sufficient to derive the term structure and prices for some term structure contingent claims. Still, pricing formulae would include preference parameters and model term structures would fail to replicate observed terms structures.

Both shortcomings were addressed in the seminal articles of Ho and Lee (1986), Heath, Jarrow and Morton (1990) and Heath, Jarrow and Morton (1992). Taking the observed term structure as an input, these models are attractive for the purpose of pricing term structure contingent claims in that the underlying zero coupon bonds are priced correctly. Equally important, the dynamics of the stipulated state variables, and consequently the model term structure and model prices, are fully parameterized in terms of volatility parameters, estimation of which is easier than estimation of preference parameters.

A related part of the literature is concerned with the term structures corresponding to debt obligations which are subject to credit risk, the review of which is the subject of the following sections.

2.1.1. General Considerations

The probability of default of a particular debt instrument will rise in a situation in which the issuing firm is experiencing financial distress. In principle, the presence of financial distress has the potential to free resources to be moved to higher valued uses by forcing managers and directors to reduce capacity and to rethink operating policy and strategy decisions. In an all equity firm this kind of organizational change is unlikely to occur because poor performance does not lead to financial distress. However, given that a firm has chosen a particular liability structure[4], Wruck (1990) notes that "it is financial distress that gives creditors a legal right to demand restructuring."

The final payoff on a given debt instrument will depend on the source of value from which liability claims are met, the specific events triggering default, the nature of the ensuing restructuring and the final settlement between the parties involved.

A debt obligation entitles the holder to receive a stream of payouts defined in and subject to contingencies set out in its covenants (e.g. Smith and Warner (1979)). The source of value from which payouts are met are the real and financial assets of the instrument's issuer. As will be shown below, established models differ in the extent to which firm value is explicitly modeled and linked to the value of specific debt issues.

Financial distress occurs when cash flow is insufficient to cover current obligations. Default is defined as occurring when principal or interest payments under borrowing agreements are not met. The occurrence of technical default, understood as the violation of debt covenants other than the ones stipulating repayment schedules for principal and interest repayments[5], can be a warning sign that distress is imminent. Financial distress generally leads to negotiations with at least one of the firm's creditors.

Payoffs on bonds which have defaulted are potentially subject to the complex bargaining processes which are initiated when a firm is precipitated into financial distress. Models may either aim to replicate this process or sidestep the issue by assuming an exogenous payoff in the event of bankruptcy. Depending on the nature of the firm's insolvency[6], the resolution of distress will take different forms.

In a private workout, the firm and its creditors renegotiate their contracts without recourse to the legal system; typical settlements involve either waivers of specific payments or complete restructurings of all liabilities involving offers to exchange some of the distressed instruments for new debt securities or equity claims.

Alternatively, firms can file for bankruptcy protection, subsequent to which negotiations to rewrite contracts with creditors take place in a court supervised setting.

Finally, a firm may file for liquidation, in which case a bankruptcy court oversees the distribution of the firm's assets to claimholders in order of their claim's

[4] A number of authors discuss strategies available to equityholders to effect wealth transfers among claimholders and possible defensive mechanisms against such actions (see Fama (1978), Kalay (1982), Black (1976), Brickley (1983), Dann (1982), Easterbrook (1984), Gertner and Scharfstein (1991), Masulis (1980)). There is a substantial literature addressing the problem of adverse incentives between managers, equityholders and bondholders in the presence of credit risky debt and proposing liability designs which maximize firm value (e.g. Fama (1978), Jensen and Meckling (1976), Myers (1977), Myers and Majluf (1984), Titman (1984)).

[5] E.g. minimum net worth requirements or working capital constraints.

[6] Insolvency and distress are used synonymously; see Wruck (1990) for for definitions of stock based and flow based definitions of insolvency.

priority. The nature of a firm's financial- and asset structure, the constituency of its creditors and the legal environment in which the firm operates will determine the nature of the process by which distress is resolved[7]. Given the complexity of this process and the fact that strict absolute priority rules only provide an approximative indication of actual distributions to claimholders[8], final payoffs in the event of default on a given debt claim must be considered highly unpredictable.

2.1.2. The Capital Structure Approach

Building on the insight by Black and Scholes (1973) that corporate liabilities can be viewed as contingent claims on firm value, the capital structure approach to credit risk places the analysis of individual debt claims issued by a specific firm within the framework of its complete asset liability structure, which is explicitly modelled. Most generally, default is defined as the moment at which this structure violates contractual constraints, usually emanating from covenants in the firms outstanding debt. Formalizing this idea in its full generality is usually not feasible and most approaches in this category define the time of default as the moment at which firm value hits a boundary subsuming the firm's payment obligations associated with outstanding liabilities.

The first model of this type was formalized by Merton (1974). Assuming frictionless markets and constant interest rates, he stipulates that firm value follows a mean reverting diffusion process[9]:

$$dV = (\alpha V - C)dt + \sigma dz \tag{1}$$

where α is the instantaneous expected rate of return on the firm's assets, C denotes the net flows to or from the firm[10], σdt is the instantaneous volatility of the firm's assets and dz is a Brownian increment. It is further assumed that investors can take positions both in the firm and in another security, the price F of which is assumed to be a function of firm value and time only. By taking positions in this security, in the firm and in a riskless debt instrument, Merton demonstrates that investors can construct a zero net wealth portfolio, the value of which is free of uncertainty. This

[7] See Easterbrook (1990) and Gertner and Scharfstein (1991) for a discussion of the merits of different mechanisms used to resolve financial distress.

[8] Weiss (1990) and Franks and Torous (1989) provide empirical evidence of significant deviations from strict absolute priority rules for bankruptcy cases under U.S. law.

[9] Notation in the current chapter corresponds to the notation in the referenced articles and will be explicitly referred to only as far as warranted for the purpose of discussion.

[10] If positive, it denotes the the firm's payouts to its claimholders in the form of dividends or interest payments; if negative, it denotes injections from new financing.

consideration is equivalent to the price of the firm value contingent claim to satisfy the following partial differential equation:

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (rV - C)F_V - rF + F_t + C_y = 0 \tag{2}$$

where C_y is the dollar payout per unit time on the contingent security. Merton considers a firm whose capital structure is given by a single zero coupon issue, default on which occurs upon failure to make the principal payment at the maturity date, and a residual equity claim. Default results in a transfer of ownership of the firm's assets from equity holders to bond holders and entails no loss of value. Therefore, the value of the firm is identical to the sum of the value of the equity- and bond holder's claims:

$$V = F(V,t) + f(V,t) \tag{3}$$

where f is the value of the equity claim. The price of the zero coupon bond must satisfy:

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (rV - C)F_V - rF + F_t = 0 \tag{4}$$

If at the maturity of the debt firm value is insufficient to repay the bond's principal, control of the company is transferred to the debt holders and the value of the debt drops to firm value; otherwise bond holders receive the principal B[11]

$$F(V,0) = (V,B)^- \tag{5}$$

Given its definition as the residual claim, the value of the equity claim at the bond's maturity date is given by:

$$f(V,0) = (0, V - B)^+ \tag{6}$$

Given (6), Merton argues that the value of the equity claim is identical to the value of a call option on the firm with exercise set to the bond's principal, and using (1) and (3) he derives a closed form solution for the value of credit risky debt:

$$\begin{aligned} F(V, T-t) &= Be^{-r(T-t)} N\left(\frac{-\sigma^2(T-t)/2 - \ln(d)}{\sigma\sqrt{T-t}}\right) \\ &\quad + Be^{-r(T-t)} \frac{1}{d} N\left(\frac{-\sigma^2(T-t)/2 + \ln(d)}{\sigma\sqrt{T-t}}\right) \end{aligned} \tag{7}$$

where $d = \frac{B}{V}e^{-r(T-t)}$. Using the definition of yields $R(T-t)$:

$$F = Be^{-R(T-t)\times(T-t)} \tag{8}$$

(7) can be restated as follows:

$$R(T-t) - r = -\frac{1}{T-t}\log N\left(\frac{-\sigma^2(T-t)/2 - \ln(d)}{\sigma\sqrt{T-t}}\right) \tag{9}$$

[11] The following notation is used throughout: $(a,b)^+ = max\,(a,b)$, $(a,b)^- = min\,(a,b)$

$$-\frac{1}{T-t}\log\frac{1}{d}N\left(\frac{-\sigma^2(T-t)/2+\ln(d)}{\sigma\sqrt{T-t}}\right)$$

Merton denotes $R(T-t)-r$ the risk premium and refers to (9) as defining a risk structure of interest rates. For a given maturity T, this risk structure of interest rates can be seen to be a function of the volatility of the firm's value, and the ratio of the discounted promised principal to the current value of the firm.

A number of researchers have provided useful extensions to Merton's models. Black and Cox (1976) consider valuation subject to inclusion of more realistic indenture provisions and demonstrate that existence of safety covenants, subordination arrangements and restrictions on the financing of interest and dividend payments increase the value of credit risky bonds. Allowance for stochastic interest rates is made by Shimko et al. (1993), who stipulate a lognormal process for firm value:

$$\frac{dV}{V}=\alpha dt+\sigma_v dz_1 \tag{10}$$

and a mean reverting process for interest rates:

$$dr=k(\gamma-r)dt+\sigma_r dz_2 \tag{11}$$

The authors consider the pricing of a default risky zero coupon bond on which default is declared at the maturity date if firm value is insufficient to cover the principal repayment. They propose a formula for the price of a credit risky zero coupon bond as:

$$F=V-VN(h_1)+BP(T-t)N(h_2) \tag{12}$$

where h_1 and h_2 are functions depending on, among other things, the correlation between firm value and interest rates. (13) is similar to (7), except that the deterministic risk free discount function $Be^{-r(T-t)}$ has been replaced by its stochastic equivalent, $P(T-t)$. The authors provide a constraint which must be satisfied by credit and interest rate contingent claims in the presence of stochastic firm value and interest rates:

$$\frac{\sigma_v^2}{2}V^2F_{VV}+\rho\sigma_v\sigma_r VF_{Vr}+\frac{\sigma_r^2}{2}F_{rr}+F_r(k(\gamma-r)-\lambda)+F_t+rVF_V-rF=0 \tag{13}$$

An extension to Merton's model to coupon bearing debt is provided by Kim et al. (1993), who examine the possibility that default is triggered prior to maturity by equity holder's inability to meet coupon payments. Specifically, bankruptcy is triggered when $\gamma V<c$, where c is the coupon on the default risky bond and γV is the net cash outflow from the firm. If this is the case, the payoff to bond holders is assumed to be:

$$(\delta(\tau)B(r,\tau;c),V^*)^- \tag{14}$$

where V^* is the lower reorganization boundary for the firm's value, $\delta(\tau)$ is a positive fraction and $B(r, \tau; c)$ is the price of a treasury obligation free of default risk. At the maturity date, $\delta(0) = 1$, and bond holders receive the minimum of firm value and principal amount. The authors demonstrate that at reasonable parameter values, their specification is able to generate the empirically observed magnitude of yield spreads on corporate bonds. A rather stylized model nesting the above specifications is provided by Longstaff and Schwartz (1995a). The authors propose a model in which firm value and interest rates evolve according to the following processes:

$$dV = \mu V dt + \sigma V dZ_1 \tag{15}$$

$$dr = (\zeta - \beta r)dt + \eta dZ_2 \tag{16}$$

Default is defined as the time at which firm value reaches an exogenous threshold level K; this condition is equivalent to the process $X = V/K$ reaching unity from above. This specification allows default to be triggered prior to maturity. The fixed threshold level K is consistent with default being triggered by the inability to meet coupon payments, but potentially permits a wider range of interpretations. In the event of default, bondholders sustain a fractional loss w on the principal value of their instruments. The authors derive a partial differential equation which must be satisfied by securities subject to firm and interest rate risk:

$$\frac{\sigma^2}{2} V^2 H_{VV} + \rho\sigma\eta V H_{Hr} + \frac{\eta^2}{2} H_{rr} + rVH_V + (\alpha - \beta r)H_r - rH = H_T \tag{17}$$

where α represents ζ and a constant representing the market price of risk. The authors demonstrate that the value of the credit risky zero coupon bond in their framework is given by

$$P(X, r, T) = D(r, T) - wD(r, T)Q(X, r, T) \tag{18}$$

where w is the fractional loss in the event of default, $D(r, T)$ is the price of a risk free zero coupon security and $Q(X, r, T)$ is the probability of default of the credit risky instrument. The value of a credit risky floating rate payment is shown to be

$$F(X, r, t, T) = D(r, T)R(r, t, T) - wD(r, T)G(X, r, \gamma, T) \tag{19}$$

where $R(r, t, T) = E_t[r_T]$ is the expected value of the floating rate payment, $G(X, r, \gamma, T) = E_t\left[r_T \cdot I_{\{\gamma \leq T\}}\right]$ corresponds to the same expectation conditional on default occurring and expectations are evaluated relative to the risk adjusted processes

$$d \ln X = (r - \frac{\sigma^2}{2} - \rho\sigma\eta B(T - t))dt + \sigma dZ_1 \tag{20}$$

$$dr = (\alpha - \beta r - \eta^2 B(T - t))dt + \eta dZ_2 \tag{21}$$

The approach is interesting to the extent that it does not assume constant interest rates and introduces an explicit role for the correlation between interest rates and firm value. The model allows for default to occur prior to maturity. The use of a threshold level triggering default does not differ from Merton's approach, where the corresponding boundary is given by the principal on the outstanding bond issue. Thus, the threshold level can be found explicitly in the bond's indenture provisions, whereas in Longstaff and Schwartz's approach K must be constructed as an aggregate default trigger from the contractual terms of all outstanding liabilities.

All of the above models assume that both the default trigger and the fractional loss are exogenous constants . This may not be the most satisfying way of capturing either the events that precipitate a firm into bankruptcy or anticipating the outcome of the complex bargaining process that might ensue. A significant shortcoming of models incorporating such a feature is that their structure is not compatible with the observation of nontrivial spreads at short maturities: Since credit risk is modeled as a continuous process and the default triggering boundary is fixed, at short maturities the time of default becomes predictable and the risk of default becomes degenerate.

Given the time varying nature of not just firm value but also outstanding liabilities and covenant structure, the desirability of attaching uncertainty to the reorganization boundary is apparent. Such an extension was proposed by Nielsen et al. (1993), whose specification of firm value and interest rates is consistent with earlier models:

$$dr(t) = a_r \left[\bar{b}_r - r(t)\right] dt + \sigma_r dZ_r(t) \tag{22}$$

$$\frac{dV(t)}{V(t)} = (r(t) + \bar{\mu})dt + \sigma_v dZ_v(t) \tag{23}$$

In contrast to earlier specifications, allowance is made for lognormal dynamics of the reorganization boundary:

$$\frac{d\overline{K}(t)}{\overline{K}(t)} = [r(t) + d]dt + \sigma_{kr} dZ_r(t) + \sigma_{kv} dZ_v(t) \tag{24}$$

If default occurs, bond holders are assumed to recover a constant fraction α of the principal value of the defaulted bonds. The price of a credit risky zero coupon bond in this model is given by:

$$P_C(t,T) = P(t,T_1) - \Delta(t,T_1,V(t),r(t),\overline{K}(t)) \tag{25}$$

where

$$\begin{aligned} \Delta(t,T_1,V(t),r(t),\overline{K}(t)) &= Q(T_k \leq T_1) \\ &\quad \times E_* \left[(1-\alpha)P(T,T)e^{\int_t^{T_k} r(s)ds} \mid T_k \leq T_1\right] \end{aligned} \tag{26}$$

is the term capturing the loss incurred by bond holders in the event of default. This approach must be credited with formalizing the idea of the default boundary itself being stochastic, adding another element of realism to the capital structure paradigm. Nevertheless, there is no economic motivation of the change in the level of the default boundary. Most recently, a number of authors have sought to extend the capital structure approach with a view of determining the reorganization boundary endogenously. A common element in these models is that explicit allowance is made for bargaining between bond holders and equity holders. One of the first models in this category was proposed by Mella-Barral and Perraudin (1996), who model the liquidation decision of a firm operating with constant flow costs w but with a stochastically varying output price p. The event of liquidation entails a loss of value in that the firms earnings fall to $\zeta_1 p - \zeta_0 c$, where $\zeta_1 \leq 1$ and $\zeta_0 \geq 1$. The timing of liquidation is determined by the decision of equity holders to cease injecting capital. The authors demonstrate that when the firm experiences financial distress, equity holders may chose to engage is strategic debt service, delivering debt service payments lower than contracted and thereby extracting concessions from bond holders.

The pricing of infinite maturity debt subject to default in the presence of bankruptcy costs and tax benefits to issuing debt is considered by Leland (1994). Bankruptcy on so-called unprotected debt is triggered when equityholders cannot meet the required coupon payment by issuing additional equity. This will be the case when the value of equity falls to zero. Leland imposes a standard smooth pasting condition relating to the value of the equity claim in order to identify the level of firm value at which default will occur:

$$\frac{dE}{dV}\,|_{V=V_B} = 0 \tag{27}$$

The value thus determined will be the lowest possible value subject to the requirement that the value of equity stays positive. Leland subsequently compares the behavior of unprotected debt with that of protected debt, for which default is assumed to occur when firm value falls below the value of the debt's principal, and derives the firm's optimal capital structure for both debt contracts. The infinite maturity assumptions serves to derive closed form solutions and is relaxed in a subsequent paper by Leland and Toft (1996). Here, equityholders have to choose not only the amount but also the maturity of debt to be issued. A condition equivalent to (18) endogenously determines the bankruptcy triggering asset value V_B. The authors show that when $V = V_B$, the appreciation of equity satisfies the following dynamics:

$$dE = ((1-\tau)C + p)\,dt - (d(V_B; V_B, T) + \delta V_B)\,dt \tag{28}$$

The above relates the change in the equity at the bankruptcy triggering asset level to the net cash flow required to be paid from equity holders to bond holders to service the debt. This cash flow consists of the costs of debt service-the after-tax coupon expense plus the principal expense-less the sum of the revenues from selling new bonds and the cash flow generated by the firm's operations. The model generates a number of interesting results. For long term debt, the endogenous bankruptcy triggering asset value will frequently lie below the principal value of debt and the firm will continue to service the debt despite having negative net worth. For short term debt, bankruptcy will occur despite net worth being positive. Both of these results are ultimately driven by the fact that bankruptcy is triggered not because asset value falls below principal value but because the anticipated equity appreciation does not warrant the additional contribution required by equity holders to avoid default on bond service payments. Finally, Anderson and Sundaresan (1996) consider the optimal design and valuation of default risky debt contracts in a bargaining framework. Debt contracts are assumed to experience a fixed loss of value K in the event of default. Equity holders chose a level of debt service. Faced with a given level of debt service, debt holders chose whether or not to liquidate the firm. The authors demonstrate the existence of an equilibrium in which equity holders will service the debt strategically and show that their model generates yield spreads consistent with empirical facts for reasonable parameter values.

The primary merit of the capital structure approach is conceptual: It explicitly links default to an imbalance between assets and liabilities. Recent contributions in this area enrich the economic intuition of this approach by extending the analysis past the time of default to include bargaining scenarios between equity holders and bond holders. Such extensions have shown some promise to improve the poor empirical performance of early implementations of models in this category. However, due to the need to model the firm's complete liability structure and the difficulty in modelling firm value, implementations remain difficult.

2.1.3. The Reduced Form Approach

The requirement to model credit risk without having to specify the complete liability structure of the issuer under consideration has led researchers to explore a more parsimonious class of models: The focus is on a subset of the firm's outstanding claims and there is no explicit role for the value of the firm's assets.

This approach was originated by Jarrow (1993) and Jarrow and Turnbull (1994), who propose a model in which both treasury and credit risky corporate bonds are traded. Forward rates for a default free treasury instrument evolve according to the following process:

$$df_0(t,T) = \alpha_0(t,T)dt + \sigma(t,T)dW_1(t) \tag{29}$$

while the equivalent process for the credit risky discount instrument is specified to be:

$$df_1(t,T) = \alpha_0(t,T)dt + \sigma(t,T)dW_1(t) + \theta(t,T)\left(I_{\{t=\tau_1^*\}} - \lambda_1 I_{\{t<\tau_1^*\}}\right) \tag{30}$$

where τ_1^* is the time of default and λ_1 is the probability of default, assumed to be deterministic. This latter instrument, denominated in a hypothetical "XYZ" currency, is assumed to be risk free: Investors with long positions in this instrument will receive the principal on this security with certainty. The authors stipulate existence of an exchange rate $e_1(t)$ which serves to convert risk free payoffs in "XYZ" currency into credit risky payoffs in dollars:

$$e_1(t) = I_{\{t<\tau_1^*\}} + I_{\{t\geq\tau_1^*\}}\delta_1 \tag{31}$$

where δ_1 is the fixed recovery rate on the corporate bonds. The model allows for default to occur prior to maturity; nevertheless, default is triggered with an exogenously specified probability which bears no link to any concept of fundamental value. The dollar price of the credit risky instrument is then given by the exchange rate adjusted "XYZ" price of the same asset:

$$v_1(t,T) = p_1(t,T)e_1(t) \tag{32}$$

Invoking the analogy to the framework for pricing of foreign exchange contingent claims developed in Amin and Jarrow (1991), the authors derive a pricing measure under which normalized security prices are martingales. Under this measure, the dollar price of the credit risky instrument is given as:

$$\begin{aligned} v_1(t,T) &= B(t) \times \widetilde{E}_t\left[\frac{e_1(T)}{B(T)}\right] \\ &= p_0(t,T) \times \left(e^{-\lambda_1\mu_1(T-t)} + \delta_1(1 - e^{-\lambda_1\mu_1(T-t)})\right) \end{aligned} \tag{33}$$

where $p_0(t,T)$ is the price of the risk free discount bond and $\lambda_1\mu_1$ is the instantaneous probability of default under the pricing measure. The value of this instrument is thus seen to be the discounted value of its default state weighted payoffs. This decomposition is possible due to the assumption that the default process is uncorrelated to interest rates. The above approach has the conceptual merit of facilitating derivation of the spread between risky and risk free bonds under the assumption that the risky bond will be subject to an explicit capital loss with a positive probability. Nevertheless, both the default probability as well as the recovery rate in this model are deterministic. In effect, credit risk is not subject to uncertainty, which limits the appeal of this model.

A variation of the above model is given in Madan and Unal (1996), who consider the case of recovery rates being stochastic, but fail to address issues relating to hedgeability and market completeness. The process of making risk adjustments and explicitly deriving a pricing measure is thus neglected. Assuming existence of a pricing measure, the authors derive the price of the credit risky discount bond as:

$$v(t,T) = B(t) \times E_t^Q \left[\frac{D(T)}{B(T)} \right] \tag{34}$$

where $D(T)$ is a random process denoting the recovery rate. Further models in this category include Duffie et al. (1995) and Duffie and Singleton (1995), who assume existence of a pricing measure without consideration of issues relating to market completeness. While Lando (1994), assuming a stochastic probability of default, derives an expression similar to (33):

$$v(t,T) = \delta p(t,T) + I_{\{\tau > t\}}(1-\delta)E_t \left[\exp\left(-\int_t^T (R(X_s) + \lambda(X_s))ds \right) \right] \tag{35}$$

and notes that due to the expectation, the price of this instrument will potentially be a function of the correlation between interest rates and the probability of default, none of the existing models in this category make explicit allowance for interaction between credit risk and interest rates.

While the default state space in the above specifications is limited to a default and a no default state, researchers have sought to enrich the no default state to include distinct credit ratings (see Lando (1994), Das and Tufano (1996), Jarrow et al. (1994)). Although parameterization of such models will be complex, the suggestion that changes in credit quality prior to the event of default occur in a discontinuous fashion is an interesting one.

2.1.4. The Spread Approach

While both the capital structure approach as well as the reduced form approach make some attempt at specifying the components of credit risk, the spread approach considers that for some purposes specification of the spread between two instruments is sufficient to capture their differing credit characteristics.

An early illustration of the this approach is offered by Ramaswamy and Sundaresan (1986), who argue that the expected return on credit risky seasoned instruments should be identical to the return on newly issued instruments in the same credit risk class:

$$E_t\left[dF\right] + (r(t) + \pi)dt = F\{r(t) + p(t)\} \tag{36}$$

where π is the coupon on the former security, set at the time of issue, $p(t)$ is the market premium on the latter instrument and $r(t)$ is the rate of return on the risk free instrument. Assuming that interest rates and the default premium follow mean reverting square root processes,

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dz \tag{37}$$

$$dp = \kappa_p(\mu_p - p)dt + \sigma_p\sqrt{p}dz_p \tag{38}$$

The authors show that interest and credit contingent claims must satisfy the following partial differential equation

$$\begin{aligned} 0 &= \frac{1}{2}F_{rr}\sigma^2 r + \frac{1}{2}F_{pp}\sigma_p^2 p + F_{rp}\sigma_{rp}^2\sqrt{r}\sqrt{p} \\ &\quad + F_r\kappa(\mu - r) + F_p\kappa_p(\mu_p - p) + (r + \pi) - F(r + p) \end{aligned} \tag{39}$$

subject to the boundary condition $F(r, p, 0; \pi) = 1$, where F is the price of the credit contingent claim. Longstaff and Schwartz (1995) stipulate mean reverting normal specifications for both the credit spread and interest rates:

$$dX = (a - bX)dt + sdZ_1 \tag{40}$$

$$dr = (\alpha - \beta r)dt + \sigma dZ_2 \tag{41}$$

and derive an expression similar to (39):

$$\begin{aligned} 0 &= \frac{s^2}{2}F_{XX} + \frac{\sigma^2}{2}F_{rr} + F_{Xr}\rho\sigma s \\ &\quad + F_r(\alpha - \beta r) + F_X(a - bX) - rF - F_t \end{aligned}$$

The spread approach is less than ambitions in that no attempt is made to provide an articulation of the nature of credit risk. Nevertheless, the approach might prove fruitful in situations in which the credit risk is negligible relative to other risks or information required to employ more elaborate approaches is not available.

Conclusion

Depending on the objectives of the analysis, the merit of distinct modelling approaches might be evaluated by the application of three criteria: The extent to which it reflects economic mechanisms generating and information flows disseminating credit risk, the degree of explicitness and level of complexity to which components of credit risk are modelled and finally the relative ease of implementation and ability of models to generate results consistent with empirical facts. While the capital structure approach has been credited with capturing intuition by explicitly modelling firm value, analysis of individual securities is not possible without consideration of the complete set of the issuer's outstanding liabilities; this requirement makes implementation difficult. The definition of default as the moment at which firm value, which is invariably modelled as a continuous process, reaches a fixed reorganization boundary implies that the transmission of information regarding the credit quality of a particular issuer to investors is free of frictions and the time of default becomes predictable. Because of this feature spreads generated by this model are bound to be at variance with market spreads at short maturities.

The reduced form approach fails to motivate credit risk in terms of firm value insufficient to cover liabilities. The assumption that the default state is entered with an exogenously specified probability captures the intuition that the time of default cannot be foreseen by investors, who effectively face an information barrier regarding the issuer's credit quality. It provides explicit roles for the probability of default and the recovery rate. The importance of addressing specification of these components has been underlined by investor's demands to transfer specific constituents of credit risk, although existing specifications fail to generate sufficient complexity in order to address these requirements.

The spread approach must be credited with providing a useful tool for the purposes of exploring empirical properties of credit risky securities, but might prove too parsimonious for some applications.

2.2. Swaps: A Review of the Issues

Introduction

Swaps are one of the most successful financial innovations of the last decade. The pace of innovation in these markets has been paralleled by increased academic interest in these instruments. The literature has focused on two complementary areas. Considerable attention has been given to issues related to the pricing and hedging of swaps. Valuation of swaps has been based on the argument that the standard swap agreement is equivalent to the exchange of two bonds. A number of authors have recognized that this characterization is incomplete since it neglects the credit risk to which participants in a swap contract are exposed.

A second area of analysis has focused on trying to identify the causes underlying the phenomenal growth of the swaps market. Early theories were based on capital market imperfections and comparative advantages among different borrowers in these markets. These arguments imply that all counterparties should benefit from entering into a swap, a situation the continued existence of which is inconsistent with the assumption of complete and integrated swap markets. Agency costs, informational asymmetries and liquidity considerations have been adduced as possible motivations for investors to enter into swaps, and the relevant arguments will be discussed below.

2.2.1. Swap Pricing

In its most general form, a swap consists of an agreement between two parties to exchange two streams of cash flows; it is the definition of these cash flows which define the particular swap. While a detailed categorization of all possible structures[12] is beyond the scope of the present review, suffice it to say that the majority of arrangements can be classified according to the following characterization. In an interest rate swap, streams based on one interest rate index[13] are exchanged for streams based on another index; both indices as well as cash flows are denominated in a single currency. In a currency swap, flows based on an index in one currency are exchanged for flows based on an index in another currency, the respective payments being made in the two distinct currencies.

[12] For examples of possible variations, see for Turnbull (1993), Babbs (1994), Jamshidian (1993) and Wei (1994)).

[13] Typical indices include Libor, T-bill rates, commercail paper rates, federal funds rates and prime rates (see Fabozzi (1991)).

2.2.1.1. Pricing of default-free Swaps

In the absence of credit risk, a standard interest rate swap can be characterized as an exchange of two bonds. The party agreeing to make fixed payments exchanges a fixed coupon bond for a floating rate note issued by the party having contracted to make floating payments. Given that coupon and maturity dates on the bonds correspond to the contractual terms of the swap, the cash flows from bonds replicate the cash flows from the swap agreement. In the absence of credit risk and barring non-standard reset provisions, the value of the swap to the floating rate payer is equal to the difference between the coupon bond and the floating rate note. The equilibrium swap rate denotes the coupon on the fixed coupon bond which results in the value of the swap having zero market value.

This approach to valuing swaps is consistent with industry practice (e.g. Henderson and Price (1988) or Miron and Swanell (1991)): A zero coupon swap term structure is obtained by interpolation from quoted swap spreads, Eurodollar futures and recently issued government securities. This zero curve is used to value the coupon bond and the floating rate instrument, assumed to sell at par at the payment dates; the value of the swap is the difference between these two instruments. This approach values the cash flows under the contract as if they are free of credit risk , using a zero coupon curve which includes a spread over the government yield.

2.2.1.2. Pricing of default-risky Swaps

In quantifying and pricing credit risk in swaps, researchers have generally relied on models originally developed to price credit risky debt instruments.

Wealth transfers and swap spreads in the presence of default risk are analyzed by Cooper and Mello (1991) within a capital structure framework. They stipulate existence of a credit risky counterparty whose firm value evolves according to

$$dV = (\alpha V - C)dt + \sigma_V V dz_V \tag{42}$$

where α is the instantaneous expected rate of return on the firm's assets, C represents the payouts per unit of time and σ_V is the instantaneous volatility of the firm's assets. Before entering into a swap, the firm has raised funds of an amount B in the variable debt market by issuing a floating rate note with face value F; the remainder of the firm is equity financed. Swaps are assumed to be subordinated to debt; in the event of default the net value of the swap will be paid to the counterparty whose net position is positive, regardless of the identity of the defaulting party. Assuming that interest rates follow a lognormal specification

$$dr = m(\mu - r)dt + \sigma_r r dz_r \tag{43}$$

the authors show that a coupon bond issued by the credit risky counterparty must satisfy

$$\begin{aligned} 0 &= (rV - C)B_V + m(\mu - r)B_r - rB + F_t \\ &\quad + C + \frac{1}{2}(\sigma_V^2 V^2 B_{VV} + 2\rho\sigma_V\sigma_r r V B_{Vr} + \sigma_r^2 r^2 B_{rr}) \end{aligned} \tag{44}$$

subject to the boundary condition

$$B(V, r, T, C) = (F, V)^- \tag{45}$$

It is assumed that default is triggered if the value of the firm at any of the coupon dates falls below the coupon level. The authors show that a risk free coupon bond must satisfy the following partial differential equation:

$$m(\mu - r)G_r + \frac{\sigma_r^2 r^2 G_{rr}}{2} - rG + G_t + C = 0 \tag{46}$$

subject to the condition that at maturity, $G(r, T) = F$. The par spread S_X for credit risky floating rate bonds is defined to be the fixed mark up over the index rate r that equates the value of the instrument with its face value: $B(V, r, t, (r + S_X)F) = F$. The par spread S_F for credit risky coupon bonds is defined to be the fixed mark up over the coupon g that equates the value of the instrument with its face value: $B(V, r, t, (g + S_F)F) = F$. Once the firm enters into a swap, its net liability is to pay a fixed coupon of $g + S_X$, to be divided between the bondholders and the counterparty, assumed to be free of credit risk. Ex post, shareholders will be worse off if the firm ends up being a net payers, while shareholders will gain if the firm happens to be a net receiver. Ex ante, since the present value of the floating rate bond and the fixed component of the swap are identical, shareholders are indifferent as to whether the firm enters into the swap. Bondholders, on the other hand, gain because the cash flow to service the debt will be increased when the firm is due to receive payments under the swap. Therefore, whereas the unswapped floating rate note will sell at par, the equivalent instrument in combination with a swap will be worth more than par. Cooper and Mello demonstrate that the price of the swapped note must satisfy the following partial differential equation:

$$\begin{aligned} 0 &= B_V(rV - (g + S_F)F) + m(\mu - r)B_r - rB + F_t \\ &\quad + (r + S_X)F + \frac{1}{2}(\sigma_V^2 V^2 B_{VV} + 2\rho\sigma_V\sigma_r r V B_{Vr} + \sigma_r^2 r^2 B_{rr}) \end{aligned} \tag{47}$$

Default will occur when the firm is unable to make the promised payment of $(r+S_X)F$ to the bondholders. However, cash flow to service the debt is available from two sources: The firm's own assets as well as payments under the swap, if positive. This is reflected in the boundary condition for the swapped floating rate note. If the firm is receiving a swap payment, $r + S_X \leq g + S_F$, default occurs if its assets and the swap payment are insufficient to cover the coupon payment:

$$B(V, r, t, r + S_X) = (V, (r + S_X)F)^- \tag{48}$$

If the firm is due to make a swap payment, $r + S_X > g + S_F$, it will default if its assets alone do not cover the coupon payment on the debt:

$$B(V, r, t, r + S_X) = (V + (r + S_X - g - S_F)F, (r + S_X)F)^- \tag{49}$$

However, this equal value swap will not be an equilibrium transaction: Since shareholders will be indifferent towards the transaction and bondholders gain from entering into the swap, the contract must have negative value to the risk free counterparty. Consequently, the firm must increase the fixed payment $g + S_F$ it offers in return for the variable payment $r + S_X$ by a strictly positive amount S'_S and the equilibrium fixed swap rate is given by

$$c_s = g + S_F + S'_S \tag{50}$$

The equilibrium swap spread is defined as the fixed rate that must be added to the risk free fixed rate in exchange for a risk free variable rate r:

$$S_S = c_s - g - S_X \tag{51}$$

Using (50), the swap spread can be expressed as

$$S_S = S_F - S_X + S'_S \tag{52}$$

Thus, the swap spread is composed of the difference between the fixed and floating debt market spreads and the specific swap spread; this latter component is added to the debt market spread differential to compensate the risk free counterparty for the fact that, in the case that it is due to make payments under the agreement, it will not be able to withhold payments if the credit risky counterparty has defaulted.

A similar analysis is provided by Abken (1993), who examines a swap contract between a risk free swap dealer and a credit risky counterparty. Default is triggered if the counterparty is unable to meet its obligations under the swap. It is assumed that interest rates and firm value of the credit risky counterparty evolve according to the following specifications,

$$dr_0 = (\alpha - \kappa r_0)dt + \sigma_r \sqrt{r_0}dz_1 \tag{53}$$

$$dV = \left[\mu_V V - \sum_{i=1}^{n} CF(r_0)_i \delta(t - t_i) \right] dt + \sigma_V r_0 V dz_2 \tag{54}$$

where the drift in (54) contains an adjustment due to the payments under the swap contract. If the credit risky counterparty has contracted to pay floating rates, its firm value is liable to drop by

$$CF(r_0)_i = (r_{t_i} - r_0)^+ \tag{55}$$

In case this counterparty has contracted to receive floating rates, firm value will decrease by

$$CF(r_0)_i = (r_0 - r_{t_i})^+ \tag{56}$$

Decomposing the counterparties' positions under the swap into positions in caps and floor, Abken notes that the component owned by the risk free counterparty will only pay off if the value of the assets of the credit risky counterparty are sufficient to make the contractual payment. If the credit risky counterparty has contracted to pay floating rates, the boundary condition for the cap the risk free counterparty is holding is given by:

$$C(r_T, V, 0, \hat{r}) = \left(V(r_T), \left(r_T - \hat{r} \right)^+ \right)^- \tag{57}$$

If the credit risky counterparty has contracted to pay fixed rates, the equivalent condition for the floor is given by:

$$F(r_T, V, 0, \hat{r}) = \left(V(r_T), \left(\hat{r} - r_T \right)^+ \right)^- \tag{58}$$

Abken uses the yield option model proposed by Longstaff to identify equilibrium swap rates in this framework by finding the fixed rate which equates the value of the respective cap and floor, adjusted for the credit risk of the risky counterparty.

An application of the reduced form model to swaps is given by Duffie and Huang (1996), who demonstrate that credit risk can be incorporated into the pricing of swaps by discounting payments under the contract at the risk free rate and the credit spreads of the respective counterparty.

The spread approach to credit risk in swaps is chosen by Babbs (1991), who constructs a model for the joint evolution of the risk free term structure and the term structure of swap rates. The rate of return on risk free assets is assumed to evolve according to

$$\frac{dB_g(M,t)}{B_g(M,t)} = \{r_g(t) + s(M,t) + \theta_g(t)\sigma_g(M,t)\}dt + \sigma_g(M,t)dZ_g(t) \tag{59}$$

while returns on the credit risky asset from which Libor rates are specified by the following process:

$$\frac{dB_p(M,t)}{B_p(M,t)} = \{r_p(t) + \theta_p(t)\sigma_p(M,t)\}dt + \sigma_p(M,t)dZ_p(t) \tag{60}$$

Expected returns on these two assets differ by two components: $s(M,t)$ captures the maturity dependent return differential between the two instruments, while the spread between the instantaneous spot rates $r_g(t)$ and $r_p(t)$ reflects the maturity independent component. Due to the possibility of default, investors require a positive yield spread on the credit risky instrument and Babbs demonstrates that this differential must satisfy

$$D(M,t) = r_p(t) - r_g(t) - s(M,t) \tag{61}$$

if arbitrage is to be precluded. The model formalizes the idea that the two term structures should differ by a deterministic but possibly maturity dependent spread. This is reasonable in light of the fact that swaps are contracts contingent on future interbank borrowing rates; such rates do not correspond to future borrowing rates of specific firms but refer to rates obtainable by members of a pool of firms, the constituency of which is continuously updated with a view of preserving a given credit quality.

2.2.2. Swap Theories

2.2.2.1. Market Imperfections

Market imperfections and the existence of quality spread differentials[14] were the first arguments advanced to explain the existence of the swap market. Bicksler and Chen (1986) and Henderson and Price (1988) argue that high credit quality borrowers have a comparative cost advantage in the fixed rate markets, while companies of lower credit quality might have a comparative advantage in the floating rate markets. Both firms can issue bonds in the market in which they hold a favorable borrowing position and subsequently enter into a swap. Net payments under this combined transactions will be lower than net payments under a similar borrowing strategy which does not

[14] A quality spread is the difference between interest rates at which issuers of distinct credit quality can borrow.

make use of the swap; the net reduction in combined borrowing costs can be shared by the counterparties.

In the absence of market frictions a situation in which all participants gain from a given transaction under all possible scenarios seems inconsistent with equilibrium, and other authors have sought to identify the apparent inconsistency in the above argument. Cooper and Mello (1991) demonstrate that if the swap agreement is valued as an exchange of two debt instruments and the swap rate is set to equate the market value of these securities, wealth transfers from a high credit quality counterparty to the debtholders of a low credit quality counterparty can occur. This result underlines that in the absence of externalities, not all parties can gain from entering into a swap. The equilibrium swap rate at which wealth transfers are precluded is shown to be higher than the rate at which the value of the two debt instruments is equated. Rendleman (1992) argues that while swap transactions can reduce promised interest payments for both counterparties, they cannot reduce expected interest savings unless, due to market imperfections or other externalities, swaps create real benefits that would not accrue under conventional borrowing arrangements.

2.2.2.2. Agency Cost Explanations

Given that swaps should be a zero sum game in the absence of market imperfections, various authors have sought to identify externalities specific to swap arrangements.

According to Rendleman (1992), companies will have incentives to reduce agency costs associated with financial distress by hedging their interest rate exposure. While the issuance of short term debt resolves the agency problem, it leaves the firm exposed to interest rate risk. Callable bonds set a ceiling on the firms borrowing costs and allow the firm to exploit a drop in market rates; however, the firm must pay for the option to call the bonds through the payment of a higher coupon rate. The combination of short term borrowing and swaps, on the other hand, eliminates both the interest rate exposure as well as the agency costs.

2.2.2.3. Asymmetric Information and Incomplete Markets

An alternative motivation for the existence of externalities to counterparties entering into a swap is given by Arak et al. (1988), who argue that heterogeneous expectations and asymmetric information between lenders and credit risky borrowers regarding the latter's future credit quality could provide a preference for swap financing relative to the issue of straight bonds. These authors decompose the credit risky firm's borrowing costs into four components: The risk free rate, the risk premium associated with the future uncertainty of that rate, the credit premium attached the specific borrower and a risk premium associated with the uncertainty of this credit premium. For fixed rate debt, the total cost of borrowing per unit principal is given by

$$K_L = \prod_{i=1}^{n} \left(1 + r_{i,M} + p_{i,M}^{r} + c_{i,M} + p_{i,M}^{c}\right) \tag{62}$$

where the subscript M refers to the fact that in the case of fixed rate debt, all components are fixed over the life of the contract, based on the market's expectation at the time the contract is initiated. For floating rate debt, the total borrowing cost is given by:

$$K_v = \prod_{i=1}^{n} \left(1 + r_{i,B} + p_{i,B}^{r} + c_{i,M} + p_{i,M}^{c}\right) \tag{63}$$

The risk free rate components of this cost is based on the borrower's expectation, whereas the credit components are fixed over the life of the contract on the basis of the market's assessment: Floating rate debt fixes the credit component while allowing the rate component to fluctuate. Finally, the cost of rolling over short term debt is entirely based on the borrower's assessment of all cost components and thus allows both the credit and rate components to float:

$$K_S = \prod_{i=1}^{n} \left(1 + r_{i,B} + p_{i,B}^{r} + c_{i,B} + p_{i,B}^{c}\right) \tag{64}$$

The authors subsequently consider a strategy of borrowing in the short term market and entering into an agreement to swap floating for fixed payments. The borrower would expect to receive a floating rate of $k_{i,v} = r_{i,B} + p_{i,B}^{r}$ per period, pay a fixed rate of $k_{i,f} = r_{i,M} + p_{i,M}^{r}$ and face an expected cost of $k_{i,s} = r_{i,B} + p_{i,B}^{r} + c_{i,B} + p_{i,B}^{c}$ on the short term borrowing for a combined cost of

$$k_{i,Swap} = \prod_{i=1}^{n} \left(r_{i,M} + p_{i,M}^{r} + c_{i,B} + p_{i,B}^{c}\right) \tag{65}$$

Through this strategy, the borrower has replaced the floating risk free rate component $r_{i,B}+p_{i,B}^{r}$ with the fixed risk free rate component $r_{i,M}+p_{i,M}^{r}$; the net borrowing costs is thus given by the risk free rate components, set according to market expectations, and variable credit components, based on the borrowers own assessment. Borrowers will prefer synthetic floating rate borrowing using the swap to other borrowing strategies if the net costs of this strategy are lower than under alternative borrowing arrangements. From (62), (63), (64) and (65), this will be the case if any of the following restrictions are satisfied:

$$r_{i,M} + p^r_{i,M} + c_{i,B} + p^c_{i,B} < r_{i,M} + p^r_{i,M} + c_{i,M} + p^c_{i,M} \tag{66}$$

$$r_{i,M} + p^r_{i,M} + c_{i,B} + p^c_{i,B} < r_{i,B} + p^r_{i,B} + c_{i,M} + p^c_{i,M} \tag{67}$$

$$r_{i,M} + p^r_{i,M} + c_{i,B} + p^c_{i,B} < r_{i,B} + p^r_{i,B} + c_{i,B} + p^c_{i,B} \tag{68}$$

Consequently, the swap strategy enables the borrower to fully hedge his interest exposure while leaving the credit component of his borrowing costs unhedged, allowing him to take advantage of private information concerning his future credit quality. This will be the case if at least one of the following conditions hold:

- the borrower's expectations of future risk free rates are higher than the market's.
- the borrower is more risk averse than the market with respect to risk free rates.
- the borrower's expectations of its future credit quality is higher than the market's.
- the borrower is less risk averse than the market with respect to his own credit quality.

A more elaborate approach to motivating swaps on the basis of asymmetric information is taken by Titman (1992), who argues that in the absence of a swap market, the presence of financial distress costs arising from interest rate exposure creates a dilemma for firms of high credit quality: A strategy of long term borrowing forces them to pool with issuers of lower credit quality, requiring them to pay coupons which are high relative to their own credit quality, while short term borrowing subjects the firm to interest rate exposure and thus enhances the likelihood of financial distress. Titman demonstrates that the introduction of a swap market facilitates construction of borrowing strategies permitting low risk firms to both hedge their interest rate exposure as well as obtain better financing terms by permitting them to signal their credit quality.

2.2.2.4. Swaps and Financial Intermediation

A distinct line of enquiry has aimed to rationalize swap activity on the basis of financial intermediaries' comparative advantage in managing financial assets. Campbell and Kracaw (1991) argue that financial intermediaries face lower transaction costs and realize returns to scale in hedging interest rate risk. The fact that a significant fraction of swaps are contracted between endusers and intermediaries is adduced as a manifestation of those advantages.

2.2.2.5. Liquidity Motivations

Some authors dismiss credit risk as a dominating factor in determining swap spreads. Grinblatt (1991) argues that there is a liquidity advantage to holding treasury securities relative to holding Libor or corporate assets due to the ability to execute collateralized borrowing strategies in the repo market at very favorable rates. This idea is modelled as being equivalent to investors receiving a positive convenience yield on long treasury positions. Grinblatt stipulates existence of two mean reverting correlated state variables:

$$dr = \kappa(r^* - r)dt + \sigma_r dz \tag{69}$$

$$dx = \theta(x^* - x)dt + \sigma_x dw \tag{70}$$

$$dw = \rho dz + \sqrt{1-\rho^2}du \tag{71}$$

The convenience yield $y(t)$ is assumed to be a linear combination of the two state variables:

$$y(t) = \beta r(t) + x(t) \tag{72}$$

Grinblatt considers a strategy of taking long positions in an interest rate swap, receiving fixed and paying floating rates, and a Libor deposit which is being rolled over at the payment dates of the swap; the price L of this strategy must satisfy the following partial differential equation:

$$0 = L_t + L_r(\kappa r^* - \lambda - \kappa r) + \frac{\sigma_r^2}{2}L_{rr} - rL \tag{73}$$

The price G of the treasury security is shown to be constrained by the following equation:

$$\begin{aligned} 0 \;=\; & G_t + G_r(\kappa r^* - \lambda - \kappa r) + G_x(\theta x^* - \mu - \theta x) \\ & + \frac{1}{2}\left(\sigma_r^2 G_{rr} + 2\rho\sigma_r\sigma_x G_{rx} + \sigma_x^2 G_{xx}\right) + (r-y)G \end{aligned} \tag{74}$$

Grinblatt argues that the present value of the swap spread cash flows is equivalent to the liquidity benefit on the treasury instrument, the value of which he shows to be:

$$PV_{LB} = \tilde{E}_t\left[\int_0^T y(t)\exp\left(-\int_0^t r(s)ds\right)dt\right] \tag{75}$$

While categorical exclusion of default risk from swap spreads might be premature, introduction of spread components unrelated to credit risk is warranted.

Conclusion

The literature on swaps has evolved considerably since academics began to take interest in these issues. With regard to credit risk, applications of approaches from the credit risky debt literature has met with some success but remains subject to the shortcomings of those models as discussed above. One area which has received considerable attention by practitioners is the development of credit risk mitigation techniques, a subject which so far has generated only limited formal analysis among academics. With regard to externalities, I believe that the efficiencies afforded to endusers by the presence of intermediaries in conjunction with the flexibility of the instrument itself provide the most convincing rationale for the existence of the swap market.

References

Abken, P. (1993): "Valuation of default-risky interest rate swaps," *Advances in Futures and Options Research*, 6, 93-116.

Ahn, C. and H. Thompson (1988): "Jump-diffusion processes and the term structure of interest rates," *The Journal of Finance*, 43/1, 155-174.

Anderson, R. and S. Sundaresan (1996): "Design and Valuation of Debt Contracts," *Review of Financial Studies*, 9, 37-68.

Amin, K. and R. Jarrow (1991): "Pricing foreign currency options under stochastic interest rates," *Journal of International Money and Finance*, 10, 310-329.

Arak, M., A. Estrella, L. Goodman and A. Silver (1988): "Interest rate swaps: An alternative explanation," *Financial Management* (Summer), 12-18.

Babbs, S. (1991): "Interest rate swaps and default-free bonds: A joint term structure model," *Working Paper.*

Babbs, S. (1994): "The valuation of cross currency interest-sensitive claims, with applications to diff-swaps," *Working Paper.*

Barclay, M. and C. Smith (1995a): "The maturity structure of corporate debt," *Journal of Finance*, 609-631.

Barclay, M. and C. Smith (1995b): "The priority structure of corporate debt," *Journal of Finance*, 899-917.

Bennet, P. (1984): "Applying portfolio theory to global bank lending," *Journal of Banking and Finance*, 153-169.

Bicksler, J. and A. Chen (1986): "An economic analysis of interest rate swaps," *The Journal of Finance*, 41, 645-656.

Black, F. (1976): "The Dividend Puzzle," *Journal of Portfolio Management* (Winter), 2, 5-8.

Black, F. (1976): "The pricing of commodity contracts," *Journal of Financial Economics*, 3, 167-179.

Black, F. and J. Cox (1976): "Valuing corporate securities: Some effects of bond indenture provisions," *The Journal of Finance*, 31/2, 351-367.

Black, F. and E. Derman and W. Toy (1990): "A one-factor model of interest rates and its applications to treasury bond options," *Financial Analysts Journal* (Jan-Feb), 33-39.

Black, F. and M. Scholes (1973): "The pricing of options and corporate liabilities," *Journal of Political Economy*, 81, 637-659.

Brace, A. and M. Musiela (1994): "A multifactor Gauss-Markov implementation of Heath, Jarrow, and Morton," *Mathematical Finance*, 4/3, 259-283.

Breeden, D. (1979): "An intertemporal asset pricing model with stochastic consumption and investment opportunities," *Journal of Financial Economics*, 7, 265-296.

Brennan, M. and E. Schwartz (1982): "An equilibrium model of bond pricing and a test of market efficiency," *Journal of Financial and Quantitative Analysis*, 17/3, 301-329.

Brenner, R. and R. Jarrow (1993): "A simple formula for options on discount bonds," *Advances in Options and Futures Research*, 6, 45-51.

Brickley, J. (1983): "Shareholder wealth, information signalling and specially designated dividend: An empirical study," *Journal of Financial Economics* (August), 12/2, 187-209.

Brown, K., W. Harlow and D. Smith (1994): "An empirical analysis of interest rate swap spreads," *The Journal of Fixed Income* (March), 61-78.

Brown, R. and S. Schaefer (1994): "Interest rate volatility and the shape of the term structure," *Phil. Trans. R. Soc. Lond.*, 347, 563-576.

Brown, K. and D. Smith (1993): "Default risk and innovations in the design of interest rate swaps," *Financial Management* (Summer), 94-105.

Campbell, T. and W. Kracaw (1991): "Intermediation and the market for interest rate swaps," *Journal of Financial Intermediation*, 1. 362-384.

Chance, D. (1990): "Default risk and the duration of zero coupon bonds," *The Journal of Finance*, 45/1, 265-274.

Cooper, I. and M. Martin (1996): "Default risk and derivative products", *Applied Mathematical Finance*, 3, 53-74.

Cooper, I. and A. Mello (1991): "The default risk of swaps," *The Journal of Finance*, 46/2, 597-620.

Cox, J., J. Ingersoll and S. Ross (1979): "Duration and the measurement of basis risk," *The Journal of Business*, 52, 51-62.

Cox, J., J. Ingersoll and S. Ross (1981a): "A re-examination of the traditional hypothesis of the term structure of interest rates," *The Journal of Finance*, 36, 769-799.

Cox, J., J. Ingersoll and S. Ross (1981b): "The relation between forward prices and futures prices," *Journal of Financial Economics*, 9, 321-346.

Cox, J., J. Ingersoll and S. Ross (1985a): "A theory of the term structure of interest rates," *Econometrica*, 53, 385-467.

Cox, J., J. Ingersoll and S. Ross (1985b): "An intertemporal general equilibrium model of asset prices," *Econometrica*, 53, 363-467.

Cox, J. and M. Rubinstein (1985): *Options markets* Prentice Hall.

Das, S. and P. Tufano (1996): "Pricing credit-sensitive debt when interest rates, credit ratings and credit spreads are stochastic," *The Journal of Financial Engineering*, 5/2, 161-198.

Dann, L. (1982): "Common stock repurchases: An analysis of returns to bondholders and stockholders," *Journal of Financial Economics* (June), 9/2, 113-138.

Dothan, U. (1978): "On the term structure of interest rates," *Journal of Financial Economics*, 6, 59-69.

Duffie, D. (1992): *Dynamic Asset Pricing Theory* Princeton University Press.

Duffie, D. and M. Huang (1996): "Swap rates and credit quality," *The Journal of Finance*, 51/3, 921-949.

Duffie, D. and R. Kan (1996): "A yield factor model of interest rates," *Mathematical Finance* (October), 6/4, 379-406.

Duffie, D. and K. Singleton (1995): "Modelling term structures of defaultable bonds," *Working Paper.*

Duffie, D., M. Schroder and C. Skiades (1995): "Recursive valuation of defaultable securities and the timing of resolution of uncertainty," *Working Paper.*

Dybvig, P., J. Ingersoll and S. Ross (1996): "Long forward and zero-coupon rates can never fall," *Journal of Business*, 69/1, 1-25.

Easterbrook, F. (1984): "Two agency cost explanations of dividends," *American Economic Review* (September), 74/4, 650-659.

Easterbrook, F. (1990): "Is corporate bankruptcy efficient ?," *Journal of Financial Economics*, 27, 411-417.

Fabozzi, F. (1991): *Handbook of Fixed Income Securities*, Irwin

Fama, E. (1970): "Efficient capital markets: A review of theory and empirical work," *The Journal of Finance*, 25, 383-417.

Fama, E. (1978): "The effect of a firm's investment and financing decisions on the welfare of its security holders," *American Economic Review*, 68/3, 248-272.

Fama, E. (1991): "Efficient capital markets: II," *The Journal of Finance*, 46/5, 1575-1617.

Flesaker, B. and L. Hughston (1996): "Positive interest," *Risk*, 9/1.

Fons, J. and L. Carty (1995): "Probability of default: A derivatives perspective," in: *Derivative Credit Risk*, Risk Publications.

Franks, J. and W. Torous (1989): "An empirical investigation of U.S. firms in reorganization," *The Journal of Finance*, 44, 747-769.

Franks, J. and W. Torous (1994): "A comparison of financial recontracting in distressed exchanges and Chapter 11 reorganization," *Journal of Financial Economics*, 35, 349-370.

Gertner, R. and D. Scharfstein (1991): "A theory of workouts and the effects of reorganization law," *The Journal of Finance*, 46/4, 1189-1222.

Geske, R. (1977): " The valuation of corporate liabilities as compound options," *Journal of Financial and Quantitative Analysis*, 541-552.

Giberti, D., M. Mentini and P. Scabellone (1993): "The valuation of credit risk in swaps: Methodological issues and empirical results," *The Journal of Fixed Income* (March), 24-36.

Goodman, L. (1990): "The use of interest rate swaps in managing corporate liabilities," *Journal of Applied Corporate Finance* (Winter), 35-47.

Grinblatt, M. (1993): "An analytic solution for interest rate swap spreads," *Working Paper.*

Heath, D., R. Jarrow and A. Morton (1990): "Bond pricing and the term structure of interest rates: A discrete time approximation," *Journal of Financial and Quantitative Analysis*, 25/4, 419–440.

Heath, D., R. Jarrow and A. Morton (1992): "Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation," *Econometrica*, 60, 77 105.

Henderson, S. and J. Price (1988): *Currency and Interest Rate Swaps* Butterworths.

Ho, T. and S. Lee (1986): "Term structure movements and the pricing of interest rate claims," *The Journal of Finance*, 41, 1011-1029.

Hogan, M. (1993): "Problems in certain two-factor term structure models," *The Annals of Applied Probability*, 3/2, 576-581.

Hull, J. and A. White (1990): "Pricing interest rate derivative securities," *The Review of Financial Studies*, 3/4, 573-592.

Hull, J. and A. White (1993): "Bond option pricing based on a model for the evolution of bond prices," *Advances in Futures and Options Research*, 6, 1-13.

Jamshidian, F. (1989): "An exact bond option formula," *The Journal of Finance*, 44, 205-209.

Jamshidian, F. (1990): "The preference free determination of bond and option prices from the spot interest rate," *Advances in Futures and Options Research*, 4, 51-67.

Jamshidian, F. (1994): "Hedging quantos, differential swaps and ratios," *Applied Mathematical Finance*, 1, 1-20.

Jamshidian, F. (1995): "A simple class of square root interest rate models," *Applied Mathematical Finance*, 2, 61-72.

Jamshidian, F. (1996): "Bond, futures and option evaluation in the quadratic interest rate model," *Applied Mathematical Finance*, 3, 93-115.

Jarrow, R. (1993): "An integrated approach to hedging and pricing Eurodollar derivatives," *Working Paper.*

Jarrow, R. (1996): *Modelling fixed income securities and interest rate options* McGraw-Hill.

Jarrow, R. and S. Turnbull (1995): "Pricing derivatives on financial securities subject to credit risk," *The Journal of Finance*, 50/1, 53-85.

Jarrow, R., D. Lando and S. Turnbull (1994): "A Markov model for the term structure of credit risk spreads," *Working Paper.*

Jensen, M. (1986): "Agency cost of free cash flow, corporate finance, and takeovers," *American Economic Review* (May), 76/2, 323-329.

Jensen, M. and W. Meckling (1976): "Theory of the firm: Managerial behaviour, Agency costs and ownership structure," *Journal of Financial Economics*, 3, 305-360.

Johnson, H. (1987): "Options on the maximum or the minimum of several assets," *Journal of Financial and Quantitative Analysis*, 22/3, 277-283.

Johnson, H. and R. Stulz (1987): "The pricing of options with default risk," *The Journal of Finance*, 42/2, 267-280.

Kalay, A. (1982): "Stockholder-bondholder conflict and dividend constraints", *Journal of Financial Economics*, 10/2, 211-233.

Kealhofer, S. (1995): "Managing default risk in portfolios of derivatives", in: *Derivative Credit Risk*, Risk Publications.

Kim, E., K.Ramaswamy and S. Sundaresan (1993): "Does default risk in coupons affect the valuation of corporate bonds?," *Financial Management* (Autumn), 117-131.

Lando, D. (1994): "Three essays on contingent claims pricing," *PhD thesis*, Cornell University.

Leland, H. (1994): "Risky debt, bond covenants and optimal capital structure," *The Journal of Finance*, 49/4, 1213-1252.

Leland, H. and K. Toft (1996): "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads," *The Journal of Finance*, 51/3, 987-1019.

Litzenberger, R. (1992): "Swaps: Plain and fanciful," *The Journal of Finance*, 47/3, 831-850.

Litzenberger, R. and C. Huang (1988): *Foundations for Financial Economics* North-Holland.

Longstaff, F. (1990): "The Valuation of options on yields," *Journal of Financial Economics*, 26, 97-121.

Longstaff, F. and E. Schwartz (1992): "Interest rate volatility and the term structure: A two-factor general equilibrium model," *The Journal of Finance*, 47/4, 1259-1281.

Longstaff, F. and E. Schwartz (1995): "A simple approach to valuing risky fixed and floating rate debt," *The Journal of Finance*, 50/3, 789-819.

Longstaff, F. and E. Schwartz (1995): "Valuing credit derivatives," *The Journal of Fixed Income* (June), 6-12.

Madan, D. and H. Unal (1996): "Pricing the risks of default," *Working Paper.*

Maghsood, Y. (1996): "Solution of the extended CIR term strucutre and bond option valuation", *Mathematical Finance*, 6/1, 89-109.

Margrabe, W. (1978): "The option to exchange one asset for another," *Journal of Finance*, 33/1, 177-186.

Masulis, R. (1980): "The effects of capital structure change on security prices: A study of exchange offers," *Journal of Financial Economics* (June), 8/2, 139-177.

Mella-Barral, P. (1996): "The dynamics of corporate debt forgiveness and contract renegotiation," *Working Paper.*

Mella-Barral, P. and W. Perraudin (1996): "Strategic debt service," *Working Paper.*

Mella-Barral, P. and P. Tychon (1996): "Default risk and asset pricing," *Working Paper.*

Merton, R. (1973): "Theory of rational option pricing," *Bell Journal of Economics and Management Science*, 4, 141-183.

Merton, R. (1974): "On the pricing of corporate debt: The risk structure of interest rates," *The Journal of Finance*, 29.

Merton, R. (1977): "On the pricing of contingent claims and the Modigliani-Miller theorem," *Journal of Financial Economics*, 5, 241-249.

Merton, R. (1990): *Continuous Time Finance* Blackwell.

Milgrom, L. and P. Glosten (1985): "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," *Journal of Financial Economics*, 14, 71-100.

Miller, M. (1977): "Debt and taxes," *The Journal of Finance*, 32/2, 261-275.

Miron, P. and P. Swannell (1991): *Pricing and Hedging Swaps* Euromoney Publications.

Musiela, M. (1993): "Stochastic PDE's and term structure models," *Working Paper.*

Musiela, M. and K. Sondermann (1993): "Different dynamical specifications of the term structure of interest rates and their implications," *Working Paper.*

Myers, S. (1977): "Determinants of corporate borrowing", *Journal of Financial Economics*, 5, 147-175.

Myers, S. and N. Majluf (1984): "Corporate financing and investment decisions when firms have information that investors do not have", *Journal of Financial Economics*, 13, 187-221.

Nielsen, L., J. Saa-Requejo and P. Santa-Clara (1993): "Default risk and interest rate risk: The term structure of default spreads," *Working Paper.*

Nelson, L. (1997): "Modelling and pricing credit risk," *Financial Stability Review* (Spring), 2, 11-17.

Ramaswamy, K. and S. Sundaresan (1986): "Valuation of floating rate instruments," *Journal of Financial Economics*, 17, 251-272.

Rendleman, R. (1992): "How risks are shared in interest rate swaps," *Journal of Financial Services Research*, 5-34.

Richard, S. (1978): "An arbitrage model of the term structure of interest rates," *Journal of Financial Economics*, 6, 33-57.

Schaefer, S. (1985): "Immunisation and duration: A review of theory, performance and applications," *Midland Corporate Finance Journal* (Fall), 2/2, 41-59.

Schwartz, A. (1993): "Bankruptcy workouts and debt contracts," *Journal of Law and Economics* (April), 36, 595-632.

Shimko, D., N. Tejima and D. van Deventer (1993): "The pricing of risky debt when interest rates are stochastic", *Journal of Fixed Income* (September), 58-65.

Smith, C., C. Smithson and L. Wakeman (1988): "The market for interest rate swaps,"*Financial Management* (Winter), 34-44.

Smith, C. and J. Warner (1979): "On financial contracting," *Journal of Financial Economics*, 7, 117-161.

Solnik, B. (1990): "Swap pricing and default risk," *Journal of International Financial Management and Accounting*, 2/1, 79-91.

Sun, T., S. Sundaresan and C. Wang (1995): "Interest rate swaps: An empirical investigation," *Journal of Financial Economics*, 34, 77-99.

Sundaresan, S. (1991): "Futures prices on yields, forward prices, and implied forward prices from term structure," *Journal of Financial and Quantitative Analysis*, 26/3, 412-426.

Titman, S. (1984): "The effect of capital structure on a firm's liquidation decision," *Journal of Financial Economics*, 137-151.

Titman, S. (1992): "Interest rate swaps and corporate financing choices," *The Journal of Finance*, 47/4, 1503-1516.

Turnbull, S. (1987): "Swaps: A zero sum game ?," *Financial Management*, 15, 15-21.

Turnbull, S. (1993): "Pricing and hedging diff-swaps," *Working Paper.*

Vasicek, O. (1977): "An equilibrium characterization of the term structure," *Journal of Financial Economics*, 5, 177-188.

Wakeman, L. (1981): "The real function of bond rating agencies", *Chase Financial Quarterly* (Fall), 1/1, 18-26.

Wall, L. (1989): "Interest rate swaps in an agency theoretic model with uncertain interest rates," *Journal of Banking and Finance*, 13, 261-270.

Warner, J. (1977): "Bankruptcy costs: Some evidence", *Journal of Finance* (May), 32/2, 18-26.

Wei, J. (1994): "Valuing differential swaps," *The Journal of Derivatives* (Spring), 64-76.

Weiss, L. (1990): "Bankruptcy resolution: Direct costs and violation of priority of claims," *Journal of Financial Economics*, 27, 285-314.

Wruck, K. (1990): "Financial distress, reorganization and organizational efficiency," *Journal of Financial Economics*, 27, 419-444.

Chapter 3: A No-Arbitrage Framework for the Pricing of Credit Contingent Claims

Introduction

The objective of the current chapter is to elaborate a framework for the pricing of credit contingent claims. The modelling approach chosen here is consistent with the reduced form approach in that explicit specifications for the probability of default and a recovery rate process are made. In other models specification of these components was sufficient to comprehensively characterize credit risk. I demonstrate that this is not the case when both components are stochastic: In this case comprehensive characterization of credit risk requires specification of the correlation structure between credit risk and market risk as well as the maturity dependence of credit risk in the form of a forward probability of default. While existence of these elements has been alluded to, I am unaware of any model providing an explicit specification for either of these components. I propose a security market model in which discount bonds issued by issuers from three distinct risk classes are traded is a single currency environment: Treasury bonds, interbank deposits and credit risky corporate bonds. Payoffs on these instruments are characterized in terms of their respective contractual and non contractual payoffs. The contingency of default of the interbank deposit is assumed to be negligible.. This is consistent with the fact that empirically the interbank market is free of default. Instead, the yield spread between treasury and interbank assets is motivated in terms of institutional features of these markets. Given the fact that both markets are empirically free of default, the existence of non-trivial spreads requires that in order to preclude arbitrage between the treasury and the interbank market instruments, restrictions must be placed on permissible trading strategies. Payoffs from credit risky corporate bonds are assumed to be subject to default. Existence of a strictly positive probability that actual payoffs on these instruments fall short of contractual payoffs generates a yield spread between interbank and corporate assets. Having formulated the market structure, I turn to the derivation of pricing measures under which normalized assets are martingales. A very significant technical point arises when stochastic processes are considered under the pricing measures. It turns out that under the respective martingale measures, the dynamics of the stochastic processes driving market and credit risk are fully specified in terms of volatility parameters. The analysis can thus be interpreted as extending the approach of Heath, Jarrow and Morton from market risk to credit risk. Given that volatility parameters are much easier to estimate than drift parameters, the current analysis therefore improves the prospects of making the pricing of credit risk more operational.
This chapter is structured as follows. Section 1 contains a characterization of the time at which default occurs. The requirement to specify a forward probability of

default is also motivated. Section 2 contains the description of the traded discount instruments in terms of their contractual and non-contractual payoffs It is shown that the possibility of default can be modelled as a negative non-contractual dividend process attached to the instrument which is subject to default. The derivation of pricing measures is the subject of section 3. Different distributional specifications for recovery rates are examined and restrictions on permissible specifications necessary for the use of the no-arbitrage paradigm are identified. Martingale measures associated with the use of a risk free money market account and a risk free deposit of a given maturity as numeraires are identified. I show that absence of arbitrage implies certain drift restrictions such that processes under the martingale measure are specified in terms of volatility parameters. Section 5 elaborates on the extension of the derived pricing operators to derivative securities.

3.1. Modelling the Time of Default

The first credit risk component to be modelled here is the time of default, τ. As noted above, one of the weaknesses of the capital structure approach to credit risk was that default, being triggered upon continuously evolving firm value hitting a constant reorganization boundary, could be predicted by investors. This is somewhat equivalent with credit risk being locally deterministic, a characterization fundamentally at odds with the existence of credit spreads at short maturities. Capital structure model based on endogenous reorganization boundaries may generate improvements relative to earlier models, but sofar no empirical evidence as to their performance is available. The current modelling approach is consistent with reduced form models of credit risk as proposed, among others, by Duffie and Huang (1986) and Lando (1994). In these models, the time of default is driven by a doubly stochastic Poisson process with a stochastic intensity, which has the interpretation of being the instantaneous probability of default. The structure of such a process is briefly reviewed here. Furthermore, I will motivate why the specification of a stochastic default intensity generates the requirement to specify a forward probability of default. This is the point at which the current analysis surpasses established approaches.

I consider a continuous time securities market subject to a fixed trading interval $[0, 1]$. Uncertainty is characterized by a fixed probability space $(\Omega, \mathbb{F}= (\mathcal{F}_t)_{t=0}^{1}, P)$, where Ω is the state space, $\mathcal{F}$ is the sigma algebra representing measurable events and P is a given probability measure. Information evolves according to the right continuous

augmented filtration[15] $\mathcal{F}_t$. There is no explicit role for information asymmetries in the present setup and the information $\mathcal{F}_t$ is assumed to be available to all investors.
At any given point in time it is assumed that observation of $\mathcal{F}_t$ is sufficient to determine whether or not default has occurred. The mathematical equivalent of this requirement is that the time of default is a stopping time: $\{t \geq \tau\} \in \mathcal{F}_t$. The set of stopping times of $\mathcal{F}_t$ can be decomposed into a set of accessible and inaccessible stopping times, the former capturing the idea of events associated with stopping times being predictable and the latter set denoting the complementary concept[16]. The requirement that uncertainty of credit risk be preserved locally thus translates into the specification of the time of default as an inaccessible stopping time of the filtration $\mathcal{F}_t$. Associated with τ is the default indicator

$$N_t = I_{\{t \geq \tau\}} \tag{1}$$

where $N_t = 1$ if $\{t \geq \tau\}$ and $N_t = 0$ if $\{t < \tau\}$. Assuming that τ admits a density $G(dt)$, Jacod and Shiryaev (1987)demonstrate that the compensator[17] of (1) is given by the following process:

$$\Lambda_t = \int_0^{t \wedge \tau} \frac{G(ds)}{G([s, \infty])} \tag{2}$$

Example 3.1 If τ is exponential with parameter λ_t, a deterministic function, Λ_t is given by:

$$\Lambda_t = \int_0^{t \wedge \tau} \lambda_s ds \tag{3}$$

This is the specification used by Jarrow and Turnbull (1995).

Example 3.2 If τ is uniform with parameter 1, Λ_t is given by:

$$\Lambda_t = -\log\left(1 - (t \wedge \tau)\right) \tag{4}$$

In the present context I assume that the time of default is distributed exponentially with a stochastic parameter $\lambda_t(\omega) : [0,1] \times \Omega \to \mathbb{R}^+$, itself a measurable process adapted to $\mathcal{F}_t$. In order to economize notation, I will use λ_t to denote $\lambda_t(\omega)$ in the following sections. Note that the filtration generated by this process is defined as $\mathcal{F}_t^\lambda = \sigma(\lambda_s; 0 \leq s \leq t)$. (1) in conjunction with the exponential specification for

[15] An augmented filtration contains all null sets of a given measure (Karatzas and Shreve (1989), p.89). The filtration is therefore assumed to satisfy the socalled "usual coditions" (e.g. Protter (1990), p.3 or Jacod and Shiryaev (1989), p.10), a technical requirement neccessary for the application of many results in the theory of stochastic integration.

[16] A stopping time T is inaccessible if $P(\omega : T(\omega) = S(\omega) < \infty) = 0$ for every predictable stopping time S (e.g. Protter (1990), p.99 or Jacod and Shiryaev (1989), p.20).

[17] Given a process X, its compensator X' is characterized by the fact that $X - X'$ is a martingale.

$G(dt)$ characterizes N_t as a Poisson process. Given the assumption that the parameter $\lambda_t(\omega)$ is stochastic, the following definition, taken from Bremaud (1981), p.22, will facilitate further derivations:

Definition 3.1 If N_t' is a conditional Poisson process[18], the following holds:

$$E\left[\exp\left(iv(N_t' - N_s') \mid \mathcal{F}_t^\lambda\right]\right. = \exp\left((e^{iv}-1)\int\limits_s^t \lambda_u du\right) \tag{5}$$

(5) is the characteristic function of a conditional Poisson increment $N_t' - N_s'$, where the prime is used to distinguish the process from (1), which is subject to a single jump only. Expanding both sides of (5) yields the following:

$$\begin{aligned} &\sum_{k=0}^{\infty} P\left[N_t' - N_s' = k \mid \mathcal{F}_t^\lambda\right] \times \exp\left(iv(N_t' - N_s')\right) \qquad (6) \\ = \quad &\exp\left(-\int\limits_s^t \lambda_u du\right) \sum_{k=1}^{\infty} \frac{e^{ivk}}{k!} \left(\int\limits_s^t \lambda_u du\right)^k \end{aligned}$$

Therefore, the probability that a conditional Poisson process will incur k jumps over the interval $[s,t]$ is given by

$$P\left[N_t' - N_s' = k \mid \mathcal{F}_t^\lambda\right] = \frac{1}{k!}\exp\left(-\int\limits_s^t \lambda_u du\right)\left(\int\limits_s^t \lambda_u du\right)^k \tag{7}$$

This expression can be used to verify the following lemma:

Lemma 3.1 The compensator of a conditional Poisson process is given by:

$$\Lambda_t = \int\limits_0^{t\wedge\tau} \lambda_s ds = \int\limits_0^t I_{\{s\leq\tau\}} ds \tag{8}$$

Proof:

We need to verify the martingale property of the compensated process $N_t' - \Lambda_t$:

$$E\left[N_t' - N_s' - \Lambda_t + \Lambda_s \mid \mathcal{F}_t^\lambda\right] = 0 \tag{9}$$

Using (7),

$$\begin{aligned} E\left[N_t' - N_s' \mid \mathcal{F}_t^\lambda\right] &= \sum_{k=0}^{\infty} P\left[N_t' - N_s' = k \mid \mathcal{F}_t^\lambda\right] \cdot k \\ &= \exp\left(-\int\limits_s^t \lambda_u du\right) \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(\int\limits_s^t \lambda_u du\right)^k \end{aligned}$$

[18] Also denoted doubly stochastic or compound poisson process.

Noting that

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(- \int_s^t \lambda_u du \right)^k = \left(\int_s^t \lambda_u du \right) \sum_{k=0}^{\infty} \frac{1}{k!} \left(\int_s^t \lambda_u du \right)^k$$

and

$$\Lambda_t - \Lambda_s = \int_s^t \lambda_u du$$

yields (8).□

Note that an alternative motivation of (9) is possible using (2) and

$$G(dt) = \lambda_t e^{-t\lambda_t} dt \tag{10}$$

Applied to the process N_t, (9) facilitates interpretation of λ_t as the intensity of the single jump process (1):

$$\lambda_t = \lim_{u \to 0} \frac{1}{u} E\left[N_{t+u} - N_t \mid \mathcal{F}_t^{\lambda}\right] \tag{11}$$

as well as the instantaneous probability of default occurring over the infinitesimal interval $[t, t+u]$:

$$\lambda_t = \lim_{u \to 0} \frac{1}{u} P\left(t < \tau \leq t + u \mid \mathcal{F}_t^{\lambda}\right) \tag{12}$$

The next lemma collects useful results regarding probabilities relating to default occurring over given time intervals.

Lemma 3.2

$$P(t < \tau) = E\left[\exp\left(-\int_0^t \lambda_s ds\right)\right] \tag{13}$$

$$P\left(t < \tau \mid \mathcal{F}_t^{\lambda}\right) = \exp\left(-\int_0^t \lambda_s ds\right) \tag{14}$$

$$P(t \geq \tau) = E\left[1 - \exp\left(-\int_0^t \lambda_s ds\right)\right] \tag{15}$$

$$P\left(t \geq \tau \mid \mathcal{F}_t^{\lambda}\right) = 1 - \exp\left(-\int_0^t \lambda_s ds\right) \tag{16}$$

Proof:

While (14)-(18) could be proven using (7), I provide an alternative derivation. Following Karatzas and Shreve (1987), p.369, define the stopping time

$$\tau = \inf\left\{t > 0; \int_0^t \lambda_s ds \geq \tau'\right\} \tag{17}$$

where τ' is a Poisson random variable with unit intensity. Using the law of iterated expectations and (19) facilitates the following manipulations:

$$\begin{aligned} P(t<\tau) &= E\left[I_{\{t\leq\tau\}}\right] \\ &= E\left[E\left[I_{\{t\leq\tau\}} \mid \mathcal{F}_t^{\lambda}\right]\right] \\ &= E\left[E\left[I_{\{\Lambda_t<\tau'\}} \mid \mathcal{F}_t^{\lambda}\right]\right] \\ &= E\left[\int_0^{\Lambda_t} e^{-s}ds\right] \end{aligned}$$

Evaluating the integral yields (14). (12)-(18) are derived in a similar manner.□

The final result in the present section regards a technical point the relevance of which will become apparent at a later stage.

Lemma 3.3

$$E\left[N_T-\Lambda_T \mid \mathcal{F}_T^{\lambda}\right] = E\left[N_T-\Lambda_{T-} \mid \mathcal{F}_{T-}^{\lambda}\right] \tag{18}$$

Proof:

Using Protter (1990), p.101, the following holds

$$P\left[\triangle\Lambda_T \neq 0\right] = 0 \tag{19}$$

where $\triangle\Lambda_T = \Lambda_T - \Lambda_{T-}$ and $\Lambda_{T-} = \lim_{s\uparrow T} \Lambda_s$. (21) is equivalent to

$$P\left[\Lambda_T = \Lambda_{T-}\right] = 1 \tag{20}$$

Assuming the filtration generated by the intensity process contains the null sets of P, (22) can be used to establish left continuity of $\mathcal{F}_T^{\lambda}$:

$$\mathcal{F}_T^{\lambda} = \mathcal{F}_{T-}^{\lambda} \tag{21}$$

completing the proof.□

The above lemma demonstrates that the martingale nature of the compensated default indicator process is preserved if the intensity process is stopped just prior to the time of default. From a modelling perspective this means that specification of the intensity process up to $\tau-$, as opposed to τ, is sufficient to characterize the uncertainty over the interval $[0,\tau]$ relating to the time of default. The compensator (9) of (1) is thus equivalent to

$$\Lambda_t = \int_0^{t\wedge\tau-} \lambda_s ds = \int_0^{t} I_{\{s<\tau\}} ds \tag{22}$$

The notation "$t\wedge\tau-$" is somewhat tedious, and for the remainder of the analysis, I will use "$t\wedge\tau$" with the implicit understanding that processes associated with such a notation exist at most up to time $\tau-$, not τ. Having characterized uncertainty relating to the time of default in terms of its density and a stochastic intensity, it must be noted that such a specification necessitates introduction of a maturity dependence of the probability of default. In analogy with interest rates, let $\lambda_{t,T}$ denote the instantaneous forward probability of default, at time t, over the infinitesimal interval $[T, T + dt]$, where $t < T$. Consider a model in which the probability of default is deterministic. Specification of λ_t is sufficient to characterize the probability of default over the interval $[T, T + dt]$, because $\lambda_{t,T} = \lambda_T$ and there is no need to separately specify a process $\lambda_{t,T}$. Now assume that λ_t is stochastic. Knowledge of λ_t only characterizes credit risk dynamics over the infinitesimal interval $[t, t + dt]$ and $P(\lambda_T = \lambda_{t,T}) = 0$. For any interval $[T, T + dt]$ with $t < T$, a function $\lambda_{t,T}$ must be specified which characterizes the probability, at time t, that default will occur over the infinitesimal interval $[T, T + dt]$, where $t < T$. Based on the intuition that the credit quality varies over time in a smooth but stochastic fashion, we impose a diffusion specification for the dynamics of the forward probability of default.

Definition 3.2 The forward probability of default, characterizing, at time t, the instantaneous default intensity at some future date T, is given by

$$d\lambda_{t,T} = \alpha^{\lambda}(\omega, t, T)dt + \sigma^{\lambda}(\omega, t, T)dW_t^{\lambda} \tag{23}$$

The coefficient functions are assumed to satisfy conditions necessary for existence and uniqueness of strong solutions to the above differential equation.

Remark 3.1 Note that due to the nature of the variable modelled, the process in the above definition must not become negative. Such a requirement could be achieved by defining the forward probability of default in terms of a new variable $\lambda'_{t,T} = (\lambda_{t,T})^{+}$or $\lambda'_{t,T} = |\lambda_{t,T}|$; in the first case, $\lambda'_{t,T}$ is simply kept at zero when negative values of the process in (23) indicate very high levels of credit quality; the second case postulates a reflecting boundary for the forward probability of default, which could be more difficult to justify in economic terms (why should $\lambda'_{t,T}$ bounce off zero once this value is touched and not be able to stay there over non zero time intervals ?). While conditions guaranteeing non-negativity of the process in (23) is of fundamental importance, it is not the objective of the current analysis to explicitly identify such conditions. Rather it is assumed that the dependence of the coefficient functions on the current state implicitly subsumes such conditions. Furthermore, in the absence of such explicit conditions, these considerations demonstrate that state dependency is indeed necessary in (23) in order to refer to $\lambda_{t,T}$ as a probability. In order to economize notation and unless otherwise stated, I will suppress the parameter ω in the coefficient functions of the process in (23).

Remark 3.2 The forward probability of default in the current context is an exogenously specified variable. While it is very similar to instantaneous forward interest rates, it is more difficult to express this variable in terms of others. Note that given the instantaneous forward rate $f^d_{t,T}$ on a credit risky zero coupon bond, the value of this bond is given by

$$B^d_{t,T} = exp\left(- \int_t^T f^d_{t,u} du \right) \tag{24}$$

Vice versa, given the value of the zero coupon bond, the forward rates can be defined as

$$f^d_{t,T} = -\frac{\partial}{\partial T} ln B^d_{t,T} \tag{25}$$

Assuming a preference for bond prices rather than rates as the fundamental economic concept, such rates can thus be expressed as functions of prices. The difficulty in extending this idea to the forward probability of default is that instantaneous credit risky forward rates are nontrivial functions of credit risk free forward rates, the process describing recovery value and the forward probability of default. Once the nature of this interdependence has been clarified, it is possible that the forward probability of default can be expressed as a function of risk free and credit risky zero coupon prices. The current analysis considers rates, not prices, as the fundamental economic variables. It is therefore consistent to specify forward default rates exogenously and not in terms of other variables. Nevertheless one might gain further intuition as to what this variable captures by considering a contract which makes a payment of one at the maturity date if default has not occurred; assuming zero interest rates, the value of this contract would be given by

$$B^d_{t,T} = exp\left(- \int_t^T \lambda_{t,u} du \right) \tag{26}$$

In such an environment and assuming that $B^d_{t,T}$ was given, the forward probability of default could be defined as

$$\lambda_{t,T} = -\frac{\partial}{\partial T} ln B^d_{t,T} \tag{27}$$

I have demonstrated that my modelling approach regarding the time of default is consistent with a number of established models. Nevertheless, it is demonstrated that established approached leave certain credit risk elements unspecified. More precisely, credit risk models using the reduced form approach are characterized by their specification of the probability of default and a recovery rate process. Regarding a framework in which the probability of default is stochastic, the present considerations demonstrate that specification of the latter two components is not sufficient to comprehensively characterize credit risk. Specification of a forward intensity is warranted. The next section will introduce an explicit specification for both the spot and forward intensity of default, facilitating incorporation of a correlation structure between credit risk and market risk.

3.2. Characterization of Discount Instruments

In order to derive no-arbitrage pricing rules for contracts contingent on credit risk, the simplest possible contracts subject to credit risk must be characterized. The contracts chosen for this purpose are zero coupon bonds. The following subsections are devoted to the economic characterization of both contractual and non contractual payoffs associated with traded discount instruments. Specifically, the components of credit risk will be embedded in a market structure. It is assumed that available for trading is a continuum of discount instruments, indexed by maturity, issued by the treasury, interbank market participants and corporations, respectively. Assets issued by interbank market participants will also be denoted as Libor assets.

3.2.1. Contractual Payoffs

The current section discusses characterization of contractual payoffs to traded discount securities. In particular, it is demonstrated how processes for recovery rates and forward probability of default are integrated into a forward rate process associated with the contractual component of the credit risky corporate discount security. Furthermore, explicit specification of a correlation structure between market risk and credit risk is achieved in the market structure outlined below.

By definition, contractual payoffs to discount instruments are given by the security's principal value, normalized to unity in the present context. Dynamics of these contractual components can be characterized through specification of the respective instantaneous forward rate processes which are the object of the following set of assumptions.

Assumption 3.1 Instantaneous forward rates on the contractual component of the treasury discount bond evolve according to:

$$df^g_{t,T} = \alpha^g(t,T)dt + \sigma^g(t,T)dW^g_t \tag{28}$$

Assumption 3.2 Instantaneous forward rates on the contractual component of the Libor discount bond evolve according to:

$$df^l_{t,T} = \alpha^l(t,T)dt + \sigma^l(t,T)dW^l_t \tag{29}$$

(25) and (26) are consistent with standard Heath-Jarrow-Morton diffusion specifications for forward rate processes. While forward rates on the corporate asset will also follow a diffusion, their dynamics will include components arising from the presence of credit risk. It was noted above that the assumption of the instantaneous probability of default being stochastic generates the requirement to separately specify a process relating to the forward probability of default. Recalling the relevant definition ion the previous section, the dynamics of the forward probability of default $\lambda_{t,T}$ is given by:

$$d\lambda_{t,T} = \alpha^\lambda(t,T)dt + \sigma^\lambda(t,T)dW^\lambda_t \tag{30}$$

This explicit specification of the dynamics of the forward probability in terms of drift and volatility functions of default marks the departure from established models in this area. $\alpha^\lambda(t,T)$ is the maturity dependent drift of the forward probability of default, and $\sigma^\lambda(t,T)$ is the instantaneous volatility of the same process. Implicit in (27) is the understanding that changes in credit quality evolve in a continuous fashion. As will become apparent below, the mechanism translating credit quality as summarized by $\lambda_{t,T}$ into the contingency of default is a discontinuous one. In order to make allowance for stochastic recovery rates, I add the following assumption:

Assumption 3.3 Conditional on default having occurred at time t, the recovery rate $R_t(x) : [0,1] \times E \rightarrow [0,1]$ is a measurable adapted function; the random variable x admits a distribution $\Phi_t(x)$.

Note that the recovery rate is a function of a random variable x taking values in a mark space $(E, \mathcal{E})$, where $\mathcal{E} = \mathcal{B}(E)$ is the Borel set of events in E. A realization of $R_t(x) = 0$ corresponds to a complete loss; a realization of $R_t(x) = 1$ corresponds to the contingency of complete recovery; intermediate realizations correspond to partial recovery. Integrating over the mark space E, the expected recovery rate R_t is given by

$$R_t = \int_E R_t(x)\Phi_t(x)dx \tag{31}$$

(27) and (28) will be integrated into the diffusion specification for corporate forward rates.

Assumption 3.4 The process for instantaneous forward rates associated with the contractual component of the corporate discount bond is given by:

$$df^d_{t,T} = \alpha^l(t,T)dt + \sigma^l(t,T)dW^l_t + (R_t - 1)d\lambda_{t\wedge\tau,T} \tag{32}$$

Remark 3.3 In contrast to the original specification of Heath, Jarrow and Morton (1992), forward rates in the present context are driven by a single diffusion term. This fact precludes articulation of interaction between rates, at different maturities, on a given term structure but facilitates specification of interaction between rates on different term structures by simplifying notation.

Instantaneous forward rates on the corporate bond include a drift and diffusion component identical to the credit risk free benchmark asset, in the present framework given by the Libor instrument; in addition, there are components relating to the dynamics of the forward probability of default and recovery rates. Defining $w_t = 1 - R_t$ as the expected loss rate, (29) is equivalent to the following process:

$$df^d_{t,T} = \alpha^l(t,T)dt + \sigma^l(t,T)dW^l_t + w_t d\lambda_{t\wedge\tau,T} \tag{33}$$

(30) formalizes the integration of the processes relating to the forward probability of default and recovery rates into a process for the instantaneous credit risky forward rate relating to the contractual component of the credit risky zero coupon bond. This specification is an innovative alternative to the specification by Jarrow and Turnbull (1995), whose process of the contractual component of forward rates includes a jump process. Forward rate dynamics in (30) are specified in differential form; for future reference, the respective integral representation is given by:

$$\begin{aligned} f^d_{t,T} &= f^d_{0,T} + \int_0^t \alpha^l(s,T)dt + \int_0^t \sigma^l(s,T)dW^l_s \\ &\quad - \int_0^t I_{\{s<\tau\}} w_s \alpha^\lambda(s,T)dt - \int_0^t I_{\{s<\tau\}} w_s \sigma^\lambda(s,T)dW^\lambda_{s\wedge\tau} \end{aligned} \tag{34}$$

Given that $r^d_t = f^d_{t,t}$, spot rates associated with the contractual component of the credit risky corporate discount instrument are given by:

$$\begin{aligned} r^d_t &= f^d_{0,t} + \int_0^t \alpha^l(s,t)dt + \int_0^t \sigma^l(s,t)dW^l_s \\ &\quad - \int_0^t I_{\{s<\tau\}} w_s \alpha^\lambda(s,t)dt - \int_0^t I_{\{s<\tau\}} w_s \sigma^\lambda(s,t)dW^\lambda_{s\wedge\tau} \end{aligned} \tag{35}$$

There are no reduced form models which make explicit allowance for a correlation structure between credit risk and market risk. Given the diffusion specification for the forward probability of default, articulation of such a correlation structure is straightforward.

Assumption 3.5 The correlation structure of

$$\mathbf{W}_t = (W_t^g, W_t^l, W_t^\lambda)^\top \tag{36}$$

is given by the symmetric, positive semidefinite matrix

$$\mathbf{C}_t = \left(\rho_t^{i,j}\right)_{i,j=1}^{3} \tag{37}$$

where, for $i, j \in \{g, l, \lambda\}$,

$$\rho_t^{i,j} = \frac{d}{dt}\langle W^i, W^j\rangle_t \tag{38}$$

Given that (34) is symmetric and positive semidefinite, it can be diagonalized by an orthogonal matrix $\widetilde{\mathbf{Q}}_t = (\widetilde{q}_t^{ij})_{i,j=1}^3$:

$$\mathbf{C}_t = \widetilde{\mathbf{Q}}_t \mathbf{N}_t \widetilde{\mathbf{Q}}_t^\top \tag{39}$$

where $\mathbf{N}_t$ is a diagonal matrix. By the definition of orthogonality of $\widetilde{\mathbf{Q}}_t$,

$$\widetilde{\mathbf{Q}}_t \widetilde{\mathbf{Q}}_t^\top = \widetilde{\mathbf{Q}}_t^\top \widetilde{\mathbf{Q}}_t = \mathbf{I} \tag{40}$$

and

$$\sum_{k=1}^{3} \widetilde{q}_t^{ik} \widetilde{q}_t^{jk} = \sum_{k=1}^{3} \widetilde{q}_t^{ki} \widetilde{q}_t^{kj} = \delta_{ij} \tag{41}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Defining

$$\mathbf{Q}_t = \widetilde{\mathbf{Q}}_t \sqrt{\mathbf{N}_t} \tag{42}$$

(36) is equivalent to

$$\mathbf{C}_t = \mathbf{Q}_t \mathbf{Q}_t^\top \tag{43}$$

Making explicit the elements of $\mathbf{Q}_t$:

$$\mathbf{Q}_t = \begin{pmatrix} q_t^{g1} & q_t^{g2} & q_t^{g3} \\ q_t^{l1} & q_t^{l2} & q_t^{l3} \\ q_t^{\lambda 1} & q_t^{\lambda 2} & q_t^{\lambda 3} \end{pmatrix} \tag{44}$$

(34) and (40) yield the following set of equations:

$$\sum_{i=1}^{3} q_t^{gi} q_t^{li} = \rho_t^{gl} \tag{45}$$

$$\sum_{i=1}^{3} q_t^{gi} q_t^{\lambda i} = \rho_t^{g\lambda} \tag{46}$$

$$\sum_{i=1}^{3} q_t^{li} q_t^{\lambda i} = \rho_t^{li} \tag{47}$$

While different decompositions $\mathbf{Q}_t$ of $\mathbf{C}_t$ satisfying (40) are possible, the one chosen in the present framework is the Cholesky decomposition, which is characterized by the fact that $\mathbf{Q}_t$ is lower triangular. For the remainder of the analysis, $\mathbf{Q}_t$ is thus given by (41), subject to the constraint

$$q_t^{g2} = q_t^{g3} = q_t^{l3} = 0 \tag{48}$$

Defining a vector of independent continuous martingales:

$$\mathbf{M}_t^c = (M_t^{1,c}, M_t^{2,c}, M_t^{3,c})^\top \tag{49}$$

(33)-(35) can be summarized as

$$d\mathbf{W}_t = \mathbf{Q}_t d\mathbf{M}_t^c \tag{50}$$

(47) is consistent with the representation results in Karatzas and Shreve (1987), p. 170, in that the correlated continuous martingale $\mathbf{W}_t$ can be represented as stochastic integral with respect to a vector $\mathbf{M}_t^c$ of independent continuous martingales:

$$W_t^i = \sum_{j=1}^{3} \int_0^t q_t^{i,j} dM_t^{j,c} \tag{51}$$

for $i, j \in \{g, l, \lambda\}$.

Given the specifications for forward rate processes and correlation structure, it is possible to derive the return processes for the three discount instruments. The return process for the treasury and Libor instrument, respectively, is given by

$$\frac{dB_{t,T}^g}{B_{t,T}^g} = (r_t^g + b^g(t,T))dt + a^g(t,T)dM_t^{1,c} \tag{52}$$

$$\frac{dB_{t,T}^l}{B_{t,T}^l} = (r_t^l + b^l(t,T))dt + a^l(t,T)\sum_{i=1}^{2} q_t^{li} dM_t^{i,c} \tag{53}$$

where, for $i \in \{g, l\}$,

$$b^i(t,T) = \frac{1}{2}a^i(t,T)^2 - \int_t^T \alpha^i(t,u)du \tag{54}$$

and

$$a^i(t,T) = -\int_t^T \sigma^i(t,u)du \tag{55}$$

(49) and (50) are the familiar expression of zero coupon bond return processes, augmented for the existence of an explicit correlation between the factors driving the uncertainty. Derivation of the equivalent expression for the corporate bond is rele-

gated to the appendix; the return process on this instrument is given by

$$\begin{aligned}\frac{dB^d_{t,T}}{B^d_{t,T}} &= (r^d_t + b^l(t,T) + I_{\{t<\tau\}}b^\lambda(t,T))dt \\ &\quad -I_{\{t<\tau\}}\rho^{l,\lambda}_t a^l(t,T)w_t a^\lambda(t,T)dt \\ &\quad +a^l(t,T)dW^l_s + I_{\{t<\tau\}}w_t a^\lambda(t,T)W^\lambda_s\end{aligned} \tag{56}$$

where

$$b^\lambda(t,T) = \frac{1}{2}w_t^2 a^\lambda(t,T)^2 - w_t \int_t^T \alpha^\lambda(t,u)du \tag{57}$$

$$a^\lambda(t,T) = -\int_t^T \sigma^\lambda(t,u)du \tag{58}$$

and the remaining terms are defined in (51) and (52). (53) describes the return process for the credit risky zero coupon bond. In addition to terms arising from uncertainty in risk free forward rates, the analysis demonstrates how the return process is affected by the presence of uncertainty in the process describing the forward probability of default. Using (47), (53) is equivalent to

$$\begin{aligned}\frac{dB^d_{t,T}}{B^d_{t,T}} &= (r^d_t + b^l(t,T) + I_{\{t<\tau\}}b^\lambda(t,T))dt \\ &\quad -I_{\{t<\tau\}}a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q^{li}_t q^{\lambda i}_t dt \\ &\quad +a^l(t,T)\sum_{i=1}^{2} q^{li}_t dM^{i,c}_t + I_{\{t<\tau\}}w_t a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t dM^{i,c}_t\end{aligned} \tag{59}$$

This concludes specification of the contractual components of traded discount securities. It was demonstrated how explicit specifications for processes relating to recovery rates and forward probability of default can be embedded into a market structure. Furthermore, an explicit correlation structure between market risk and credit risk was incorporated.

3.2.2. Non-Contractual Payoffs

Contractual components of discount securities return the principal value at the maturity date with certainty. Characterization of this component is insufficient in order to capture the payoffs associated with the traded instruments. The fact that three different assets with the same contractual payoffs continue to be traded, and assuming that no arbitrage opportunities exists between these assets, means that these securities must differ in terms of features not captured by contractual specifications. The

following section is devoted to the specification of non contractual payoffs associated with different discount securities in the present market model.
In order to accommodate non contractual payoffs in the subsequent analysis, the concept of trading gains is introduced.

Definition 3.3 Trading gains associated with a given security reflect the cumulative benefits or losses to investors with long positions in these assets.

Trading gains subsume both contractual and non-contractual payoffs form the instruments they are associated with. No restrictions are placed on the time at which relevant payoffs occur. Benefits to investors with long positions in the Libor asset consist entirely in the principal payment at the maturity date. With regard to the treasury instrument and the corporate bond, it is argued below that institutional features in the treasury market as well as the presence of default risk in the corporate bond market can be captured by recasting these features as non contractual payoffs associated with instruments traded in these markets. I discuss non contractual payoffs associated with each security in turn.

3.2.2.1. Non-Contractual Payoffs associated with the Treasury Instrument

The interbank market is empirically free of default. Absence of arbitrage between interbank and treasury securities requires motivation of why assets with identical contractual specifications and apparently identical credit quality continue to be traded. In the present model, this fact is explained in terms of a non contractual benefit associated with positions in the treasury security, coupled with certain trading restrictions as outlined below.
With regard to treasury securities, a number of institutional features exist which influence payoffs on these instruments. Treasury securities are often used as collateral, both in the repo market[19] as well as for purposes of margin requirements in futures markets. If the rate of return on non-treasury securities for purposes of collateralization is lower than the rate of return on treasuries, such return differential will accrue to investors with long positions in treasuries. Tax arguments not formalized here could be adduced to argue that investors benefit from holding treasury securities[20]. Furthermore, treasury securities are subject to lower capital requirements

[19] See Duffie (1996).

[20] The evidence regarding tax advantages to holding treasury securities is not conclusive (e.g. Green and Oedegaard (1995) and Ronn and Shin (1993)).

than non-treasury issues and classes of market participants may be subject to statutory requirements to invest a certain fractions of their wealth in government bonds, benefitting investors with long positions in these securities.

Presence of such institutional features amounts to investors with long positions in treasuries earning an actual rate of return higher than expected, given the contractual payoffs on those instruments. This non-contractual return differential can be interpreted as a convenience yield, formalized in the following assumption

Assumption 3.6 Investors with long positions in the treasury security receive a non-contractual deterministic convenience yield y_t, proportional to the market value of the treasury position. The value of a fund accruing at this yield follows the process

$$D_t^g = 1 + \int_0^t D_s^g y_s ds \tag{60}$$

the solution to which is given by

$$D_t^g = \exp \int_0^t y_s ds \tag{61}$$

Remark 3.4 The current specification allows for the convenience yield to be a function of time. Babbs (1991) allows for additional maturity dependence by defining a convenience yield of the type $y_{t,T}$; such a characterization does not add additional insight in the present context and is not pursued here.

Remark 3.5 Extension of the above specification to make allowance for the convenience yield to be stochastic is straightforward, assuming that y_t would be driven by elements of $\mathbf{W}_t$. A mean reverting normal specification of the following type

$$dy_t = \kappa(\theta - y_t)dt + \sigma dW_t \tag{62}$$

would be consistent with the framework proposed by Grinblatt (1993). Specification of uncertainty components associated with y_t independent of $\mathbf{W}_t$ would require introduction of additional traded assets. Either of these extensions would necessitate substantial additional notation and neither of them is pursued here.

Given the assumption of the convenience yield accruing in proportion to the market value of the treasury position held, the value at time t of a treasury position, maturing at time T, incurred at time 0, $V_{t,T}^g$, is given by

$$V_{t,T}^g = B_{t,T}^g D_t^g \tag{63}$$

Using (49) and (56), the rate of return on this process must follow:

$$\frac{dV^g_{t,T}}{V^g_{t,T}} = (r^g_t + y_t + b^g(t,T))dt + a^g(t,T)dM^{1,c}_t \tag{64}$$

Investors with long positions in the treasury asset can realize arbitrage profits by shorting treasuries and taking long positions in the Libor assets. The following assumption is added to preclude this strategy.

Assumption 3.7 Short sales of treasury instruments are disallowed.

While formalization of market frictions generating short sale constraints is beyond the scope of the present analysis, the necessity of the above assumption will be further motivated below.

With regard to the Libor asset, it is noted that no non contractual payoffs are associated with this asset. Therefore the return process relating to trading gains is identical to the one relating to the contractual component:

$$\frac{dV^l_{t,T}}{V^l_{t,T}} = \frac{dB^l_{t,T}}{B^l_{t,T}} \tag{65}$$

This concludes the specification of non contractual payoffs associated with treasury and interbank assets.

3.2.2.2. Non-Contractual Payoffs associated with the Corporate Instrument

Non contractual payoffs on the credit risky corporate bond are negative in value and arise from the possibility of actual payoffs falling short of promised payoffs. It is this non contractual payoff which distinguishes the corporate security from the interbank asset. The particular probabilistic structure of the recovery rate on the corporate instrument in the event of default is of particular interest because only a subset of possible specifications permits identification of unique pricing measures. Conversely, uniqueness of pricing measures is equivalent to markets being complete, and only in this case one can expect to be able to hedge the risk arising from the contingency of default. I consider three specifications for the distribution of recovery rates: A degenerate, a finite valued and a continuous distribution.

3.2.2.2.1. Degenerate Recovery Rate Distribution

The specification of non contractual payoffs based on a degenerate recovery rate distribution is formalized as follows:

Assumption 3.8 For the case of a degenerate recovery rates distribution, the mark space consists of a nonnegative constant k: $E = \{x\}$ and $x = k$. Recovery rates are given by

$$R_t(x) = e^{-kh_t} \tag{66}$$

and the proportional dividend process capturing the credit risk embedded in the corporate bond is given by

$$D_t^d = 1 + \int_0^t D_s^d(R_s(x) - 1)dN_s \tag{67}$$

Note that the function h_t captures a possible time dependency of recovery rates. Derivation of the solution of (64) is facilitated by the following manipulations:

$$\begin{aligned} D_t^d &= 1 + \int_0^t D_s^d(e^{-kh_s} - 1)dN_s \\ &= \prod_{0<s\leq t} (1 + (e^{-kh_s} - 1)\Delta N_s) \\ &= \prod_{0<s\leq t} (1 + e^{-kh_s} - 1)^{\Delta N_s} \end{aligned}$$

which finally yields

$$D_t^d = \exp\left(-k\int_0^t h_s dN_s\right) \tag{68}$$

It is noted that for any maturity date $T \in [t, 1]$,

$$D_T^d = e^{-kh_T} I_{\{\tau\leq T\}} + I_{\{\tau>T\}} \tag{69}$$

The value at time t of a position in the corporate security incurred at time 0, $V_{t,T}^d$, based on the assumption of a degenerate recovery rate distribution, is given by

$$V_{t,T}^d = B_{t,T}^d D_t^d \tag{70}$$

Using the stochastic equivalent of the chain rule for(67)

$$dV_{t,T}^d = B_{t-,T}^d dD_t^d + D_{t-}^d dB_{t,T}^d + d[B_{t,T}^d, D_t^d] \tag{71}$$

where $[.,.]$ denotes the quadratic covariation process (e.g.Protter (1990), p.58). The use of (69), in conjunction with (53), permits derivation of the return process for trading gains associated with the corporate instrument under the assumption of a degenerate recovery rate distribution:

$$\frac{dV^d_{t,T}}{V^d_{t,T}} = (r^d_t + b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T) - I_{\{t<\tau\}} \rho^{l,\lambda}_t a^l(t,T) a^\lambda(t,T))dt \quad (72)$$
$$+a^l(t,T)dW^l_s + I_{\{t<\tau\}} a^\lambda(t,T) W^\lambda_s + w_t \left(dN_t - I_{\{t<\tau\}} \lambda_t dt\right)$$

(58) describes the rate of return process on the credit risky corporate asset under the assumption that the recovery rate is deterministic but possibly time dependent.

3.2.2.2.2. Finite-valued Recovery Rate Distribution

I consider the case in which the recovery rate takes one of a finite number of possible realizations.

Assumption 3.9 For the case of the distribution of recovery rates being finite-valued, $E = \{x^1 ... x^m\}$. For a realization x^i with $i \in [1...m]$, the recovery rate is given by

$$R_t(x^i) = e^{-x^i h_t} \quad (73)$$

and the proportional dividend process capturing the credit risk embedded in the corporate bond is given by

$$D^d_t = 1 + \int_0^t D^d_s (R_s(x) - 1) dN_s \quad (74)$$

with $x \in E = \{x^1 ... x^m\}$.

(59) is equivalent to

$$D^d_t = 1 + \sum_{i=1}^m \int_0^t D^d_s (R_s(x^i) - 1) dN_s(i) \quad (75)$$

where, for $i \in [1...m]$,

$$N_t(i) = I_{\{t \geq \tau\}} I_{\{x = x^i\}} \quad (76)$$

are point processes with respective intensities $\lambda_t(i)$. The probability of a particular realization is

$$\Phi(x^i) = P\left[x = x^i\right] = \lambda_t(i)/\lambda_t \quad (77)$$

where the probability of default occurring is given as the sum of the probabilities relating to the component processes:

$$\lambda_t = \sum_{i=1}^m \lambda_t(i) \quad (78)$$

For any maturity date $T \in [t, 1]$,

$$D_T^d = e^{-x' h_T} I_{\{\tau \leq T\}} + I_{\{\tau > T\}} \tag{79}$$

(53) and (60) permits derivation of the return process for the trading gains associated with positions in the corporate security under the assumption of a finite-valued recovery rate distribution:

$$\begin{aligned}
\frac{dV_{t,T}^d}{V_{t,T}^d} &= (r_t^d + b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T) - I_{\{t<\tau\}} \rho_t^{l,\lambda} a^l(t,T) a^\lambda(t,T))dt \qquad (80)\\
&+ a^l(t,T) dW_s^l + I_{\{t<\tau\}} a^\lambda(t,T) W_s^\lambda \\
&+ \sum_{i=1}^m \left(e^{-x^i h_t} - 1\right) \left(dN_t(i) - I_{\{t<\tau\}} \lambda_t(i) dt\right) \qquad (81)
\end{aligned}$$

where the following manipulation has been employed:

$$\begin{aligned}
w_t \lambda_t &= \lambda_t \left(\int_E e^{-x' h_t} \Phi(x) dx - 1 \right) \\
&= \lambda_t E\left[e^{-x h_t} - 1\right] \\
&= \lambda_t \sum_{i=1}^m \left(e^{-x^i h_t} - 1\right) P\left[x = x^i\right] \\
&= \sum_{i=1}^m \left(e^{-x^i h_t} - 1\right) \lambda_t(i)
\end{aligned}$$

Continuous Recovery Rate Distribution

Specification of non contractual payoffs based on a continuous recovery rate distribution is formalized as follows:

Assumption 3.10 For the case of recovery rates being continuously distributed between zero and one, the proportional dividend process capturing the credit risk embedded in the corporate bond is given by

$$D_t^d = 1 + \int_0^t \int_E D_s^d (R_s(x) - 1) \mu(ds \times dx) \tag{82}$$

where $E = [0, \infty]$ and $\mu(dt \times dx)$ is a non negative measure on $([0,1] \times E, \mathcal{B}([0,1]) \times \mathcal{E})$. The compensator $\nu(ds, dx)$ of $\mu(dt \times dx)$ separates into an intensity and a recovery rate distribution:

$$\nu(ds, dx) = I_{\{t<\tau\}} \lambda_t dt \Phi(x) dx \tag{83}$$

The random variable x is normally distributed $x \sim N\left(\mu_x, \sigma_x^2\right)$ and $\Phi(x)$ is given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{1}{2}\left(\frac{x - \mu_x}{\sigma_x}\right)^2\right) \tag{84}$$

The normal assumption for x results in recovery rates being lognormally distributed; (67) is chosen for tractability and other distributions are possible. Using the definition of the loss rate $w_t(x) = 1 - R_s(x)$, (67) is equivalent to

$$D_t^d = 1 + \int_0^t \int_E D_s^d w_s(x) \mu(ds \times dx) \tag{85}$$

For any maturity date $T \in [t, 1]$,

$$D_T^d = e^{-xh_T} I_{\{\tau \leq T\}} + I_{\{\tau > T\}} \tag{86}$$

with $x \in E = [0, \infty]$. (53) in conjunction with (67) permits identification of the return process of the trading gains associated with the corporate instrument under the assumption of a continuous distribution for recovery rates:

$$\begin{aligned}
\frac{dV_{t,T}^d}{V_{t,T}^d} &= (r_t^d + b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T) - I_{\{t<\tau\}} \rho_t^{l,\lambda} a^l(t,T) a^\lambda(t,T))dt \qquad (87) \\
&\quad + a^l(t,T) dW_s^l + I_{\{t<\tau\}} a^\lambda(t,T) W_s^\lambda \\
&\quad + \int_E w_t(x) \left(\mu(ds, dx) \times \nu(ds, dx)\right)
\end{aligned}$$

This concludes the specification of contractual and non contractual payoffs associated with the discount securities which are being traded in the current market structure. Given that return processes relating to the complete set of payoffs from the securities have been explicitly specified, the next step in the analysis focuses on the identification of probability measures under which arbitrage between traded assets is precluded.

3.3. Absence of Arbitrage and Martingale Measures

Given the specification of traded assets in the current market model, the following assumption is added.

Assumption 3.11 There are no arbitrage opportunities in the zero coupon market.

No effort is made here to formalize the content of the above assumption, which is intended to capture the requirement that any trading strategy which is profitable ex ante will be subject to uncertainty. The assumption that markets are free of arbitrage translates into the requirement that normalized trading gains are martingales under a probability measure yet to be identified. While precursory arguments relating to this fact were formulated by Cox and Ross (1976), the formal links between absence of arbitrage and martingale theory were expounded in a string of seminal articles by Harrison and Kreps (1979), Kreps (1981), Harrison and Pliska (1981) and Harrison and Pliska (1983). Subsequently a number of authors have sought to establish similar equivalence propositions using generalized concepts of no arbitrage (e.g. No Free Lunch, No Free Lunch with Bounded Risk, No Free Lunch with Vanishing Risk) as well as generalized market structures. Contributions in this literature (see Back and Pliska (1991), Dalang, Morton and Willinger (1990), Delbaen (1994), Delbaen and Schachermayer (1994a), Delbaen and Schachermayer (1994b), Dybvig and Huang (1988), Jarrow and Madan (1991), Jarrow and Madan (1994), Stricker (1990), Schachermayer (1994) and Lakner (1993)) differ with respect to their economic content and are not reviewed here. The probability measure of interest has been denoted as the risk neutral, or equivalent martingale, measure in the literature.

Definition 3.4 A probability measure P^* is called an equivalent martingale measure if the probability measure is equivalent to the original measure: $P^* \sim P$ and normalized security prices are P^*-martingales.

The first condition in the above definition formalizes the requirements that both probability measures attribute positive mass to given events; the null sets of the two measures must be identical. The second condition specifies that on average, investors should expect to earn zero profits on their trading strategies. The remainder of the current section is devoted to the identification of equivalent martingale measures associated with the use of different numeraire securities. Eligibility of a security as a numeraire assets is determined by the requirement that the expected rate of return on this asset must be given by the rate at which investors can borrow and lend freely without incurring any credit risk. Given the assumption of short sale constraints on treasury securities in the present framework, Libor instruments of differing maturities will serve as numeraire assets for the purpose of normalizing security prices.

3.3.1. Spot Risk Measure

The current section is devoted to the derivation of a probability measure under which discount securities, normalized by the value of a money market deposit, are martingales.

It is assumed that a money market account, accruing at the instantaneous expected rate of return on the Libor asset, is available to investors. The value of this deposit,

A_t^l, is given by

$$A_t^l = \exp \int_0^t r_s^l ds \tag{88}$$

Using (61), (62) and (73), normalized return processes for trading gains associated with treasury and Libor assets are given as follows:

$$\frac{d\left(V_{t,T}^g / A_t^l\right)}{V_{t,T}^g / A_t^l} = (r_t^g + y_t - r_t^l + b^g(t,T))dt + a^g(t,T) q_t^{g1} dM_t^{1,c} \tag{89}$$

$$\frac{d\left(V_{t,T}^l / A_t^l\right)}{V_{t,T}^l / A_t^l} = b^l(t,T)dt + a^l(t,T) \sum_{i=1}^{2} q_t^{li} dM_t^{i,c} \tag{90}$$

The respective processes associated with trading gains from corporate positions corresponding to degenerate, finite-valued and continuous recovery rate distributions are given as follows:

$$\begin{aligned}
\frac{d\left(V_{t,T}^d / A_t^l\right)}{V_{t,T}^d / A_t^l} &= (b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T) + a^l(t,T) \sum_{i=1}^{2} q_t^{li} dM_t^{i,c} \\
&\quad - I_{\{t<\tau\}} a^l(t,T) w_t a^\lambda(t,T) \sum_{i=1}^{3} q_t^{li} q_t^{\lambda i})dt \\
&\quad + I_{\{t<\tau\}} w_t a^\lambda(t,T) \sum_{i=1}^{3} q_t^{\lambda i} dM_t^{i,c} + w_t \left(dN_t - \lambda_{t\wedge\tau} dt\right)
\end{aligned} \tag{91}$$

$$\begin{aligned}
\frac{d\left(V_{t,T}^d / A_t^l\right)}{V_{t,T}^d / A_t^l} &= (b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T)dt \\
&\quad - I_{\{t<\tau\}} a^l(t,T) w_t a^\lambda(t,T) \sum_{i=1}^{3} q_t^{li} q_t^{\lambda i} dt \\
&\quad + a^l(t,T) \sum_{i=1}^{2} q_t^{li} dM_t^{i,c} + I_{\{t<\tau\}} w_t a^\lambda(t,T) \sum_{i=1}^{3} q_t^{\lambda i} dM_t^{i,c} \\
&\quad + \sum_{i=1}^{m} \left(e^{-x^i h_t} - 1\right) \left(dN_t(i) - \lambda_{t\wedge\tau}(i) dt\right)
\end{aligned} \tag{92}$$

$$\begin{aligned}
\frac{d\left(V_{t,T}^d / A_t^l\right)}{V_{t,T}^d / A_t^l} &= (b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T)dt \\
&\quad - I_{\{t<\tau\}} a^l(t,T) w_t a^\lambda(t,T) \sum_{i=1}^{3} q_t^{li} q_t^{\lambda i} dt \\
&\quad + a^l(t,T) \sum_{i=1}^{2} q_t^{li} dM_t^{i,c} + I_{\{t<\tau\}} w_t a^\lambda(t,T) \sum_{i=1}^{3} q_t^{\lambda i} dM_t^{i,c} \\
&\quad + \int_E w_t(x) \left(\mu(ds,dx) \times \nu(ds,dx)\right)
\end{aligned} \tag{93}$$

(76)-(78) demonstrate how the distribution of credit losses impact the return distribution of the credit risky zero coupon bond. In the first case, the normalized return process is subject to a single jump. (77) expands the recovery rate distribution to allow for a finite number of jumps and (78) generalizes the other two cases by allowing for an arbitrary support for the recovery rate distribution. Corresponding to each of the processes (76)-(78), the question of existence and uniqueness of martingale measures will be examined below.

3.3.1.1. Density Processes

I consider the structure of martingale measure densities corresponding to (74)-(78). Questions of existence and uniqueness of such densities relate to the hedgeability of risks embedded in the tradeable securities. It must be noted that in addition to the market risk component, the corporate instrument is subject to credit risk arising from two components. Prior to default, the corporate instrument is subject to uncertainty arising from random changes in the probability of default. At the time of default, investors face uncertainty arising from stochastic recovery rates. These facts generate the seeming contradiction that both components of credit risk must be hedged with one assets, a requirement which is inconsistent with intuition. The key to the resolution of this puzzle is the fact that the components of credit risk generate uncertainty on the non-overlapping time sets $\{t < \tau\}$ and $\{t \geq \tau\}$; by considering analysis on each of these subsets in turn, issues relating to the hedgeability of credit risk components can be illuminated.

3.3.1.1.1. Degenerate Recovery Rate Distribution

The following assumptions relate to the existence of market prices of risk prior to default and at the time of default.

Assumption 3.12 (Existence of market prices of risk prior to default) On $\{t < \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{94}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) \\ b^l(t,T) \\ b^l(t,T) + b^\lambda(t,T) - a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{li} q_t^{\lambda i} - w_t\lambda_t \end{pmatrix} \tag{95}$$

is the matrix of excess returns on the three discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & 0 & 0 \\ 0 & a^l(t,T) & 0 \\ 0 & a^l(t,T) & w_t a^\lambda(t,T) \end{pmatrix} \tag{96}$$

$$\mathbf{\Pi}_3 = \mathbf{Q}_t \tag{97}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \\ \theta_t^{3,c} \end{pmatrix} \tag{98}$$

contains the market prices of risk arising from the presence of uncertainty generated by the continuous martingale vector $\mathbf{M}_t^c$.

Assumption 3.13 (Existence of market prices of risk at the time of default) On $\{t = \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds:

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{99}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) \\ b^l(t,T) \\ b^l(t,T) + b^\lambda(t-,T) - a^l(t-,T)w_{t-}a^\lambda(t-,T)\sum_{i=1}^{3} q_{t-}^{li} q_{t-}^{\lambda i} - w_{t-}\lambda_{t-} \end{pmatrix} \tag{100}$$

is the matrix of excess returns on the three discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & 0 & 0 \\ 0 & a^l(t,T) & 0 \\ 0 & a^l(t,T) & w_t \end{pmatrix} \tag{101}$$

$$\mathbf{\Pi}_3 = \begin{pmatrix} q_t^{g1} & 0 & 0 \\ q_t^{l1} & q_t^{l2} & 0 \\ q_t^{l1} & q_t^{l2} & 1 \end{pmatrix} \tag{102}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \\ \lambda_t \theta_t^d \end{pmatrix} \tag{103}$$

contains the market prices of risk arising from the presence of uncertainty generated by the martingale vector $\left(M_t^{c,1}, M_t^{c,2}, M_t^d\right)$, where $M_t^d = dN_t - \lambda_t dt$.

The market price of risk at the time of default involves terms relating to the recovery rate as well as terms relating to volatility parameters of the default intensity and, because of the explicit correlation modelling, volatility parameters of the risk free forward rate process. Such a result is consistent with recent no arbitrage approaches of bond pricing in the presence of jumps, such as Shimko (1995) or Jarrow and Madan (1995).

Assumption 3.14 (Existence of market prices of risk after the time of default) On $\{t > \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{104}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) \\ b^l(t,T) \end{pmatrix} \tag{105}$$

is the matrix of excess returns on the treasury and Libor discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & 0 \\ 0 & a^l(t,T) \end{pmatrix} \tag{106}$$

$$\mathbf{\Pi}_3 = \begin{pmatrix} q_t^{g1} & 0 \\ q_t^{l1} & q_t^{l2} \end{pmatrix} \tag{107}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \end{pmatrix} \tag{108}$$

contains the market prices of risk arising from the presence of uncertainty generated by the continuous martingale vector $\left(M_t^{c,1}, M_t^{c,2}\right)$.

Note that after the time of default there is no uncertainty relating to credit risk; the only source of uncertainty is given by the market risk embedded in the treasury and Libor security. (79) and (89) can be expanded as follows:

$$b^g(t,T) + a^g(t,T)q_t^{g1}\theta_t^{1,c} = 0 \tag{109}$$

$$b^l(t,T) + a^l(t,T)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} = 0 \tag{110}$$

$$\begin{aligned} 0 &= b^l(t,T) + b^\lambda(t,T) - a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda i} - w_t\lambda_t \\ &\quad + a^l(t,T)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{\lambda i}\theta_t^{i,c} \end{aligned} \tag{111}$$

$$
\begin{aligned}
0 &= b^l(t,T) + b^\lambda(t-,T) - a^l(t-,T)w_{t-}a^\lambda(t-,T)\sum_{i=1}^{3} q_{t-}^{li} q_{t-}^{\lambda i} - w_{t-}\lambda_{t-} \\
&\quad + a^l(t,T)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + w_t a^\lambda(t,T)\lambda_t\theta_t^d \qquad (112)
\end{aligned}
$$

Differentiating (89)-(92) with respect to maturity yields arbitrage-free forward rate drift restrictions for the three discount securities:

$$
\alpha^g(t,T) = -\sigma^g(t,T)\left(q_t^{g1}\theta_t^{1,c} + a^g(t,T)\right) \qquad (113)
$$

$$
\alpha^l(t,T) = -\sigma^l(t,T)\left(\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + a^l(t,T)\right) \qquad (114)
$$

On $\{t < \tau\}$, drift restrictions relating to instantaneous corporate forward rates are given by

$$
\begin{aligned}
\alpha^\lambda(t,T) &= -\sigma^\lambda(t,T)\left(\sum_{i=1}^{3} q_t^{\lambda i}\theta_t^{i,c} + w_t a^\lambda(t,T)\right) + \qquad (115) \\
&\quad \left(\sum_{i=1}^{3} q_t^{\lambda i} q_t^{li}\right)\left(a^l(t,T)\sigma^\lambda(t,T) + \sigma^l(t,T)a^\lambda(t,T)\right)
\end{aligned}
$$

whereas on $\{t = \tau\}$, the corresponding restrictions are given by

$$
\begin{aligned}
\alpha^\lambda(t-,T)w_{t-} &= -\sigma^\lambda(t,T)w_t\lambda_t\theta_t^d + w_{t-}^2 a^\lambda(t-,T)\sigma^\lambda(t-,T) + \qquad (116) \\
&\quad w_{t-}\left(\sum_{i=1}^{3} q_t^{\lambda i} q_t^{li}\right)\left(a^l(t-,T)\sigma^\lambda(t-,T) + \sigma^l(t-,T)a^\lambda(t-,T)\right)
\end{aligned}
$$

Consider (100) and (10); these equations formalize drift restrictions relating to the drift of the forward probability of default. Coupled with (99), these restrictions mean that, under the risk neutral measure, the drift of the instantaneous forward rates of the corporate assets are fully specified in terms of volatility parameters (note that w_t is like a volatility parameter for the Poisson process). This result can be interpreted as extending the analysis of Heath, Jarrow and Morton from market risk to credit risk.

While market prices relating to the various sources of risk may exist, they may not be unique. The following assumption must be added.

Assumption 3.15 (Uniqueness of Market Prices of Risk) The matrix products $\mathbf{\Pi}_2\mathbf{\Pi}_3$ in (79), (84) and (89) are nonsingular.

The above assumption guarantees that the relevant matrices can be inverted and unique solutions for the market prices of risk as in (83), (98) and (93) exist. Given the assumptions relating to existence and uniqueness of the market prices of risk, (74)-(76) can be expressed as follows:

$$\frac{d\left(V^g_{t,T}/A^l_t\right)}{V^g_{t,T}/A^l_t} = a^g(t,T)q^{g1}_t\left(dM^{1,c}_t - \theta^{1,c}_t dt\right) + (r^g_t + y_t - r^l_t)dt \tag{117}$$

$$\frac{d\left(V^l_{t,T}/A^l_t\right)}{V^l_{t,T}/A^l_t} = a^l(t,T)\sum_{i=1}^{2} q^{li}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \tag{118}$$

$$\begin{aligned}\frac{d\left(V^d_{t,T}/A^l_t\right)}{V^d_{t,T}/A^l_t} &= a^l(t,T)\sum_{i=1}^{2} q^{li}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \\ &\quad + w_t\left(dN_t - \lambda_{t\wedge\tau}\theta^d_t dt\right) \\ &\quad + I_{\{t<\tau\}}a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \\ &= a^l(t,T)\sum_{i=1}^{2} q^{li}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \\ &\quad + I_{\{t<\tau\}}a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \\ &\quad + w_t\lambda_{t\wedge\tau}(1-\theta^d_t)dt + w_t dM^d_t\end{aligned} \tag{119}$$

where $dM^d_t = dN_t - \lambda_{t\wedge\tau}\theta^d_t dt$. (102)-(106) illustrate the familiar notion that expected returns on normalized assets are given by the sum of the risk premia corresponding to the various sources of risk. The following proposition identifies the density process defining the equivalent martingale measure subject to the assumptions of the current section.

Proposition 3.1 There exists a unique equivalent martingale measure P^* under which the processes $V^g_{t,T}/A^l_t$, $V^l_{t,T}/A^l_t$ and $V^d_{t,T}/A^l_t$ in (102)-(106) are martingales. The density process defining this measure is given by

$$\begin{aligned}Z^*_t &= Z^*_0 \times \\ &\quad \mathcal{E}\left(\sum_{i=1}^{2}\int_0^t Z^*_{s-}\theta^{i,c}_s dM^{i,c}_s + \int_0^t I_{\{s<\tau\}}Z^*_s\theta^{3,c}_s dM^{3,c}_s + \int_0^t Z^*_{s-}(\theta^d_s - 1)dM^d_s\right)\end{aligned} \tag{120}$$

where $\mathcal{E}(.)$ is the Doleans-Dade exponential. The solution to (107) is given by

$$\begin{aligned}Z^*_t &= Z^*_0 \\ &\quad \times\exp\left(\int_0^t I_{\{s<\tau\}}v(\theta^d_s - 1)\lambda_t ds + \int_0^t I_{\{s<\tau\}}\ln\theta^d_s dN_s\right) \\ &\quad \times\exp\left(\sum_{i=1}^{2}\int_0^t \theta^{i,c}_s dM^{i,c}_s - \frac{1}{2}\sum_{i=1}^{2}\int_0^t (\theta^{i,c}_s)^2 ds\right)\end{aligned} \tag{121}$$

$$\times \exp\left(\int_0^t I_{\{s<\tau\}} Z^*_{s-}\theta^{3,c}_s dM^{3,c}_s - \frac{1}{2}\int_0^t I_{\{s<\tau\}}(\theta^{3,c}_s)^2 ds\right)$$

and defines the spot risk measure through

$$\frac{dP^*}{dP} = Z^*_1 \tag{122}$$

Furthermore, the following restriction holds:

$$r^g_t + y_t - r^l_t = 0 \tag{123}$$

Proof: See appendix.□

Remark 3.6 It is easily verified that the density process identified in the above proposition facilitates expression of expected excess returns on traded asset as the negative of the covariation of asset returns and the return process associated with the martingale density:

$$E\left[\frac{d(V^k_{t,T}/A^l_t)}{V^k_{t,T}/A^l_t}\right] = -[\frac{dV^k_{t,T}}{V^k_{t,T}}, \frac{dZ^*_t}{Z^*_t}] \tag{124}$$

where $k \in \{g, l, d\}$ and $[.,.]$ is the optional quadratic covariation process, extending similar results from Back (1991) and Jarrow and Madan (1994) to the case of pricing operators associated with credit risk.

Under the spot risk measure return processes of trading gains associated with the three discount instruments are given by

$$\frac{dV^g_{t,T}}{V^g_{t,T}} = r^l_t dt + a^g(t,T)q^{g1}_t dM^{1,c,*}_t \tag{125}$$

$$\frac{dV^l_{t,T}}{V^l_{t,T}} = r^l_t dt + a^l(t,T)\sum_{i=1}^{2} q^{li}_t dM^{i,c,*}_t \tag{126}$$

$$\begin{aligned}\frac{dV^d_{t,T}}{V^d_{t,T}} &= r^l_t dt + a^l(t,T)\sum_{i=1}^{2} q^{li}_t dM^{i,c,*}_t \\ &\quad + I_{\{t<\tau\}} a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t dM^{i,c,*}_t + w_t dM^{d,*}_t\end{aligned} \tag{127}$$

where the sources of risk are given by

$$M^{i,c,*}_t = M^{i,c}_t - \int_0^t \theta^{i,c}_s ds \tag{128}$$

$$M^{d,*}_t = M^d_t - \int_0^t \left(\theta^d_s - 1\right)\lambda_s ds \tag{129}$$

(125)-(127) reflect the fact that under the spot risk measure, expected instantaneous returns on assets are given by the risk free rate. The normalized processes (102)-(106)

evolve according to

$$\frac{d\left(V^g_{t,T}/A^l_t\right)}{V^g_{t,T}/A^l_t} = a^g(t,T)q^{g1}_t dM^{1,c,*}_t \tag{130}$$

$$\frac{d\left(V^l_{t,T}/A^l_t\right)}{V^l_{t,T}/A^l_t} = a^l(t,T)\sum_{i=1}^{2} q^{li}_t dM^{i,c,*}_t \tag{131}$$

$$\begin{aligned}\frac{d\left(V^d_{t,T}/A^l_t\right)}{V^d_{t,T}/A^l_t} &= a^l(t,T)\sum_{i=1}^{2} q^{li}_t dM^{i,c,*}_t \\ &\quad + I_{\{t<\tau\}}a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t dM^{i,c,*}_t + w_t dM^{d,*}_t\end{aligned} \tag{132}$$

For future reference, (130)-(132) can be integrated to identify normalized prices as

$$\frac{V^g_{T,T}}{A^l_T} = \frac{V^g_{t,T}}{A^l_t} \times \exp\left(\int_t^T a^g(s,T)q^{g1}_s dM^{1,c,*}_s - \frac{1}{2}\int_t^T a^g(s,T)^2 ds\right) \tag{133}$$

$$\frac{V^l_{T,T}}{A^l_T} = \frac{V^l_{t,T}}{A^l_t} \times \exp\left(\sum_{i=1}^{2}\int_t^T a^l(s,T)q^{li}_s dM^{i,c,*}_s - \frac{1}{2}\sum_{i=1}^{2}\int_t^T a^l(s,T)^2 ds\right) \tag{134}$$

$$\begin{aligned}\frac{V^d_{T,T}}{A^l_T} &= \frac{V^d_{t,T}}{A^l_t} \\ &\times \exp\left(\sum_{i=1}^{3}\int_t^{T\wedge\tau} a^\lambda(s,T)a^l(s,T)q^{\lambda i}_s q^{li}_s - \int_t^T I_{\{s<\tau\}}w_s\lambda^*_s ds - k\int_t^T h_s dN_s\right) \\ &\times \exp\left(\sum_{i=1}^{2}\int_t^T a^l(s,T)q^{li}_s dM^{i,c,*}_s - \frac{1}{2}\sum_{i=1}^{2}\int_t^T a^l(s,T)^2 ds\right) \\ &\times \exp\left(\sum_{i=1}^{3}\int_t^T I_{\{s<\tau\}}a^\lambda(s,T)q^{\lambda i}_s dM^{i,c,*}_s - \frac{1}{2}\sum_{i=1}^{2}\int_t^T I_{\{s<\tau\}}a^\lambda(s,T)^2 ds\right)\end{aligned} \tag{135}$$

The latter expressions will prove useful for the purpose of pricing specific contracts. While the probability measure identified in the above proposition will subsequently serve as the main pricing tool, the impact of alternative specifications for the recovery rate distribution is briefly considered.

3.3.1.1.2. Finite-valued Recovery Rate Distribution

When the recovery rate distributions permits a finite number m of possible realizations, a sufficient number of securities spanning this uncertainty must be available. I assume this is the case.

Assumption 3.16 There exists a number m of zero coupon bonds $B^{i,d}_{t,T}$, $i \in \{1...m\}$, issued by the same corporation who issued $B^d_{t,T}$. The normalized return process associated with trading gains on bond i is given by

$$
\begin{aligned}
\frac{d\left(V_{t,T}^{i,d}/A_t^l\right)}{V_{t,T}^{i,d}/A_t^l} &= (b^l(t,T) + I_{\{t<\tau\}}b^\lambda(t,T)dt \\
&\quad -I_{\{t<\tau\}}a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda i}dt \\
&\quad +a^l(t,T)\sum_{i=1}^{2} q_t^{li}dM_t^{i,c} + I_{\{t<\tau\}}w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{\lambda i}dM_t^{i,c} \\
&\quad \left(e^{-x^i h_t} - 1\right)(dN_t(i) - \lambda_{t\wedge\tau}(i)dt)
\end{aligned} \tag{136}
$$

The above assumption is a precondition to the existence of market prices of risk for the various sources of uncertainty.

Assumption 3.17 (Existence of market prices of risk prior to default) On $\{t < \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds

$$
\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{137}
$$

The matrix

$$
\underset{(m+2)\times 3}{\mathbf{\Pi}_1} = \begin{pmatrix} b^g(t,T) \\ b^l(t,T) \\ b^l(t,T) + b^{\lambda(1)}(t,T) - a^l(t,T)w_t(1)a^{\lambda(1)}(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda(1)i} - w_t(1)\lambda_t(1) \\ \vdots \\ b^l(t,T) + b^{\lambda(m)}(t,T) - a^l(t,T)w_t(m)a^{\lambda(m)}(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda(m)i} - w_t(m)\lambda_t(m) \end{pmatrix} \tag{138}
$$

is the matrix of excess returns on the treasury and Libor instruments and the m discount assets from (136); the corresponding volatility matrix is given by

$$
\underset{(m+2)\times 3}{\mathbf{\Pi}_2} = \begin{pmatrix} a^g(t,T) & 0 & 0 \\ 0 & a^l(t,T) & 0 \\ 0 & a^l(t,T) & w_t(1)a^{\lambda(1)}(t,T) \\ \vdots & \vdots & \vdots \\ 0 & a^l(t,T) & w_t(m)a^{\lambda(m)}(t,T) \end{pmatrix} \tag{139}
$$

$$
\mathbf{\Pi}_3 = \mathbf{Q}_t \tag{140}
$$

is the correlation matrix and

$$
\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \\ \theta_t^{3,c} \end{pmatrix} \tag{141}
$$

contains the market prices of risk arising from the presence of uncertainty generated by the continuous martingale vector $\mathbf{M}_t^c$.

Assumption 3.18 (Existence of market prices of risk at the time of default) On $\{t=\tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds:

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{142}$$

The matrix

$$\underset{(m+2)\times 3}{\mathbf{\Pi}_1} = \begin{pmatrix} b^g(t,T) \\ b^l(t,T) \\ b^l(t,T) + b^{\lambda(1)}(t-,T) - a^l(t-,T)w_{t-}(1)a^{\lambda(1)}(t-,T)\sum_{i=1}^{3} q_{t-}^{li}q_{t-}^{\lambda(1)i} - w_{t-}(1)\lambda_{t-}(1) \\ \vdots \\ b^l(t,T) + b^{\lambda(m)}(t-,T) - a^l(t-,T)w_{t-}(m)a^{\lambda(m)}(t-,T)\sum_{i=1}^{3} q_{t-}^{li}q_{t-}^{\lambda(m)i} - w_{t-}(m)\lambda_{t-}(m) \end{pmatrix} \tag{143}$$

is the matrix of excess returns on the three discount assets, the volatility matrix is given by

$$\underset{(m+2)\times 3}{\mathbf{\Pi}_2} = \begin{pmatrix} a^g(t,T) & 0 & 0 \\ 0 & a^l(t,T) & 0 \\ 0 & a^l(t,T) & w_t(1) \\ \vdots & \vdots & \vdots \\ 0 & a^l(t,T) & w_t(m) \end{pmatrix} \tag{144}$$

$$\underset{3\times(m+2)}{\mathbf{\Pi}_3} = \begin{pmatrix} q_t^{g1} & 0 & 0 & \cdots & 0 \\ q_t^{l1} & q_t^{l2} & 0 & \cdots & 0 \\ q_t^{l1} & q_t^{l2} & 1 & \cdots & 1 \end{pmatrix} \tag{145}$$

is the correlation matrix and

$$\underset{(m+2)\times 1}{\mathbf{\Pi}_4} = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \\ \lambda_t(1)\theta_t^{1,d}(x^1) \\ \vdots \\ \lambda_t(m)\theta_t^{m,d}(x^m) \end{pmatrix} \tag{146}$$

where $\theta_t^{i,d}(x^i) = \theta_t^{i,d}\theta_t^d(x^i)$, contains the market prices of risk arising from the presence of uncertainty generated by the martingale vector $\left(M_t^{c,1}, M_t^{c,2}, M_t^{d,1}, ..., M_t^{d,m}\right)$, where $M_t^{d,i} = dN_t(i) - \lambda_t(i)dt$.

Assumption 3.19 (Existence of market prices of risk after the time of default) On $\{t > \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{147}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) \\ b^l(t,T) \end{pmatrix} \tag{148}$$

is the matrix of excess returns on the treasury and Libor discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & 0 \\ 0 & a^l(t,T) \end{pmatrix} \tag{149}$$

$$\mathbf{\Pi}_3 = \begin{pmatrix} q_t^{g1} & 0 \\ q_t^{l1} & q_t^{l2} \end{pmatrix} \tag{150}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \end{pmatrix} \tag{151}$$

contains the market prices of risk arising from the presence of uncertainty generated by the continuous martingale vector $\left(M_t^{c,1}, M_t^{c,2}\right)$.

Note that after the time of default there is no uncertainty relating to credit risk; the only source of uncertainty is given by the market risk embedded in the treasury and Libor security. (79) and (89)can be expanded as follows:

$$b^g(t,T) + a^g(t,T)q_t^{g1}\theta_t^{1,c} = 0 \tag{152}$$

$$b^l(t,T) + a^l(t,T)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} = 0 \tag{153}$$

On $\{t < \tau\}$, (137) yields, for $i \in \{1...m\}$,

$$\begin{aligned} 0 \;=\;& b^l(t,T) + b^{\lambda(i)}(t,T) - a^l(t,T)w_t(i)a^{\lambda(i)}(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda(i)i} \\ & -w_t(i)\lambda_t(i) + a^l(t,T)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + w_t(i)a^{\lambda(i)}(t,T)\sum_{i=1}^{3} q_t^{\lambda(i)i}\theta_t^{i,c} \end{aligned} \tag{154}$$

whereas on $\{t = \tau\}$, (142) yields the restriction

$$\begin{aligned} 0 \;=\;& b^l(t,T) + b^{\lambda(i)}(t-,T) - a^l(t-,T)w_{t-}(i)a^{\lambda(i)}(t-,T)\sum_{i=1}^{3} q_{t-}^{li}q_{t-}^{\lambda(i)i} \\ & -w_{t-}(i)\lambda_{t-}(i) + a^l(t,T)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + w_t(i)a^{\lambda(i)}(t,T)\lambda_{t-}(i)\theta_t^{d,i}(x) \end{aligned} \tag{155}$$

Differentiating (152)-(155) with respect to maturity yields arbitrage-free forward rate drift restrictions for the case of finite-valued recovery rate distributions similar to (98)-(101) and is not pursued at this point. As for the case of fixed recovery rates, a non-degeneracy assumption must be added on order to guarantee uniqueness of the spot risk measure.

Assumption 3.20 (Uniqueness of Market Prices of Risk) The matrix products $\mathbf{\Pi}_2\mathbf{\Pi}_3$ in (137), (142) and (147) are non-singular..

The above assumption guarantees that the relevant matrices can be inverted and unique solutions for the market prices of risk as in (141), (146) and (151) exist on the relevant time sets. Given the assumptions relating to existence and uniqueness of the market prices of risk, normalized return processes can be expressed as follows:

$$\frac{d\left(V^g_{t,T}/A^l_t\right)}{V^g_{t,T}/A^l_t} = a^g(t,T)q^{g1}_t\left(dM^{1,c}_t - \theta^{1,c}_t dt\right) + (r^g_t + y_t - r^l_t)dt \tag{156}$$

$$\frac{d\left(V^l_{t,T}/A^l_t\right)}{V^l_{t,T}/A^l_t} = a^l(t,T)\sum_{i=1}^{2} q^{li}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \tag{157}$$

$$\begin{aligned}\frac{d\left(V^d_{t,T}/A^l_t\right)}{V^d_{t,T}/A^l_t} &= a^l(t,T)\sum_{i=1}^{2} q^{li}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \\ &\quad + I_{\{t<\tau\}} w_t a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t\left(dM^{i,c}_t - \theta^{i,c}_t dt\right) \\ &\quad + \sum_{i=1}^{m}\left(e^{-x' h_t} - 1\right)\left(dN_t(i) - \lambda_{t\wedge\tau}(i)\theta^{i,d}_t(x)dt\right)\end{aligned} \tag{158}$$

Proposition 3.2 There exists a unique equivalent martingale measure P^* under which the processes $V^g_{t,T}/A^l_t$, $V^l_{t,T}/A^l_t$ and $V^d_{t,T}/A^l_t$ in (156)-(158) are martingales. The density process defining this measure is given by

$$\begin{aligned} Z^*_t &= Z^*_0 \times \\ &\mathcal{E}\left(\sum_{i=1}^{2}\int_0^t Z^*_{s-}\theta^{i,c}_s dM^{i,c}_s + \int_0^t I_{\{s<\tau\}} Z^*_s \theta^{3,c}_s dM^{3,c}_s + \sum_{i=1}^{m}\int_0^t Z^*_{s-}(\theta^{i,d}_s(x) - 1)dM^{i,d}_s\right)\end{aligned} \tag{159}$$

where $\mathcal{E}(.)$ is the Doleans-Dade exponential. The solution of (159) is given by

$$\begin{aligned} Z^*_t &= Z^*_0 \times \\ &\prod_{i=1}^{m}\exp\left(\int_0^t I_{\{s<\tau\}}(\theta^{i,d}_s(x) - 1)\lambda_t(i)ds + \int_0^t I_{\{s<\tau\}}\ln\theta^{i,d}_s(x)dN_s(i)\right) \\ &\times\exp\left(\sum_{i=1}^{2}\int_0^t \theta^{i,c}_s dM^{i,c}_s - \frac{1}{2}\sum_{i=1}^{2}\int_0^t (\theta^{i,c}_s)^2 ds\right)\end{aligned} \tag{160}$$

$$\times \exp\left(\int_0^t I_{\{s<\tau\}} Z^*_{s-}\theta^{3,c}_s dM^{3,c}_s - \frac{1}{2}\int_0^t I_{\{s<\tau\}}(\theta^{3,c}_s)^2 ds\right)$$

and defines the spot risk measure through

$$\frac{dP^*}{dP} = Z^*_1 \tag{161}$$

Furthermore, the following restriction holds:

$$r^g_t + y_t - r^l_t = 0 \tag{162}$$

Proof: Identical to the proof relating to fixed recovery rates.□

Under the measure identified in the above proposition, (158) is equivalent to

$$\begin{aligned}\frac{d\left(V^d_{t,T}/A^l_t\right)}{V^d_{t,T}/A^l_t} &= a^l(t,T)\sum_{i=1}^{2} q^{li}_t dM^{i,c,*}_t \\ &+ I_{\{t<\tau\}} w_t a^\lambda(t,T)\sum_{i=1}^{3} q^{\lambda i}_t dM^{i,c,*}_t \\ &+ \sum_{i=1}^{m}\left(e^{-x^i h_t} - 1\right) dM^{i,d,*}_t\end{aligned} \tag{163}$$

where the risk sources under P^* are given by

$$dM^{i,c,*}_t = dM^{i,c}_t - \theta^{i,c}_t dt \tag{164}$$

$$dM^{i,d,*}_t = dN_t(i) - \lambda_{t\wedge\tau}(i)\theta^{i,d}_t(x^i)dt \tag{165}$$

Note that $\theta^{i,d}_t(x) = \theta^{i,d}_t\theta^d_t(x)$. $\theta^{i,d}_t$ has the interpretation of the price of uncertainty relating to the probability of a recovery rate realization of $R_t(x^i)$ occurring; $\theta^d_t(x)$ is the price of risk relating to the uncertainty as to the realization of $R_t(x^i)$ itself. The above analysis is relevant because it makes explicit that in order to hedge uncertainty in recovery rates, assets spanning this uncertainty must be traded. If recovery rates take on a finite number of values, this assumption may not be unrealistic. If the support of the recovery rate distribution becomes infinity, such an assumption is less tenable and one must conclude that credit risk in that case is not fully hedgeable.

Continuous Recovery Rate Distribution

In this section the case of a continuous distribution for recovery rates is considered. The return process relating to trading gains associated with the corporate bond under this assumption is given by

$$\frac{d\left(V^d_{t,T}/A^l_t\right)}{V^d_{t,T}/A^l_t} = (b^l(t,T) + I_{\{t<\tau\}} b^\lambda(t,T)dt \tag{166}$$

$$-I_{\{t<\tau\}}a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda i}dt$$
$$+a^l(t,T)\sum_{i=1}^{2} q_t^{li}dM_t^{i,c} + I_{\{t<\tau\}}w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{\lambda i}dM_t^{i,c}$$
$$+\int_E w_t(x)\left(\mu(ds,dx)-\nu(ds,dx)\right)$$

where $E=[0,\infty]$. Because the number of possible recovery rate realizations is infinite, existence of market prices of risk relating to uncertain recovery rates cannot be established. For illustrative purposes it was assumed above that $x \sim N\left(\mu_x, \sigma_x^2\right)$; the following theorem can then be proven.

Theorem 3.1 There exists a class of equivalent martingale measures P^*, indexed by μ_x^*, under which the processes $V_{t,T}^g/A_t^l$, $V_{t,T}^l/A_t^l$ and $V_{t,T}^d/A_t^l$ in (156), (157) and (166) are martingales. The density processes defining these measures are given by

$$\begin{aligned} Z_t^* &= Z_0^* \\ &\times\mathcal{E}\left(\sum_{i=1}^{2}\int_0^t Z_{s-}^*\theta_s^{i,c}dM_s^{i,c} + \int_0^t I_{\{s<\tau\}}Z_s^*\theta_s^{3,c}dM_s^{3,c}\right) \\ &\times\mathcal{E}\left(\sum_{i=1}^{m}\int_0^t Z_{s-}^*(\theta_s^d\theta_s(x)-1)(\mu(ds,dx)-\nu(ds,dx))\right) \end{aligned} \tag{167}$$

where $\mathcal{E}\,(.)$ is the Doleans- Dade exponential and

$$\theta_s(x)=\frac{\Phi^*(x)}{\Phi(x)}=\exp-\frac{(\mu_x^*-\mu_x)}{2\sigma_x^2}\left(\mu_x^*+\mu_x-2x\right) \tag{168}$$

The solution of (167) is given by

$$\begin{aligned} Z_t^* &= Z_0^* \\ &\times\exp\left(\int_0^t I_{\{s<\tau\}}(\theta_s^d\theta_s(x)-1)\nu(ds,dx)+\int_0^t I_{\{s<\tau\}}\ln\theta_s^d\theta_s(x)\mu(ds,dx)\right) \\ &\times\exp\left(\sum_{i=1}^{2}\int_0^t \theta_s^{i,c}dM_s^{i,c}-\frac{1}{2}\sum_{i=1}^{2}\int_0^t(\theta_s^{i,c})^2ds\right) \\ &\times\exp\left(\int_0^t I_{\{s<\tau\}}Z_{s-}^*\theta_s^{3,c}dM_s^{3,c}-\frac{1}{2}\int_0^t I_{\{s<\tau\}}(\theta_s^{3,c})^2ds\right) \end{aligned} \tag{169}$$

and defines the spot risk measure through

$$\frac{dP^*}{dP}=Z_1^* \tag{170}$$

Furthermore, the following restriction holds:

$$r_t^g+y_t-r_t^l=0 \tag{171}$$

Proof: Identical to the proof relating to fixed recovery rates.□

Under a probability measure P^* from the class of measures identified in the above theorem, the normalized return process for the trading strategy associated with the credit risky corporate instrument is given by

$$\frac{d\left(V_{t,T}^d/A_t^l\right)}{V_{t,T}^d/A_t^l} = a^l(t,T)\sum_{i=1}^{2} q_t^{li} dM_t^{i,c,*} + I_{\{t<\tau\}} w_t^* a\lambda(t,T)\sum_{i=1}^{3} q_t^{\lambda i} dM_t^{i,c,*} \tag{172}$$
$$+ \int_E w_t(x)\left(\mu(ds,dx) - \nu^*(ds,dx)\right)$$

where

$$\nu^*(dt,dx) = \lambda_t \theta_t^d dt \Phi^*(x)dx \tag{173}$$

and the expected recovery rate w_t^* under P^* is given by

$$w_t^* = \int_E w_t(x)\Phi^*(x)dx \tag{174}$$

Recovery rates under the new measure are lognormally distributed. Given that μ_x^* is arbitrary, this fact generates a whole class of measures. Lack of uniqueness of the martingale measure in this situation arises from an infinity of possible writedowns in the event of default. Unless one is willing to assume that an infinity of credit contingent hedging assets is available, pricing by arbitrage is not feasible.. The fact that markets are incomplete thus does not preclude the martingale characterization of asset prices but destroys the uniqueness properties.

3.3.1.2. Asset Values, Rate and Yield Processes under the Spot Risk Measure

In the preceding sections martingale measures under different assumptions relating to the distribution of recovery rates were examined. This analysis was relevant because in the case of a continuous recovery rate distribution, it is unlikely that the risk of uncertain recovery can be hedged and no-arbitrage arguments will fail. Having established this point, the current section is devoted to the characterization of various processes under the measure P^*. I start by considering the discount securities.

The price of the contractual component of the treasury security, to be identified by the superscript c, is given by

$$B_{t,T}^{g,c} = E_t^*\left[\exp\left(-\int_t^T r_s^g ds\right)\right] \tag{175}$$

The instrument that is available to investors who, due to some of the institutional features outlined above, are in a position to receive a convenience yield on their treasury positions, is given by:

$$B_{t,T}^{g} = E_t^{*}\left[\exp\left(-\int_t^T (r_s^l + y_s)ds\right)\right] \tag{176}$$

Although (175) and (176) have the same value, they are based on different contract specifications and are therefore denoted as different instruments.
The following lemma will clarify the payoff profile of the contractual components of the treasury security.

Lemma 3.4 (176) is equivalent to

$$B_{t,T}^{g} = E_t^{*}\left[\int_t^T y_u \exp\left(-\int_t^u r_s^l ds\right) du + \exp\left(-\int_t^T r_s^l ds\right)\right] \tag{177}$$

Proof:
Define

$$V_t^g = B_{t,T}^g / A_t^l \tag{178}$$

Taking the differential of (178) yields

$$dV_t^g = \left(dB_{t,T}^g - B_{t,T}^g r_t^l dt\right) / A_t^l \tag{179}$$

Integrating the above using (176) and taking expectations yields (177).□
While (176) and (177) have the same value, the contracts differ with respect to their payoff structure. $B_{t,T}^{g,c}$ is a discount instrument accruing at a rate r_s^g, whereas by inspection of (177), $B_{t,T}^g$ is a contract which accrues at a rate r_s^l, pays a continuous, positive dividend of y_t and makes a principal payment of unity at the maturity date.
With regard to the Libor security, trading gains are identical to contractual payoffs and we have

$$B_{t,T}^{l} = B_{t,T}^{l,c} = E_t^{*}\left[\exp\left(-\int_t^T r_s^l ds\right)\right] \tag{180}$$

The contractual component of the credit risky corporate bond is given by

$$B_{t,T}^{d,c} = E_t^{*}\left[\exp\left(-\int_t^T r_s^d ds\right)\right] \tag{181}$$

where

$$r_t^d = r_t^l - w_t \lambda_t^{*} \tag{182}$$

Note that (181) is identical to

$$B_{t,T}^{d,c} = E_t^* \left[\exp\left(-\int_t^T (r_s^l + (1 - e^{-kh_s})\lambda_s^*) ds \right) \right] \tag{183}$$

$B_{t,T}^{d,c}$ is the pre-default value of the credit risky corporate instrument. Such an instrument is not available for trading. The following lemma will help to clarify the payoff structure of the pre-default security.

Lemma 3.5 (181) is equivalent to

$$B_{t,T}^{d,c} = E_t^* \left[\int_t^T w_u \lambda_u^* \exp\left(-\int_t^u r_s^l ds \right) du + \exp\left(-\int_t^T r_s^l ds \right) \right] \tag{184}$$

Proof:

Define

$$V_t^d = B_{t,T}^{d,c} / A_t^l \tag{185}$$

Taking the differential of (185) yields

$$dV_t^d = \left(dB_{t,T}^{d,c} - B_{t,T}^{d,c} r_t^l dt \right) / A_t^l \tag{186}$$

Integrating the above using (181) and taking expectations yields (184).□

Note that $r_t^g = r_t^l + y_t$ and $r_t^d = r_t^l - w_t \lambda_t^*$; therefore, the following relations hold:

$$B_{t,T}^{d,c} \leq B_{t,T}^l \leq B_{t,T}^{g,c} \tag{187}$$

Given that $B_{t,T}^l \leq B_{t,T}^{g,c}$, and that the contract $B_{t,T}^{g,c}$ is available to a subset of investors, these investors could realize riskless profits by shorting the treasury asset and taking a long position in the Libor security for an instant profit of $B_{t,T}^{g,c} - B_{t,T}^l \geq 0$, while at maturity principal payments would offset each other. The assumption imposing short sale constraints serves precisely to disallow this strategy. On the other hand, $B_{t,T}^{d,c} \leq B_{t,T}^l$, and if $B_{t,T}^{d,c}$ was available, investors could realize riskless profits by shorting the Libor instrument and taking long positions in $B_{t,T}^{d,c}$ for an instant profit of $B_{t,T}^{d,c} - B_{t,T}^l \geq 0$ and no subsequent cashflows. While $B_{t,T}^{d,c}$ is not available, a similar trade using $B_{t,T}^d$ would generate identical profits if investors could liquidate their positions just before default occurs. However, the time of default being an inaccessible stopping time in the current framework, there is no sequence of predictable times announcing the time of default. Investors will not be able to liquidate their positions and capture the positive return differential $-w_t \lambda_t^*$ because there is no rule indicating at which time to do this.

Consider the corporate bond; under P^*, from (135):

$$V^d_{t,T} = E^*_t \left[\exp\left(-\int\limits_t^T r^l_s ds \right) V^d_{T,T} \right] \tag{188}$$

On $\{t \geq \tau\}$, (188) is identical to

$$B^d_{t,T} = e^{-xh_T} B^l_{t,T} \tag{189}$$

and e^{-xh_τ} is a realization of the stochastic recovery rate. On $\{t < \tau\}$, (188) is identical to

$$\begin{aligned}
B^d_{t,T} &= E^*_t \left[\exp\left(-\int\limits_t^T r^l_s ds \right) \frac{B^d_{T,T} D^d_T}{D^d_t} \right] \\
&= E^*_t \left[\exp\left(-\int\limits_t^T r^l_s ds \right) \exp\left(-x \int\limits_t^T h_s dN_s \right) \right] \\
&= E^*_t \left[\left(e^{-xh_T} I_{\{T \geq \tau\}} + I_{\{T<\tau\}} \right) \exp\left(-\int\limits_t^T r^l_s ds \right) \right] \\
&= E^*_t \left[\left(e^{-xh_T}(1 - I_{\{T<\tau\}}) + I_{\{T<\tau\}} \right) \exp\left(-\int\limits_t^T r^l_s ds \right) \right] \\
&= E^*_t \left[\left(e^{-xh_T} - (e^{-xh_\tau} - 1) I_{\{T<\tau\}} \right) \exp\left(-\int\limits_t^T r^l_s ds \right) \right]
\end{aligned}$$

which is identical to

$$B^d_{t,T} = E^*_t \left[e^{-xh_T} \exp\left(-\int\limits_t^T r^l_s ds \right) - (e^{-xh_T} - 1) \exp\left(-\int\limits_t^T (r^l_s + \lambda^*_s) ds \right) \right] \tag{190}$$

(190) denotes the price of a credit risky corporate bond in the presence of stochastic recovery rates and allowance for credit components to be correlated with interest rates. While desirability of such features has frequently expressed, a result of this nature is not available in the literature. In order to make explicit the role of the correlation parameter relative to other models, assume that the recovery rate distribution is degenerate and that the default intensity follows a Gaussian process. (190) is then equivalent to

$$B^d_{t,T} = e^{-kh_t} B^l_{t,T} + I_{\{t<\tau\}} E^*_t \left[w_t \exp\left(-\int\limits_t^T (r^d_s + \lambda^*_s) ds \right) \right] \tag{191}$$

Under the assumption that the recovery rate distribution is degenerate, the following proposition demonstrates the explicit impact of the correlation between credit risk and market risk on the valuation of a credit risky zero coupon bond.

Proposition 3.3 Assume that the recovery rate distribution is degenerate and that the default intensity follows a Gaussian process. (191) is then equivalent to

$$\begin{aligned} B^d_{t,T} &= B^l_{t,T} \times e^{-kh_T} + \\ & w_t B^l_{t,T} \times E^*_t \left[\exp\left(-\int_t^T \lambda^*_s ds \right) \right] \times \exp\left(\int_t^T a^\lambda(s,T) a^l(s,T) \rho^{\lambda l}_s \theta^d_s ds \right) \end{aligned} \tag{192}$$

Proof:

Note that under P^*:

$$r^l_t = f^l_{0,t} - \int_0^t \sigma^l(s,t) a^l(s,t) dt + \sum_{i=1}^{2} \int_0^t \sigma^g(s,t) q^{l_i}_s dM^{i,c,*}_s \tag{193}$$

$$\begin{aligned} \lambda^d_t &= \lambda^d_{0,t} - \int_0^t w_s \sigma^l(s,t) a^l(s,t) dt + \sum_{i=1}^{3} \int_0^t \sigma^\lambda(s,t) q^{\lambda i}_s dM^{i,c,*}_s \\ & + \int_0^t \left(\sum_{i=1}^{3} q^{\lambda_i}_s q^{li}_s \right) \left(a^l(s,t) \sigma^\lambda(s,t) + \sigma^l(s,t) a^\lambda(s,t) \right) \end{aligned} \tag{194}$$

Therefore

$$\begin{aligned} \int_t^T \left\langle r^l, \lambda^* \right\rangle_s ds &= \int_t^T \int_t^s \sigma^\lambda(u,s) \sigma^l(u,s) \theta^d_u \rho^{\lambda l}_u du ds \\ &= \int_t^T \int_u^T \sigma^\lambda(u,s) \sigma^l(u,s) \theta^d_u \rho^{\lambda l}_u ds du \\ &= \int_t^T a^\lambda(s,T) a^l(s,T) \theta^d_s \rho^{\lambda l}_s ds \end{aligned} \tag{195}$$

which, in conjunction with (191), yields (192).□

While the Gaussianity assumption is limiting, the above (192) is the first credit risky bond pricing equation providing an explicit role for a correlation between credit risk and market risk. If $\rho^{\lambda l}_s = 0$, (192) reduces to

$$\begin{aligned} B^d_{t,T} &= e^{-kh_T} B^l_{t,T} \times E^*_t \left[1 - \exp\left(-\int_t^T \lambda^*_s ds \right) \right] \\ & + B^l_{t,T} \times E^*_t \left[\exp\left(-\int_t^T \lambda^*_s ds \right) \right] \end{aligned} \tag{196}$$

If furthermore the intensity is assumed to be deterministic, one arrives at the result of Jarrow and Turnbull (1995):

$$B^d_{t,T} = B^l_{t,T} \times \left(1 - \exp\left(-\int_t^T \lambda^*_s ds \right) \right) e^{-kh_T} + B^l_{t,T} \times \exp\left(-\int_t^T \lambda^*_s ds \right) \tag{197}$$

In many contexts the value of a credit risky corporate bond has been characterized as being equivalent to a long position in the underlying credit risk free instrument and a short position in a put option on the credit risky claim. This characterization is consistent with the current framework, as demonstrated in the following proposition.

Proposition 3.4 The value of a credit risky corporate zero coupon bond is equivalent to a long position in the underlying credit risk free instrument and a short position in a put option on the credit risky claim with the exercise price being given by the principal value.

Proof:
The value of the contract described in the proposition is given by

$$\begin{aligned} B^d_{t,T} &= B^l_{t,T} - E^*_t\left[\exp\left(-\int_t^T r^l_s ds\right)\left(B^l_{T,T} - B^d_{T,T}\right)^+\right] \qquad (198) \\ &= B^l_{t,T} - E^*_t\left[\exp\left(-\int_t^T r^l_s ds\right) E^*_T\left[\left(B^l_{T,T} - B^d_{T,T}\right)^+\right]\right] \end{aligned}$$

Noting that $B^l_{T,T} - B^d_{T,T} = 0$ if no default occurs yields

$$\begin{aligned} B^d_{t,T} &= B^l_{t,T} - \qquad (199) \\ & E^*_t\left[\exp\left(-\int_t^T r^l_s ds\right)(1 - e^{-kh_T})\left(1 - \exp\left(-\int_t^T \lambda^*_s ds\right)\right)\right] \\ &= E^*_t\left[e^{-kh_T}\exp\left(-\int_t^T r^l_s ds\right) - (e^{-kh_T} - 1)\exp\left(-\int_t^T (r^l_s + \lambda^*_s) ds\right)\right] \end{aligned}$$

which is identical to the characterization of the credit risky bond in (190).□
This proposition concludes the discussion of the discount instruments and briefly a number of results concerning rate and yield processes for the case of fixed recovery rates are collected.
Using (94) and (95), instantaneous forward and spot rate processes for the treasury and Libor asset under the spot risk measure are given by

$$f^g_{t,T} = f^g_{0,T} - \int_0^t \sigma^g(s,T)a^g(s,T)ds + \int_0^t \sigma^g(s,T)q^{g1}_s dM^{1,c,*}_s \qquad (200)$$

$$f^l_{t,T} = f^l_{0,T} - \int_0^t \sigma^l(s,T)a^l(s,T)ds + \sum_{i=1}^{2}\int_0^t \sigma^g(s,T)q^{li}_s dM^{i,c,*}_s \qquad (201)$$

$$r^g_t = f^g_{0,t} - \int_0^t \sigma^g(s,t)a^g(s,t)ds + \int_0^t \sigma^g(s,t)q^{g1}_s dM^{1,c,*}_s \qquad (202)$$

$$r_t^l = f_{0,t}^l - \int_0^t \sigma^l(s,t)a^l(s,t)ds + \sum_{i=1}^{2} \int_0^t \sigma^g(s,t)q_s^{li} dM_s^{i,c,*} \tag{203}$$

On $\{t < \tau\}$, (96) facilitates identification of the dynamics of the forward probability of default under the spot risk measure in terms of pure volatility parameters:

$$\begin{aligned}\lambda_{t,T}^d &= \lambda_{0,T}^d - \int_0^t w_s \sigma^l(s,T)a^l(s,T)ds + \sum_{i=1}^{3} \int_0^t \sigma^\lambda(s,T)q_s^{\lambda i} dM_s^{i,c,*} \\ &+ \int_0^t \left(\sum_{i=1}^{3} q_s^{\lambda i} q_s^{li} \right) \left(a^l(s,T)\sigma^\lambda(s,T) + \sigma^l(s,T)a^\lambda(s,T) \right)\end{aligned} \tag{204}$$

The instantaneous probability of default satisfies

$$\begin{aligned}\lambda_t^d &= \lambda_{0,t}^d - \int_0^t w_s \sigma^l(s,t)a^l(s,t)ds + \sum_{i=1}^{3} \int_0^t \sigma^\lambda(s,t)q_s^{\lambda i} dM_s^{i,c,*} \\ &+ \int_0^t \left(\sum_{i=1}^{3} q_s^{\lambda i} q_s^{li} \right) \left(a^l(s,t)\sigma^\lambda(s,t) + \sigma^l(s,t)a^\lambda(s,t) \right)\end{aligned} \tag{205}$$

The significance of (204) and (205) is that under the risk neutral measure, dynamics relating to the probability of default are comprehensively characterized through specification of volatility parameters. In this sense the present analysis can be considered as an extension of the arbitrage pricing framework for market risk to markets in which credit contingent claims are traded.

Yield processes relating to the treasury and Libor asset are given as follows:

$$(T-t)Y_{t,T}^g = \int_t^T (r_s^l - a^g(s,T)^2)ds + \int_t^T a^g(s,T)q_s^{g1} dM_s^{1,c,*} \tag{206}$$

$$(T-t)Y_{t,T}^l = \int_t^T (r_s^l - a^l(s,T)^2)ds + \sum_{i=1}^{2} \int_t^T a^l(s,T)q_s^{li} dM_s^{i,c,*} \tag{207}$$

Regarding the corporate asset the definition of zero coupon yields states that

$$Y_{t,T}^d = -\frac{1}{T-t} \ln B_{t,T}^d \tag{208}$$

Using the generalized Ito rule and (132) yields

$$\begin{aligned}\ln B_{T,T}^d &= \ln B_{t,T}^d + \int_t^T \frac{1}{B_{s,T}^d} dB_{s,T}^d - \frac{1}{2} \int_t^T \frac{1}{B_{s,T}^d \cdot B_{s,T}^d} d[B_{s,T}^d, B_{s,T}^d]^c \\ &+ \sum_{t<s\leq T} \left(\ln B_{s,T}^d - \ln B_{s-,T}^d - \frac{1}{B_{s-,T}^d} \triangle B_{s,T}^d \right)\end{aligned} \tag{209}$$

which, together with (208), permits identification of the yield dynamics relating to the credit risky instrument under the risk neutral measure:

$$\begin{aligned}(T-t)Y^d_{t,T} &= \int_t^T (r^l_s - a^l(s,T)^2)ds + \sum_{i=1}^{2}\int_t^T a^l(s,T)q^{li}_s dM^{i,c,*}_s \\ &+ \sum_{i=1}^{3}\int_t^{T\wedge\tau} a^\lambda(s,T)q^{\lambda i}_s dM^{i,c,*}_s \\ &- \frac{1}{2}\int_t^{T\wedge\tau} a^\lambda(s,T)^2 ds + \ln w_\tau - \int_t^{T\wedge\tau} w_s\lambda^*_s ds \\ &- \sum_{i=1}^{3}\int_t^{T\wedge\tau} a^\lambda(s,T)a^l(s,T)q^{\lambda i}_s q^{li}_s ds\end{aligned} \tag{210}$$

For future reference the following results relating to yield covariances are collected.

$$cov\left(Y^g_{t,T}, Y^g_{t,M}\right) = \frac{1}{(T-t)(M-t)}\int_t^{T\wedge M} a^g(s,T)a^g(s,M)ds \tag{211}$$

$$cov\left(Y^l_{t,T}, Y^l_{t,M}\right) = \frac{1}{(T-t)(M-t)}\int_t^{T\wedge M} a^l(s,T)a^l(s,M)ds \tag{212}$$

$$\begin{aligned}cov\left(Y^d_{t,T}, Y^d_{t,M}\right) &= \beta\int_t^{T\wedge M} \rho^{\lambda l}_s\left(a\lambda(s,T)a^l(s,M) + a^\lambda(s,T)a^l(s,T)\right)ds \\ &+\beta\int_t^{T\wedge M}\left(a^\lambda(s,T)a^\lambda(s,M) + a^l(s,M)a^l(s,T)\right)ds\end{aligned} \tag{213}$$

$$cov\left(Y^g_{t,T}, Y^l_{t,M}\right) = \beta\int_t^{T\wedge M} \rho^{gl}_s a^g(s,T)a^l(s,M)ds \tag{214}$$

$$cov\left(Y^g_{t,T}, Y^d_{t,M}\right) = \beta\int_t^{T\wedge M}\left(\rho^{g\lambda}_s a^\lambda(s,M)a^g(s,T) + \rho^{gl}_s a^g(s,T)a^l(s,M)\right)ds \tag{215}$$

$$cov\left(Y^l_{t,T}, Y^d_{t,M}\right) = \beta\int_t^{T\wedge M}\left(\rho^{\lambda l}_s a\lambda(s,M) + a^l(s,M)\right)a^l(s,T)ds \tag{216}$$

where $\beta = \frac{1}{(T-t)(M-t)}$.

3.3.2. Forward Risk Measure

The measure P^* is associated with the use of the money market account A_t^l as the numeraire assets; under this measure, spot prices are martingales. An alternative set of numeraire assets is available in the form of the Libor assets, viewed as indexed by maturity. The present section is devoted to the identification of a probability measure associated with the use of a Libor instrument of some fixed maturity $T_n \in [t, 1]$ as the numeraire asset. It turns out that under this measure forward prices of maturity T_n are martingales. For this reason this measure is also called forward risk measure and denoted by P^{T_n}. Given that the impact of different specifications for the recovery rate distribution has been already been examined in the context of the spot risk measure, it is assumed in the present section that the recovery rate in the event of default is fixed at $e^{-kh\tau}$.

3.3.2.1. Density Processes

Return processes relating to the three discount securities, normalized by the Libor asset of some fixed maturity $T_n \in [t, 1]$, V_{t,T_n}^l, are given as follows:

$$\begin{aligned}\frac{d\left(V_{t,T}^g/V_{t,T_n}^l\right)}{V_{t,T}^g/V_{t,T_n}^l} &= (r_t^g + y_t - r_t^l + b^g(t,T) - b^l(t,T_n))dt + \\ &\quad \sum_{i=1}^{3}\left(q_t^{gi}a^g(t,T) - q_t^{li}a^l(t,T_n)\right)\left(dM_t^{i,c} - q_t^{li}a^l(t,T_n)dt\right)\end{aligned} \tag{217}$$

$$\begin{aligned}\frac{d\left(V_{t,T}^l/V_{t,T_n}^l\right)}{V_{t,T}^l/V_{t,T_n}^l} &= (b^l(t,T) - b^l(t,T_n))dt + \\ &\quad \sum_{i=1}^{3} q_t^{li}\left(a^l(t,T) - a^l(t,T_n)\right)\left(dM_t^{i,c} - q_t^{li}a^l(t,T_n)dt\right)\end{aligned} \tag{218}$$

$$\begin{aligned}\frac{d\left(V_{t,T}^d/V_{t,T_n}^l\right)}{V_{t,T}^d/V_{t,T_n}^l} &= (b^l(t,T) + I_{\{t<\tau\}}b^\lambda(t,T) - b^l(t,T_n))dt \\ &\quad + \sum_{i=1}^{3} q_t^{li}(a^l(t,T) - a^l(t,T_n)) \times \left(dM_t^{i,c} - q_t^{li}a^l(t,T_n)dt\right) \\ &\quad + \sum_{i=1}^{3} I_{\{t<\tau\}}q_t^{\lambda i}w_t a^\lambda(t,T) \times \left(dM_t^{i,c} - q_t^{li}a^l(t,T_n)dt\right) \\ &\quad + w_t\left(dN_t - \lambda_{t\wedge\tau}dt\right) + a^l(t,T)I_{\{t<\tau\}}w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{\lambda i}q_t^{li}dt\end{aligned} \tag{219}$$

The following assumptions relate to the existence of market prices of risk prior to default and at the time of default.

Assumption 3.21 (Existence of market prices of risk prior to default) On $\{t < \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{220}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) - b^l(t,T_n) \\ b^l(t,T) - b^l(t,T_n) \\ b^l(t,T) + b^\lambda(t,T) - b^l(t,T_n) - a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{li} q_t^{\lambda i} - w_t\lambda_t \end{pmatrix} \tag{221}$$

is the matrix of excess returns on the three discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & -a^l(t,T_n) & 0 \\ 0 & a^l(t,T) - a^l(t,T_n) & 0 \\ 0 & a^l(t,T) - a^l(t,T_n) & w_t a^\lambda(t,T) \end{pmatrix} \tag{222}$$

$$\mathbf{\Pi}_3 = \mathbf{Q}_t \tag{223}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \\ \theta_t^{3,c} \end{pmatrix} \tag{224}$$

contains the market prices of risk arising from the presence of uncertainty generated by the continuous martingale vector $\mathbf{M}_t^c$.

Assumption 3.22 (Existence of market prices of risk at the time of default) On $\{t = \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds:

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{225}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) - b^l(t,T_n) \\ b^l(t,T) - b^l(t,T_n) \\ b^l(t,T) + b^\lambda(t-,T) - b^l(t,T_n) - \\ a^l(t-,T)w_{t-}a^\lambda(t-,T)\sum_{i=1}^{3} q_{t-}^{li} q_{t-}^{\lambda i} - w_{t-}\lambda_{t-} + w_t\lambda_t\theta_t^d \end{pmatrix} \tag{226}$$

is the matrix of excess returns on the three discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & -a^l(t,T_n) & 0 \\ 0 & a^l(t,T) - a^l(t,T_n) & 0 \\ 0 & a^l(t,T) - a^l(t,T_n) & w_t \end{pmatrix} \tag{227}$$

$$\mathbf{\Pi}_3 = \begin{pmatrix} q_t^{g1} & 0 & 0 \\ q_t^{l1} & q_t^{l2} & 0 \\ q_t^{l1} & q_t^{l2} & 1 \end{pmatrix} \tag{228}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \\ \lambda_t \theta_t^d \end{pmatrix} \tag{229}$$

contains the market prices of risk arising from the presence of uncertainty generated by the martingale vector $\left(M_t^{c,1}, M_t^{c,2}, M_t^d\right)$, where $M_t^d = dN_t - \lambda_t dt$.

Assumption 3.23 (Existence of market prices of risk after the time of default) On $\{t > \tau\}$, there exists a matrix $\mathbf{\Pi}_4$ such that the following holds

$$\mathbf{\Pi}_1 + \mathbf{\Pi}_2\mathbf{\Pi}_3\mathbf{\Pi}_4 = \mathbf{0} \tag{230}$$

The matrix

$$\mathbf{\Pi}_1 = \begin{pmatrix} b^g(t,T) - b^l(t,T_n) \\ b^l(t,T) - b^l(t,T_n) \end{pmatrix} \tag{231}$$

is the matrix of excess returns on the treasury and Libor discount assets, the volatility matrix is given by

$$\mathbf{\Pi}_2 = \begin{pmatrix} a^g(t,T) & -a^l(t,T_n) \\ 0 & a^l(t,T) - a^l(t,T_n) \end{pmatrix} \tag{232}$$

$$\mathbf{\Pi}_3 = \begin{pmatrix} q_t^{g1} & 0 \\ q_t^{l1} & q_t^{l2} \end{pmatrix} \tag{233}$$

is the correlation matrix and

$$\mathbf{\Pi}_4 = \begin{pmatrix} \theta_t^{1,c} \\ \theta_t^{2,c} \end{pmatrix} \tag{234}$$

contains the market prices of risk arising from the presence of uncertainty generated by the continuous martingale vector $\left(M_t^{c,1}, M_t^{c,2}\right)$.

Note that after the time of default there is no uncertainty relating to credit risk; the only source of uncertainty is given by the market risk embedded in the treasury and

Libor security. (79) and (89) can be expanded as follows:

$$b^g(t,T) - b^l(t,T_n) + a^g(t,T)q_t^{g1}\theta_t^{1,c} - a^l(t,T_n)\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} = 0 \tag{235}$$

$$b^l(t,T) - b^l(t,T_n) + (a^l(t,T) - a^l(t,T_n))\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} = 0 \tag{236}$$

$$\begin{aligned} 0 &= b^l(t,T) + b^\lambda(t,T) - b^l(t,T_n) - a^l(t,T)w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{li}q_t^{\lambda i} \\ &\quad -w_t\lambda_t + (a^l(t,T) - a^l(t,T_n))\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + w_t a^\lambda(t,T)\sum_{i=1}^{3} q_t^{\lambda i}\theta_t^{i,c} \end{aligned} \tag{237}$$

$$\begin{aligned} 0 &= b^l(t,T) + b^\lambda(t-,T) - b^l(t,T_n) - a^l(t-,T)w_{t-}a^\lambda(t-,T)\sum_{i=1}^{3} q_{t-}^{li}q_{t-}^{\lambda i} \\ &\quad -w_{t-}\lambda_{t-} + (a^l(t,T) - a^l(t,T_n))\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + w_{t-}a^\lambda(t-,T)\sum_{i=1}^{3} q_{t-}^{\lambda i}\theta_{t-}^{i,c} + w_t\lambda_t\theta_t^d \end{aligned} \tag{238}$$

Differentiating (235)-(238) with respect to maturity yields arbitrage-free forward rate drift restrictions for the three discount securities under the forward risk measure:

$$\alpha^g(t,T) = -\sigma^g(t,T)\left(q_t^{g1}\theta_t^{1,c} + a^g(t,T)\right) \tag{239}$$

$$\alpha^l(t,T) = -\sigma^l(t,T)\left(\sum_{i=1}^{2} q_t^{li}\theta_t^{i,c} + a^l(t,T)\right) \tag{240}$$

On $\{t < \tau\}$, drift restrictions relating to instantaneous corporate forward rates are given by

$$\begin{aligned} \alpha^\lambda(t,T) &= -\sigma^\lambda(t,T)\left(\sum_{i=1}^{3} q_t^{\lambda i}\theta_t^{i,c} + w_t a^\lambda(t,T)\right) + \\ &\quad \left(\sum_{i=1}^{3} q_t^{\lambda i}q_t^{li}\right)\left(a^l(t,T)\sigma^\lambda(t,T) + \sigma^l(t,T)a^\lambda(t,T)\right) \end{aligned} \tag{241}$$

whereas on $\{t = \tau\}$, the corresponding restrictions are given by

$$\begin{aligned} \alpha^\lambda(t-,T)w_{t-} &= -\sigma^\lambda(t,T)w_t\lambda_t\theta_t^d + w_{t-}^2 a^\lambda(t-,T)\sigma^\lambda(t-,T) + \\ &\quad w_{t-}\left(\sum_{i=1}^{3} q_t^{\lambda i}q_t^{li}\right)\left(a^l(t-,T)\sigma^\lambda(t-,T) + \sigma^l(t-,T)a^\lambda(t-,T)\right) \end{aligned} \tag{242}$$

In order to guarantee uniqueness, the following assumption is added.

Assumption 3.24 (Uniqueness of Market Prices of Risk) The matrix products $\mathbf{\Pi}_2\mathbf{\Pi}_3$ in (220), (225) and (230) are non-singular..

The above assumption guarantees that the relevant matrices can be inverted and unique solutions for the market prices of risk as in (224), (229) and (234) exist. Given the assumptions relating to existence and uniqueness of the market prices of risk, (217)-(219) can be expressed as follows:

$$\frac{d\left(V^g_{t,T}/V^l_{t,T_n}\right)}{V^g_{t,T}/V^l_{t,T_n}} = (r^g_t + y_t - r^l_t)dt + \sum_{i=1}^{3}\left(q^{gi}_t a^g(t,T) - q^{li}_t a^l(t,T_n)\right)\left(dM^{i,c}_t - (\theta^{i,c}_t + q^{li}_t a^l(t,T_n))dt\right) \tag{243}$$

$$\frac{d\left(V^l_{t,T}/V^l_{t,T_n}\right)}{V^l_{t,T}/V^l_{t,T_n}} = \sum_{i=1}^{3} q^{li}_t\left(a^l(t,T) - a^l(t,T_n)\right)\left(dM^{i,c}_t - (\theta^{i,c}_t + q^{li}_t a^l(t,T_n))dt\right) \tag{244}$$

$$\begin{aligned}\frac{d\left(V^d_{t,T}/V^l_{t,T_n}\right)}{V^d_{t,T}/V^l_{t,T_n}} &= \sum_{i=1}^{3} I_{\{t<\tau\}} q^{\lambda i}_t w_t a^\lambda(t,T)\left(dM^{i,c}_t - (\theta^{i,c}_t + q^{li}_t a^l(t,T_n))dt\right) \\ &\quad + \sum_{i=1}^{3}\left(q^{li}_t(a^l(t,T) - a^l(t,T_n))\right)\left(dM^{i,c}_t - (\theta^{i,c}_t + q^{li}_t a^l(t,T_n))dt\right) \\ &\quad + w_t\left(dN_t - \lambda_{t\wedge\tau}\theta^d_t dt\right)\end{aligned} \tag{245}$$

(243)-(245) illustrate that expected returns on normalized assets are given by the sum of the risk premia corresponding to the various sources of risk, including terms arising from the risk associated with uncertainty in the value of V^l_{t,T_n}. The following proposition identifies the density process defining the equivalent martingale measure associated with the use of the Libor assets with maturity T_n as a numeraire.

Proposition 3.5 There exists a unique equivalent martingale measure P^{T_n} under which the processes $V^g_{t,T}/V^l_{t,T_n}$, $V^l_{t,T}/V^l_{t,T_n}$ and $V^d_{t,T}/V^l_{t,T_n}$ in (243)-(245) are martingales. The density process defining this measure is given by

$$\begin{aligned} Z^{T_n}_t &= Z^{T_n}_0 \\ &\quad \times\mathcal{E}\left(\int_0^t Z^{T_n}_{s-}(\theta^d_s - 1)dM^d_s \sum_{i=1}^{2}\int_0^t Z^{T_n}_{s-}\left(a^l(t,T_n)q^{li}_t + \theta^{i,c}_s\right)dM^{i,c}_s\right) \\ &\quad \times\mathcal{E}\left(\int_0^{t\wedge\tau} Z^{T_n}_{s-}\left(a^l(t,T_n)q^{l3}_t + \theta^{3,c}_s\right)dM^{3,c}_s\right)\end{aligned} \tag{246}$$

where $\mathcal{E}\,(.)$ is the Doleans-Dade exponential. (246) is equivalent to

$$\begin{aligned} Z^{T_n}_t &= Z^{T_n}_0 \times \exp\left(\int_0^{t\wedge\tau}(\theta^d_s - 1)\lambda_t ds + \int_0^{t\wedge\tau}\ln\theta^d_s dN_s\right) \\ &\quad \times\exp\left(\sum_{i=1}^{2}\int_0^t\left(a^l(t,T_n)q^{li}_t + \theta^{i,c}_s\right)dM^{i,c}_s - \frac{1}{2}\sum_{i=1}^{2}\int_0^t\left(a^l(t,T_n)q^{li}_t + \theta^{i,c}_s\right)^2 ds\right) \\ &\quad \times\exp\left(\int_0^{t\wedge\tau}\left(a^l(t,T_n)q^{l3}_t + \theta^{3,c}_s\right)dM^{3,c}_s - \frac{1}{2}\int_0^{t\wedge\tau}\left(a^l(t,T_n)q^{l3}_t + \theta^{3,c}_s\right)^2 ds\right)\end{aligned} \tag{247}$$

and defines the spot risk measure through

$$\frac{dP^{T_n}}{dP} = Z_1^{T_n} \tag{248}$$

Furthermore, the following restriction holds:

$$r_t^g + y_t - r_t^l = 0 \tag{249}$$

Proof: See appendix.□

Under the forward risk measure return processes of trading gains associated with the three discount instruments are given by

$$\frac{d\left(V_{t,T}^g/V_{t,T_n}^l\right)}{V_{t,T}^g/V_{t,T_n}^l} = \sum_{i=1}^{3}\left(q_t^{gi}a^g(t,T) - q_t^{li}a^l(t,T_n)\right) dM_t^{i,c,T_n} \tag{250}$$

$$\frac{d\left(V_{t,T}^l/V_{t,T_n}^l\right)}{V_{t,T}^l/V_{t,T_n}^l} = \sum_{i=1}^{3} q_t^{li}\left(a^l(t,T) - a^l(t,T_n)\right) dM_t^{i,c,T_n} \tag{251}$$

$$\begin{aligned}\frac{d\left(V_{t,T}^d/V_{t,T_n}^l\right)}{V_{t,T}^d/V_{t,T_n}^l} &= \sum_{i=1}^{3}\left(q_t^{li}(a^l(t,T) - a^l(t,T_n)) + q_t^{\lambda i}a^{\lambda}(t,T)\right) dM_t^{i,c,T_n} \\ &\quad + w_t\left(dN_t - \lambda_{t\wedge\tau}^{T_n}dt\right)\end{aligned} \tag{252}$$

where the sources of risk under P^{T_n} are given by

$$M_t^{i,c,T_n} = M_t^{i,c} - \int_0^t \left(\theta_s^{i,c} + q_t^{li}a^l(t,T)\right) ds \tag{253}$$

$$M_t^{i,c,T_n} = M_t^{i,c,*} - \int_0^t q_t^{li}a^l(t,T)ds \tag{254}$$

$$M_t^{d,T_n} = M_t^d - \int_0^t \left(\theta_s^d - 1\right)\lambda_s ds \tag{255}$$

Note that for any two maturities $M, T \in [0,1]$, the following relationship holds:

$$M_t^{i,c,T} = M_t^{i,c,M} - \int_0^t q_t^{li}\left(a^l(t,T) - a^l(t,M)\right) ds \tag{256}$$

Integrating (250)-(252) yields the following set of equations:

$$\begin{aligned}\frac{V_{T,T}^g}{V_{T,T_n}^l} &= \frac{V_{t,T}^g}{V_{t,T_n}^l} \times \exp\left(\sum_{i=1}^{3}\int_t^T \left(q_s^{gi}a^g(s,T) - q_s^{li}a^l(s,T_n)\right) dM_s^{i,c,T_n}\right) \\ &\quad \times \exp\left(-\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(q_s^{gi}a^g(s,T) - q_s^{li}a^l(s,T_n)\right)^2 ds\right)\end{aligned} \tag{257}$$

$$\frac{V^l_{T,T}}{V^l_{T,T_n}} = \frac{V^l_{t,T}}{V^l_{t,T_n}} \times \exp\left(\sum_{i=1}^{3}\int_t^T q^{li}_s \left(a^l(s,T) - a^l(s,T_n)\right) dM^{i,c,T_n}_s\right) \tag{258}$$
$$\times \exp\left(-\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(a^l(s,T) - a^l(s,T_n)\right)^2 ds\right)$$

$$\frac{V^d_{T,T}}{V^l_{T,T_n}} = \frac{V^d_{t,T}}{V^l_{t,T_n}} \times \exp\left(-\int_t^{T\wedge\tau} w_s\lambda^{T_n}_s ds - k\int_t^T h_s dN_s\right) \tag{259}$$
$$\times \exp\left(\sum_{i=1}^{3}\int_t^T \left(q^{li}_s(a^l(s,T) - a^l(s,T_n)) + q^{\lambda i}_s a^\lambda(s,T)\right) dM^{i,c,T_n}_s\right)$$
$$\times \exp\left(-\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(q^{li}_s(a^l(s,T) - a^l(s,T_n)) + q^{\lambda i}_s a^\lambda(s,T)\right)^2 ds\right)$$

The latter expressions will prove useful for the purpose of pricing specific contracts.

3.3.2.2. Asset Values and Rate Processes under the Forward Risk Measure

The following section is devoted to the characterization of prices and rates under the forward rate measure. Consider the martingale measure associated with the use of the Libor assets of maturity T as the numeraire asset. Under this measure, the price of the treasury instrument is given by:

$$B^g_{t,T} = B^l_{t,T} \times E^T_t\left[\exp\left(\int_t^T y_s ds\right)\right] \tag{260}$$

making explicit the nature of this asset as a claim to the accumulated convenience yield. The price of the Libor assets is given by

$$B^l_{t,T} = B^l_{t,T} \times E^T_t\left[I\right] \tag{261}$$

The price of the credit risky zero coupon bond can be expressed as:

$$B^d_{t,T} = B^l_{t,T} \times E^T_t\left[\left(1 - \exp\left(-\int_t^T \lambda^T_s ds\right)\right) e^{-kh_T}\right] \tag{262}$$
$$+B^l_{t,T} \times E^T_t\left[\exp\left(-\int_t^T \lambda^T_s ds\right)\right]$$

Note the familiar decomposition of the price of this asset given as the sum of its probability weighted payoffs. (262) is valid despite the presence of a stochastic default intensity and explicit correlation between the probability of default and interest rates and demonstrates the attractiveness of the forward risk measure. Using (239)-(242) and (254), instantaneous forward and spot rate processes for the treasury and Libor asset under the forward risk measure are given by

$$\begin{aligned} f^g_{t,T} &= f^g_{0,T} - \sum_{i=1}^{3} \int_0^t q^{g1}_s \sigma^g(s,T) q^{gi}_s (q^{gi}_s a^g(s,T) - q^{li}_s a^l(s,T_n)) ds \\ &+ \int_0^t \sigma^g(s,T) q^{g1}_s dM^{1,c,T_n}_s \end{aligned} \tag{263}$$

$$\begin{aligned} f^l_{t,T} &= f^l_{0,T} - \int_0^t \sigma^l(s,T)(a^l(s,T) - a^l(s,T_n)) ds \\ &+ \sum_{i=1}^{2} \int_0^t \sigma^l(s,T) q^{li}_s dM^{i,c,T_n}_s \end{aligned} \tag{264}$$

$$\begin{aligned} r^g_t &= f^g_{0,t} - \sum_{i=1}^{3} \int_0^t q^{g1}_s \sigma^g(s,t) q^{gi}_s (q^{gi}_s a^g(s,t) - q^{li}_s a^l(s,T_n)) ds \\ &+ \int_0^t \sigma^g(s,t) q^{g1}_s dM^{1,c,T_n}_s \end{aligned} \tag{265}$$

$$r^l_t = f^l_{0,t} - \int_0^t \sigma^l(s,t)(a^l(s,t) - a^l(s,T_n)) ds + \sum_{i=1}^{2} \int_0^t \sigma^l(s,t) q^{li}_s dM^{i,c,T_n}_s \tag{266}$$

Remark 3.7 Note that if $T_n = T$, (264) reduces to

$$f^l_{t,T} = f^l_{0,T} + \sum_{i=1}^{2} \int_0^t \sigma^l(s,T) q^{li}_s dM^{i,c,T_n}_s \tag{267}$$

Thus, the forward risk measure associated with the use of the Libor discount bond of maturity T is the measure under which, apart from normalized assets being martingales, instantaneous forward rates on this asset are also martingales. The martingale property of forward rates and yields under the forward risk measure only holds for the relevant processes associated with the numeraire asset. Inspection of (263) and (268) reveals that forward rates on the treasury instrument and the corporate asset fail to satisfy the martingale property even under the forward risk measure.

On $\{t < \tau\}$, (96) facilitates identification of the dynamics of the forward probability of default under the spot risk measure in terms of pure volatility parameters:

$$\lambda_{t,T} = \lambda_{0,T} - \sum_{i=1}^{3} \int_0^t w_s q^{\lambda i}_s \sigma^\lambda(s,T)(q^{\lambda i}_s a^\lambda(s,T) - q^{li}_s a^l(s,T_n)) ds \tag{268}$$

$$+\sum_{i=1}^{3}\int_{0}^{t}\sigma^{\lambda}(s,T)q_s^{\lambda i}dM_s^{i,c,T_n}$$

$$+\sum_{i=1}^{3}\int_{0}^{t}(q_s^{\lambda i}q_s^{li})(a^l(s,T)\sigma^{\lambda}(s,T)+\sigma^l(s,T)a^{\lambda}(s,T))ds$$

The instantaneous probability of default satisfies

$$\begin{aligned}\lambda_t &= \lambda_{0,t}-\sum_{i=1}^{3}\int_{0}^{t}w_s q_s^{\lambda i}\sigma^{\lambda}(s,t)(q_s^{\lambda i}a^{\lambda}(s,t)-q_s^{li}a^l(s,T_n))ds \\ &\quad +\sum_{i=1}^{3}\int_{0}^{t}\sigma^{\lambda}(s,t)q_s^{\lambda i}dM_s^{i,c,T_n} \\ &\quad +\sum_{i=1}^{3}\int_{0}^{t}(q_s^{\lambda i}q_s^{li})(a^l(s,t)\sigma^{\lambda}(s,t)+\sigma^l(s,t)a^{\lambda}(s,t))ds\end{aligned} \tag{269}$$

Note that for the purpose of pricing specific contracts, the dynamics of λ_t under various forward risk measures will provide useful. In such situations, the probability of default under the T_n-forward risk measure will be denoted $\lambda_t^{T_n}=\lambda_t\theta_t^d$, where λ_t is given by (269).

3.3.3. Summary of Density Processes

Application of the technology developed in the preceding sections will frequently require switching between different martingale measures. For future reference, the density processes defining various martingale measures are summarized below.

The density identifying the spot risk measure is

$$\frac{dP^*}{dP}=\mathcal{E}\left(\sum_{k=1}^{3}\int_{0}^{1}a^l(s,T_i)q_s^{lk}dM_s^{k,c}+\int_{0}^{1}(\theta_s^d-1)dM_s^d\right) \tag{270}$$

and

$$Z_t^*=E_t\left[\frac{dP^*}{dP}\right]=\frac{dP^*}{dP}\mid_{\mathcal{F}_t} \tag{271}$$

is a P-martingale. The class of densities identifying the forward risk measures, indexed by the maturity $T\in(t,1]$ of the numeraire asset, is given by

$$\frac{dP^T}{dP}=\mathcal{E}\left(\sum_{k=1}^{3}\int_{0}^{1}(\theta_s^{i,c}+a^l(s,T)q_s^{lk})dM_s^{k,c}+\int_{0}^{1}(\theta_s^d-1)dM_s^d\right) \tag{272}$$

and the process

$$Z_t^T=E_t\left[\frac{dP^T}{dP}\right]=\frac{dP^T}{dP}\mid_{\mathcal{F}_t} \tag{273}$$

is a P-martingale. To switch from the spot risk measure to the forward risk measure, the density

$$\frac{dP^T}{dP^*} = \mathcal{E}\left(\sum_{k=1}^{3}\int_0^1 a^l(s,T)q_s^{lk}dM_s^{k,c}\right) \tag{274}$$

will be used and the process

$$Z_t^{T,*} = E_t\left[\frac{dP^T}{dP^*}\right] = \frac{dP^T}{dP^*}\mid_{\mathcal{F}_t} \tag{275}$$

Finally, to switch between forward risk measures associated with different maturities M and T, the density

$$\frac{dP^T}{dP^M} = \mathcal{E}\left(\sum_{k=1}^{3}\int_0^1 (a^l(s,T) - a^l(s,M))q_s^{lk}dM_s^{k,c}\right) \tag{276}$$

will be used and the process

$$Z_t^{T,M} = E_t\left[\frac{dP^T}{dP^M}\right] = \frac{dP^T}{dP^M}\mid_{\mathcal{F}_t} \tag{277}$$

is a P^M-martingale. This concludes the summary of density processes defining different martingale measures.

Conclusion

This chapter contains the derivation of a no-arbitrage framework for the pricing of credit contingent claims. While the model derived here can be categorized as a reduced form model in that credit risk is modelled in terms of a probability of default and recovery rates, a number innovative modelling choices distinguish it from established models using this approach. Explicit allowance is made for both the probability of default as well as recovery rates to be subject to uncertainty. The concept of a stochastic instantaneous forward probability of default has been introduced,capturing the idea of default probabilities varying randomly over time and facilitating incorporation of a correlation structure between the default process and interest rates. Such a property has so far has eluded credit risk models.

The securities market model I propose integrates a number of markets which have largely been analyzed in isolation. Specifically, I assume that treasury, interbank and corporate assets are traded. Joint modelling of these assets facilitates consideration of credit related and credit unrelated spread components.

A further innovation in the preceding analysis is that asset characteristics are modelled in terms of contractual and non contractual payoffs. This approach allows certain economic features of assets under considerations to be incorporated when pricing these assets. The terminology is also more intuitive than that used in related contexts[21].

Special attention is given to the distribution of recovery rates. It is shown that only a subset of all possible specifications relating to recovery rate distributions is compatible with the use of the no arbitrage paradigm. Specifically, when the recovery rate is assumed to take on an infinity of possible values, it is unlikely that such a risk can be hedged. Ultimately, this type of consideration reflects the fact that credit risk contains substantial idiosyncratic components.

Finally, density processes defining martingale measures have been derived and discount securities under different risk neutral measures have been characterized. The following chapters contain applications of the methodology developed here. The pricing of specific credit contingent claims is the subject of the following chapter. The second application focuses on risk management issues in the current framework.

[21] Jarrow and Turnbull (1995) use the 'foreign currency' analogue in modelling credit risk; see their paper for details.

Appendix

Derivation of the return process for the default risky zero coupon bond

Integrating instantaneous forward rates as in (31):

$$\int_t^T f^d_{t,u}du = \int_t^T f^d_{0,u}du + \int_t^T\int_0^t \alpha^l(s,u)dsdu + \int_t^T\int_0^t \sigma^l(s,u)dW^l_s du - \tag{278}$$
$$\int_t^T\int_0^t w_s\alpha^\lambda(s,u)I_{\{s<\tau\}}dsdu + \int_t^T\int_0^t w_s\sigma^\lambda(s,u)I_{\{s<\tau\}}dW^\lambda_s du$$

A change of the order of integration yields:

$$\int_t^T f^d_{t,u}du = \int_t^T f^d_{0,u}du + \int_0^t\int_t^T \alpha^l(s,u)duds + \int_0^t\int_t^T \sigma^l(s,u)dudW^l_s - \tag{279}$$
$$\int_0^t\int_t^T w_s\alpha^\lambda(s,u)I_{\{s<\tau\}}duds + \int_0^t\int_t^T w_s\sigma^\lambda(s,u)I_{\{s<\tau\}}dudW^\lambda_s$$

$$\int_t^T f^d_{t,u}du = \int_0^T f^d_{0,u}du + \int_0^t\int_s^T \alpha^l(s,u)duds + \int_0^t\int_s^T \sigma^l(s,u)dudW^l_s - \tag{280}$$
$$\int_0^t\int_s^T w_s\alpha^\lambda(s,u)I_{\{s<\tau\}}duds + \int_0^t\int_s^T w_s\sigma^\lambda(s,u)I_{\{s<\tau\}}dudW^\lambda_{s\wedge\tau} -$$
$$\int_0^t f^d_{0,u}du - \int_0^t\int_s^t \alpha^l(s,u)duds - \int_0^t\int_s^t \sigma^l(s,u)dudW^l_s +$$
$$\int_0^t\int_s^t w_s\alpha^\lambda(s,u)I_{\{s<\tau\}}duds + \int_0^t\int_s^t w_s\sigma^\lambda(s\wedge\tau,u)dudW^\lambda_{s\wedge\tau}$$

$$\int_t^T f^d_{t,u}du = \int_0^T f^d_{0,u}du + \int_0^t\int_s^T \alpha^l(s,u)duds + \int_0^t\int_s^T \sigma^l(s,u)dudW^l_s \tag{281}$$
$$-\int_0^t r^d_u du +$$
$$\int_0^t\int_s^T w_s\alpha^\lambda(s,u)I_{\{s<\tau\}}duds + \int_0^t\int_s^T w_s\sigma^\lambda(s,u)I_{\{s<\tau\}}dudW^\lambda_s$$

From the functional relationship between bond prices and instantaneous forward rates,

$$\ln B^d_{t,T} = -\int_t^T f^d_{t,u} du \tag{282}$$

Using the transformation $X = \ln B^d_{t,T}$ and (282) yields (53).

Derivation of the density process for the spot risk measure

By definition of a martingale measure,

$$Z^*_t = E_t\left[\frac{dP^*}{dP}\right] \tag{283}$$

is a P-martingale with

$$Z^*_0 = E\left[\frac{dP^*}{dP}\right] = 1 \tag{284}$$

Using Karatzas and Shreve (1987), p.170, and Bremaud (1981), p.64, any P-martingale can be represented as

$$Z^*_t = Z^*_0 + \sum_{i=1}^{2}\int_0^t \phi^{i,c}_s dM^{i,c}_s + \int_0^t \phi^{3,c}_s dM^{3,c}_s + \int_0^t \phi^d_s dM^d_s \tag{285}$$

where $dM^d_t = dN_t - \lambda_t dt$ is a compensated Poisson process. Since P^* is an equivalent martingale measure, $V^g_{t,T}/A^l_t$ is a P^* martingale, implying that $V^g_{t,T}Z^*_t/A^l_t$ is a P martingale. This requirement is identical to the following process being a P martingale:

$$\begin{aligned}\frac{d\left(V^g_{t,T}Z^*_t/A^l_t\right)}{V^g_{t,T}/A^l_t} &= \frac{Z^*_{t-}}{V^g_{t-,T}}dV^g_{t,T} + dZ^*_t - \frac{Z^*_{t-}}{A^l_{t-}}dA^l_t \\ &\quad + \frac{1}{V^g_{t-,T}}[dV^g_{t,T}, dZ^*_t]^c + \frac{1}{V^g_{t-,T}}\Delta V^g_{t,T}\cdot\Delta Z^*_t\end{aligned} \tag{286}$$

(286) is equivalent to

$$\begin{aligned}\frac{d\left(V^g_{t,T}Z^*_t/A^l_t\right)}{V^g_{t,T}/A^l_t} &= \sum_{i=1}^{2}\left(\phi^{i,c}_t + Z^*_{t-}a^g(t,T)q^{gi}_t\right)dM^{i,c}_t \\ &\quad + \phi^{3,c}_{t\wedge\tau}dM^{3,c}_{t\wedge\tau} + \phi^d_t dM^d_t \\ &\quad + a^g(t,T)q^{g1}_t\left(\phi^{1,c}_t - Z^*_{t-}\theta^{1,c}_t\right)dt \\ &\quad + (r^g_t + y_t - r^l_t)dt\end{aligned} \tag{287}$$

(287) is a P martingale only if the following restrictions hold:

$$\phi^{1,c}_t = Z^*_{t-}\theta^{1,c}_t \tag{288}$$

$$r_t^g + y_t - r_t^l = 0 \tag{289}$$

Repeating the same procedure with the Libor security

$$\begin{aligned}\frac{d\left(V_{t,T}^l Z_t^* / A_t^l\right)}{V_{t,T}^l / A_t^l} &= \sum_{i=1}^{2}\left(\phi_t^{i,c} + Z_{t-}^* a^l(t,T) q_t^{li}\right) dM_t^{i,c} \\ &+\phi_{t\wedge\tau}^{3,c} dM_{t\wedge\tau}^{3,c} + \phi_t^d dM_t^d \\ &+a^g(t,T)\sum_{i=1}^{2} q_t^{l1}\left(\phi_s^{i,c} - Z_{t-}^*\theta_t^{i,c}\right) dt\end{aligned} \tag{290}$$

yields the restriction

$$\phi_t^{2,c} = Z_{t-}^*\theta_t^{2,c} \tag{291}$$

Finally,

$$\begin{aligned}\frac{d\left(V_{t,T}^d Z_t^* / A_t^l\right)}{V_{t,T}^d / A_t^l} &= \sum_{i=1}^{2}\left(\phi_t^{i,c} + Z_{t-}^* a^l(t,T) q_t^{li}\right) dM_t^{i,c} \\ &+Z_{t-}^* a^\lambda(t\wedge\tau,T)\sum_{i=1}^{3} q_{t\wedge\tau}^{\lambda i} dM_{t\wedge\tau}^{i,c} + \\ &\phi_{t\wedge\tau}^{3,c} dM_{t\wedge\tau}^{3,c} + \phi_t^d dM_t^d + Z_{t-}^* w_t\left(dN_t - \lambda_{t\wedge\tau} dt\right) \\ &+w_t\lambda_{t\wedge\tau}\left(\phi_t^d - Z_{t-}^*(\theta_t^d - 1)\right) dt \\ &+a^l(t,T)\sum_{i=1}^{2} q_t^{l1}\left(\phi_t^{i,c} - Z_{t-}^*\theta_t^{i,c}\right) dt \\ &+a^\lambda(t\wedge\tau,T)\sum_{i=1}^{3} q_{t\wedge\tau}^{\lambda 1}\left(\phi_{t\wedge\tau}^{i,c} - Z_{t-}^*\theta_{t\wedge\tau}^{i,c}\right) dt\end{aligned} \tag{292}$$

yields the restrictions

$$\phi_{t\wedge\tau}^{3,c} = Z_{t\wedge\tau-}^*\theta_{t\wedge\tau}^{3,c} \tag{293}$$

$$\phi_t^d = Z_{t-}^*(\theta_t^d - 1) \tag{294}$$

Inserting (288), (291), (293) and (294) into (285) yields

$$Z_t^* = Z_0^* + \sum_{i=1}^{2}\int_0^t Z_{s-}^*\theta_s^{i,c} dM_s^{i,c} + \int_0^t Z_{s\wedge\tau-}^*\theta_{s\wedge\tau}^{3,c} dM_s^{3,c} + \int_0^t Z_{s-}^*(\theta_s^d - 1) dM_s^d \tag{295}$$

the solution of which is given by (107).□

Derivation of the density process for the forward risk measure

In analogy to arguments in the preceding section,

$$Z_t^{T_n} = E_t\left[\frac{dP^{T_n}}{dP}\right] \tag{296}$$

is a P^{T_n}-martingale with

$$Z_0^{T_n} = E\left[\frac{dP^{T_n}}{dP}\right] = 1 \tag{297}$$

From various representation theorems for P^{T_n}-martingales,

$$Z_t^{T_n} = Z_0^{T_n} + \sum_{i=1}^{2}\int_0^t \phi_s^{i,c} dM_s^{i,c} + \int_0^t \phi_s^{3,c} dM_s^{3,c} + \int_0^t \phi_s^d dM_s^d \tag{298}$$

Since P^{T_n} is an equivalent martingale measure, $V_{t,T}^g / B_{t,T_n}^l$ is a P^{T_n}-martingale, implying that $V_{t,T}^g Z_t^{T_n} / B_{t,T_n}^l$ is a P-martingale. Noting that

$$\begin{aligned}\frac{d\left(V_{t,T}^g Z_t^{T_n} / B_{t,T_n}^l\right)}{V_{t,T}^g / B_{t,T_n}^l} &= \frac{Z_{t-}^{T_n}}{V_{t-,T}^g} dV_{t,T}^g + dZ_t^{T_n} - \frac{Z_{t-}^{T_n}}{B_{t-,T_n}^l} dB_{t,T_n}^l \\ &+ \frac{Z_{t-}^{T_n}}{B_{t-,T_n}^l \cdot B_{t-,T_n}^l}[dB_{t,T_n}^l, dB_{t,T_n}^l] \\ &+ \frac{1}{V_{t-,T}^g}[dV_{t,T}^g, dZ_t^{T_n}] - \frac{Z_{t-}^{T_n}}{V_{t-,T}^g \cdot B_{t-,T_n}^l}[dV_{t,T}^g, dB_{t,T_n}^l] \\ &- \frac{1}{B_{t-,T_n}^l}[dB_{t,T_n}^l, dZ_t^{T_n}]\end{aligned} \tag{299}$$

is identical to

$$\begin{aligned}\frac{d\left(V_{t,T}^g Z_t^{T_n} / B_{t,T_n}^l\right)}{V_{t,T}^g / B_{t,T_n}^l} &= \sum_{i=1}^{2}\left(\phi_t^{i,c} + Z_{t-}^{T_n}(a^g(t,T)q_t^{gi} - a^l(t,T_n)q_t^{li})\right) dM_t^{i,c} \\ &+ \phi_{t\wedge\tau}^{3,c} dM_{t\wedge\tau}^{3,c} + \phi_t^d dM_t^d \\ &+ \sum_{i=1}^{2}\left(a^g(t,T)q_t^{gi} - a^l(t,T_n)q_t^{li}\right) \times \\ &\left(\phi_t^{i,c} - Z_{t-}^{T_n}(a^l(t,T_n)q_t^{li} + \theta_t^{i,c})\right) dt \\ &+ (r_t^g + y_t - r_t^l)dt\end{aligned} \tag{300}$$

the above considerations require the drift of (300) to be zero. The same argument applies to the processes

$$\begin{aligned}\frac{d\left(V_{t,T}^l Z_t^{T_n} / A_t^l\right)}{V_{t,T}^l / A_t^l} &= \sum_{i=1}^{2}\left(\phi_t^{i,c} + Z_{t-}^{T_n}(a^l(t,T) - a^l(t,T_n))q_t^{li}\right) dM_t^{i,c} \\ &+ \phi_{t\wedge\tau}^{3,c} dM_{t\wedge\tau}^{3,c} + \phi_t^d dM_t^d \\ &+ \left(a^l(t,T) - a^l(t,T_n)\right) \times \\ &\sum_{i=1}^{2} q_t^{li}\left(\phi_t^{i,c} - Z_{t-}^{T_n}(a^l(t,T_n)q_t^{li} + \theta_t^{i,c})\right) dt\end{aligned} \tag{301}$$

$$\frac{d\left(V_{t,T}^d Z_t^{T_n} / A_t^l\right)}{V_{t,T}^d / A_t^l} = \sum_{i=1}^{2}\left(\phi_t^{i,c} + Z_{t-}^{T_n} a^l(t,T)\right) dM_t^{i,c} \tag{302}$$

$$
\begin{aligned}
&\sum_{i=1}^{2}\left(\phi_t^{i,c}+Z_{t-}^{T_n}(a^{\lambda}(t,T)q_t^{\lambda i}-a^{l}(t,T_n))q_t^{li})\right)dM_t^{i,c}\\
&+Z_{t-}^{*}a^{\lambda}(t\wedge\tau,T)\sum_{i=1}^{3}q_{t\wedge\tau}^{\lambda i}dM_{t\wedge\tau}^{i,c}+\phi_{t\wedge\tau}^{3,c}dM_{t\wedge\tau}^{3,c}+\phi_t^{d}dM_t^{d}\\
&+Z_{t-}^{T_n}w_t\left(dN_t-\lambda_{t\wedge\tau}dt\right)+w_t\lambda_{t\wedge\tau}\left(\phi_t^{d}-Z_{t-}^{T_n}(\theta_t^{d}-1)\right)dt\\
&+\sum_{i=1}^{3}\left(a^{l}(t,T)q_t^{li}+a^{\lambda}(t,T)q_t^{\lambda i}-a^{l}(t,T_n))q_t^{li}\right)\times\\
&\left(\phi_t^{i,c}-Z_{t-}^{T_n}(a^{l}(t,T_n)q_t^{li}+\theta_t^{i,c})\right)dt
\end{aligned}
$$

yielding the restrictions

$$\phi_t^{i,c}=Z_{t-}^{T_n}\left(\theta_t^{i,c}+a^{l}(t,T_n)q_t^{li}\right) \tag{303}$$

$$\phi_t^{d}=Z_{t-}^{T_n}(\theta_t^{d}-1) \tag{304}$$

$$r_t^{g}+y_t-r_t^{l}=0 \tag{305}$$

which can be inserted into (298) to yield

$$
\begin{aligned}
Z_t^{T_n} &= Z_0^{T_n}+\sum_{i=1}^{2}\int_0^t Z_{s-}^{T_n}\theta_s^{i,c}dM_s^{i,c}+\int_0^t Z_{s\wedge\tau-}^{T_n}\theta_{s\wedge\tau}^{3,c}dM_s^{3,c} \\
&+\int_0^t Z_{s-}^{T_n}(\theta_s^{d}-1)dM_s^{d}
\end{aligned} \tag{306}
$$

the solution of which is given by (246).□

References

Aase, K. (1988): "Contingent claims valuation when the security price is a combination of an Ito process and a random point process," *Stochastic Processes and their Applications*, 28, 185-220.

Ahn, C. (1992): "Option pricing when jump risk is systematic," *Mathematical Finance*, 2/4, 299-308.

Amihud, Y. and H. Mendelson (1991): "Liquidity, maturity and the yields on U.S. treasury securities," *The Journal of Finance*, 46/4, 1411-1425.

Amin, K. (1993): "Jump diffusion option valuation in discrete time," *The Journal of Finance*, 48/5, 1833-1863.

Amin, K. and R. Jarrow (1991): "Pricing foreign currency options under stochastic interest rates," *Journal of International Money and Finance*, 10, 310-329.

Amin, K. and R. Jarrow (1992): "Pricing options on risky assets in a stochastic interest rate economy," *Mathematical Finance*, 2/4, 217-237.

Artzner, P. and F. Delbaen (1989): "Term structure of interest rates: The martingale approach," *Advances in Applied Mathematics*, 10, 95-129.

Artzner, P. and F. Delbaen (1995): "Default risk insurance and incomplete markets," *Mathematical Finance*, 5/3, 187-195.

Artzner, P. and D. Heath (1995): "Approximate completeness with multiple martingale measures," *Mathematical Finance*, 5/1, 1-10.

Babbs, S. (1991): "Interest rate swaps and default-free bonds: A joint term structure model," *Working Paper.*

Babbs, S. and N. Webber (1994): "A theory of the term structure with an official short rate," *Working Paper.*

Back, K. and S. Pliska (1991): "On the fundamental theorem of asset pricing with an infinite state space," *Journal of Mathematical Economics*, 20, 1-18.

Back, K. (1991): "Asset pricing for general processes," *Journal of Mathematical Economics*, 20, 371-395.

Beaglehole, D. and M. Tenney (1991): "General solutions of some interest rate contingent claim pricing equations", *Journal of Fixed Income* (September), 69-83.

Bremaud, P. (1981): *Point Processes and Queues: Martingale Dynamics* Springer Verlag.

Carverhill, A. (1994): "When is the short rate Markovian ?," *Mathematical Finance*, 4/4, 305-312.

Chen, R. and L. Scott (1992): "Pricing interest rate options in a two factor Cox-Ingersoll-Ross model of the term structure," *The Review of Financial Studies*, 5/4, 613-636.

Colwell, D. and R. Elliott (1993): "Discontinous asset prices and non-attainable contingent claims," *Mathematical Finance*, 3/3, 295-308.

Conze, A. and R. Viswanathan (1991): "Probability measures and numeraires," *Working Paper.*

Cox, J. and S. Ross (1976): "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, 3, 145-166.

Dalang, R., A. Morton and W. Willinger (1990): "Equivalent martingale measures and no-arbitrage in stochastic securities market models," *Stochastics and Stochastic Reports*, 29, 185-201.

Delbaen, F. (1992): "Representing martingale measures when asset prices are continous and bounded," *Mathematical Finance*, 2/2, 107-130.

Delbaen, F. and W. Schachermayer (1994a): "A general version of the fundamental theorem of asset pricing," *Mathematische Annalen*, 300, 463-520.

Delbaen, F. and W. Schachermayer (1994b): "Arbitrage and free lunch with bounded risk for unbounded continous processes," *Mathematical Finance*, 4/4, 343-348.

Duffie, D. (1986): "Stochastic equilibria: Existence, spanning number, and the 'no expected financial gain from trade' hypothesis," *Econometrica*, 54/5, 1161-1183.

Duffie, D. (1995): "State-space models of the term structure of interest rates," *Working Paper.*

Duffie, D. (1996): "Special repo rates," *The Journal of Finance*, 51/2, 493-526.

Duffie, D. and C. Huang (1985): "Implementing Arrow-Debreu equilibria by continous trading of few long-lived securities," *Econometrica*, 53/6, 1337-1356.

Duffie, D. and Huang, C. (1986): "Multiperiod security markets with differential information," *Journal of Mathematical Economics*, 15, 283-303.

Duffie, D. and R. Kan (1995): "Multi-factor term structure models," in: *Mathematical Models in Finance*, eds.: Howison et al..

Duffie, D. and P. Protter (1992): "From discrete- to continous-time finance: Weak convergence of the financial gain process," *Mathematical Finance*, 2/1, 1-15.

Dybvig, P. and C. Huang (1988): "Nonnegative wealth, absence of arbitrage, and feasable consumption plans," *The Review of Financial Studies*, 1/4, 377-401.

El-Jahel, L., H. Lindberg and W. Perraudin (1996): "Yield curves with jump short rates," *Working Paper.*

El Karoui, N and T. Cherif (1993): "Arbitrage entre deux marches: Applications aux options quanto," *Working Paper.*

El Karoui, N. and V. Lacoste (1992): "Multifactor models of the term structure of interest rate," *Working Paper.*

El Karoui, N., R. Myneni and R. Viswanathan (1992): "Arbitrage pricing and hedging with state variables: I Theory," *Working Paper.*

El Karoui, N. and J. Rochet (1989): "A pricing formula for options on coupon bonds," *Working Paper.*

Elliott, R. (1991): "Orthogonal martingale representation," in: *Stochastic Analysis*, eds.: E. Mayer-Wolf, E. Merzbach and S. Schwartz. San Diego: Academic Press, 139-152.

Elliott, R. and P. Kopp (1990): "Option pricing and hedge portfolios for poisson processes," *Stochastic Analysis and Applications*, 8/2, 157-176.

Foellmer, H. and M. Schweitzer (1986): "Hedging of contingent claims under incomplete information," *Applied Stochastic Analysis*, eds.: Davis, M. and R. Elliott.

Foellmer, H. and D. Sondermann (1986): "Hedging of non-redundant contingent claims," *Contributions to Mathemetical Finance*, eds.: Hildenbrand, W. and A. Mas-Colell.

Frachot, A. (1995): "Factor models of domestic and foreign interest rates with stochastic volatilities," *Mathematical Finance*, 5/2, 167-185.

Frachot, A., D. Janci and V. Lacoste (1992): "Factor analysis of the term structure: A probabilistic approach," *Working Paper.*

Frachot, A. and J. Lesne (1993a): "Econometrics of linear factor models of interest rates," *Working Paper.*

Frachot, A. and J. Lesne (1993b): "Expectations hypotheses and stochastic volatilities," *Working Paper.*

Geman, H., El-Karoui, N. and C. Rochet (1995): "Changes of numeraire, changes of probability measure and option pricing," *Journal of Applied Probability*, 32, 443-458.

Garman, M. and S. Kohlhagen (1983): "Foreign currency option values," *Journal of International Money and Finance*, 2, 231-237..

Grabbe, O. (1983): "The pricing of call and put options on foreign exchange," *Journal of International Money and Finance*, 2, 239-253.

Green, R. and B. Oedegaard (1995): "Estimation of implicit tax parameters from prices of default-free bonds," *Working Paper.*

Grinblatt, M. (1993): "An analytic solution for interest rate swap spreads," *Working Paper.*

Heath, D., R. Jarrow and A. Morton (1992): "Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation," *Econometrica*, 60/1, 77-105.

Harrison, M. (1990): *Brownian Motion and Stochastic Flow Fystems* Robert Krieger Publishing.

Harrison, J. and D. Kreps (1979): "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, 20, 381-408.

Harrison, J. and S. Pliska (1981): "Martingales and stochastic integrals in the theory of continous trading," *Stochastic Processes and their Applications*, 11, 215-260.

Harrison, J. and S. Pliska (1983): "A stochastic calculus model of continous trading: Complete markets," *Stochastic Processes and their Applications*, 15, 313-316.

Hofmann, N., E. Platen and M. Schweitzer (1992): "Option pricing under incompleteness and stochastic volatility," *Mathematical Finance*, 2/3, 153-187.

Hull, J. and A. White (1987): "Hedging the risks from writing currency options," *Journal of International Money and Finance*, 6, 131-152.

Jacod, J. and A. Shiryaev (1987): *Limit Theorems for Stochastic Processes* Springer Verlag.

Jarrow, R and D. Madan (1991): "A characterization of complete security markets on a Brownian filtration," *Mathematical Finance*, 1/3, 31-43.

Jarrow, R. and D. Madan (1994): "Valuing and hedging contingent claims on semimartingales," *Working Paper.*

Jarrow, R. and D. Madan (1995): "Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns," *Mathematical Finance*, 5/4, 311-336.

Jarrow, R. and S. Turnbull (1994): "Delta, gamma and bucket hedging of interest rate derivatives," *Applied Mathematical Finance*, 1, 21-48.

Jarrow, R. and S. Turnbull (1995): "Pricing derivatives on financial securities subject to credit risk," *The Journal of Finance*, 50/1, 53-85.

Jeffrey, A. (1995): "Single factor Heath-Jarrow-Morton term structure models based on markov spot interest rate dynamics," *Journal of Financial and Quantitative*

Analysis, 30/4, 619-642.

Jolliffe, I. (1986): *Principal Component Analysis* Springer Verlag.

Jouini, E. and H. Kallal (1995a): "Martingales and arbitrage in securities markets with transaction costs," *Journal of Economic Theory*, 66, 178-197.

Jouini, E. and H. Kallal (1995b): "Arbitrage in securities markets with short-sales constraints," *Mathematical Finance*, 5/3, 197-232.

Karatzas, I. and S. Shreve (1987): *Brownian Motion and Stochastic Calculus* Springer Verlag.

Kloeden, P. and E. Platen (1991): *Numerical Solutions of Stochastic Differential Equations* Springer Verlag.

Kreps, D. (1981): "Arbitrage and equilibrium in economies with infinitely many commodities," *Journal of Mathematical Economics*, 8, 15-35.

Lakner, P. (1993): "Martingale measures for a class of right-continous processes," *Mathematical Finance*, 3/1, 43-53.

Lando, D. (1994): "Three essays on contingent claims pricing," *PhD thesis*, Cornell University.

Lepingle, D. and J. Memin (1978): "Sur l'integrabilite uniforme des martingales exponentielles," *Zur Wahrscheinlichkeitstheorie verwandte Gebiete*, 42, 175-203.

Mercurio, F. and W. Runggaldier (1993): "Option pricing for jump diffusions: Approximations and their interpretation," *Mathematical Finance*, 3/2, 191-200.

Merton, R. (1976): "Option pricing when the underlying stock returns are discontinous," *Journal of Financial Economics*, 3, 125-144.

Miltersen, K. (1995): "An arbitrage theory of the term structure of interest rates," *The Annals of Applied Probability*, 4/4, 953-967.

Mood, A., F. Graybill and D. Boes (1974): *Introduction to the Theory of Statistics* McGraw-Hill.

Naik, V. and M. Lee (1990): "General equilibrium pricing of options on the market portfolio with discontinous returns," *The Review of Financial Studies*, 3/4, 493-521.

Protter, P. (1990): *Stochastic Integration and Differential Equations.* New York: Springer Verlag.

Ritchken, P. and L. Sankarasubramanian (1995): "Volatility structures of forward rates and the dynamics of the term structure," *Mathematical Finance*, 5/1, 55-72.

Ronn, E. and Y. Shin (1993): "Tax effects in U.S. government bond markets: The tax reform acts of 1984 and 1986," *Working Paper.*

Ross, S. (1983): *Stochastic Processes* Wiley.

Royden, H. (1988): *Real Analysis* Macmillan.

Schachermayer, W. (1994): "Martingale measures for discrete-time processes with infinite horizon," *Mathematical Finance*, 4/1, 25-55.

Schaefer, S. (1981): "Measuring a tax-specific term structure of interest rates in the market for British government securities," *The Economic Journal*, 91, 415-438.

Schaefer, S. (1982): "Tax-induced clientele effects in the market for British government securities," *Journal of Financial Economics*, 10, 121-159.

Schweitzer, M. (1990): " Risk minimality and orthogonality of martingales," *Stochastics and Stochastics Reports*, 30, 123-131.

Schweitzer, M. (1991): "Option hedging for semimartingales," *Stochastic Processes and their Applications*, 37, 339-363.

Schweitzer, M. (1992): "Martingale densities for general asset prices," *Journal of Mathematical Economics*, 21, 363-378.

Shirakawa, H. (1991): "Interest rate option pricing with Poisson-Gaussian forward rate curve processes," *Mathematical Finance*, 1/4, 77-94.

Stricker, C. (1990): "Arbitrage et lois de martingale," Ann. Inst. Henri Poincare, 26/3, 451-460.

Chapter 4: Pricing of Credit Contingent Claims: Specific Contracts

Introduction

The objective of the current chapter is to derive pricing results for a number of derivative contracts within the framework elaborated in the preceding chapter. In order to clarify the impact of credit risk on the valuation of contracts, for a given contract under consideration, I will first provide pricing results for the relevant contract without default risk. This analysis will include the pricing of contracts on treasury and interbank rates and will illuminate how the presence of the convenience yield associated with treasury positions affects valuation. Subsequently, valuation proceeds to consider pricing of credit contingent claims, clarifying the impact of introducing credit risk.

I consider the value of contracts at time t with payoffs occurring at time T; in most cases payoffs will be functions of instruments maturing at time M, where $t < T < M$. Regarding pricing results for forward and futures contracts, payoffs will be functions of four underlying instruments: The treasury bond, the Libor instrument, the corporate bond and a corporate instrument whose default parameters at the maturity date of the derivative contract have been reset to satisfy a given credit quality; this latter instrument will be referred to as the refreshed corporate bond. Payoffs may be defined in terms of money market yields $L_{t,T}$ which are defined by the following relationship:

$$B_{t,T} = \frac{1}{1 + \delta_{t,T} L_{t,T}} \tag{1}$$

$B_{t,T}$ is the reference instrument and $\delta_{t,T}$ is the interest accrual factor. The interest accrual factor is necessary because money market rates are quoted on an annualized basis; given that one wants to value the a money market deposit with maturities other than one year using quoted market rates, the above relation is relevant. With regard to the analysis in the next two chapters, contracts on underlying instruments with fixed times to maturity are being valued. For example, a forward contract on a $M-T$ period money market deposit pays off at its maturity date

$$B_{T,M} - B^x = \frac{1}{1 + \delta_{T,M} L_{T,M}} - B^x \tag{2}$$

where B^x is the forward price. Given that the time to maturity of the underlying contract is fixed by the contract specification, we can drop the subscript on the interest accrual factor and set $\delta_{T,M} = \delta$, reflecting the fact that for the purpose of the contract specification, $M - T$ is fixed. As indeed this is the case for the contracts under consideration in this and the following chapter, the subscripts on the interest accrual factor will be dropped[22].

[22] Note that none of the results are affected by making the time dependency of the interest accrual

It is noted that contracts being valued in the current chapter are assumed to be free of counterparty default risk. Throughout this chapter it is assumed that volatilities and correlations are deterministic.

The current chapter is structured as follows: The first section contains general results regarding contingent claims pricing within the framework established in chapter 2. The two pricing functions are introduced and an additional general pricing function for continuously resettled claims is motivated. The second section is concerned with the pricing of forward contracts on prices and yields relating to different assets. It will be shown that the presence of credit risk generates specific convexity adjustments not present otherwise. The third section is concerned with the pricing of futures contracts and illustrates the impact of the continuous settlement feature on the valuation of credit contingent claims. Section 4 briefly considers the valuation of options on credit spreads. Section 5 is devoted to the analysis of structured products containing credit elements. Having discussed some generic structures, I will consider the valuation of specific structures that have appeared in the markets.

4.1. Contingent Claims Pricing

In order to establish the link between the framework outlined in the preceding chapter and the pricing of specific contracts, this section briefly considers general results on the pricing of contingent claims and continuously settled contracts.

Subject to regularity conditions, the uniqueness of the martingale measures identified in chapter 2 is equivalent to the markets for interest rate risk and credit risk being complete. In such a setting, any contingent claims can be synthetically constructed through trading strategies involving the discount instruments. The admission of contingent claims into the present framework must not invalidate the arbitrage free nature of the market; this requirement necessitates the imposition of additional structure on the permitted trading strategies involving the construction of synthetic claims.

Definition 4.1 Given a contingent claim with terminal value $V_T = X$, a trading strategy, characterized by its holdings φ_t^i, $i \in [g, l, d]$, in the discount instruments, is called admissible if

1.) φ_t^i, $i \in [g, l, d]$, are predictable with respect to $\mathcal{F}_t$

factor explicit, given that at the time of valuation, the value of this variable is known and can be considered fixed for the purpose of contract valuation.

2.) Under P^*, normalized value processes for synthetic claims are P^*-martingales and satisfy

$$\frac{V_t}{A_t^l} = \frac{V_0}{A_0^l} + \int_0^t \varphi_s^g d\left(\frac{V_{s,T}^g}{A_s^l}\right) + \int_0^t \varphi_s^l d\left(\frac{V_{s,T}^l}{A_s^l}\right) + \int_0^t \varphi_s^d d\left(\frac{V_{s,T}^d}{A_s^l}\right) \tag{3}$$

while price processes must satisfy

$$V_t = V_0 + \int_0^t \varphi_s^A dA_s^l + \int_0^t \varphi_s^g dV_{s,T}^g + \int_0^t \varphi_s^l dV_{s,T}^l + \int_0^t \varphi_s^d dV_{s,T}^d \tag{4}$$

3.) Under P^{T_n}, normalized value processes for synthetic claims are P^{T_n}-martingales and satisfy

$$\frac{V_t}{B_{t,T_n}^l} = \frac{V_0}{B_{0,T_n}^l} + \int_0^t \varphi_s^g d\left(\frac{V_{s,T}^g}{B_{s,T_n}^l}\right) + \int_0^t \varphi_s^l d\left(\frac{V_{s,T}^l}{B_{s,T_n}^l}\right) + \int_0^t \varphi_s^d d\left(\frac{V_{s,T}^d}{B_{s,T_n}^l}\right) \tag{5}$$

while price processes must satisfy

$$V_t = V_0 + \int_0^t \varphi_s^{T_n} dB_{s,T_n}^l + \int_0^t \varphi_s^g dV_{s,T}^g + \int_0^t \varphi_s^l dV_{s,T}^l + \int_0^t \varphi_s^d dV_{s,T}^d \tag{6}$$

4.) At the maturity of the contingent claim,

$$V_T = X_T \tag{7}$$

The first condition stipulates that portfolio selection can depend at most on information available to the market at time t. The self financing conditions (2)-(5) reflect the requirement that the change in portfolio value must be due to the capital gains/losses from the underlying securities. The last condition requires that the portfolio matches the contingent claim's cash flow at the maturity date. Note that (2) and (4) imply

$$V_t = E_t^* \left[X_T A_t^l / A_T^l\right] \tag{8}$$

$$V_t = B_{t,T}^l \times E_t^T \left[X_T\right] \tag{9}$$

(7) and (8) will serve as the main pricing tools when the valuation of specific contracts is considered. To illustrate equivalence of the two pricing functions, consider the value of a contract the payoff of which at time T is given by X_T; from (7), the value of this contract is given by

$$V_t = E_t^* \left[\frac{X_T A_t^l}{A_T^l}\right] \tag{10}$$

From Girsanov's theorem,

$$E_t^* \left[\frac{X_T A_t^l}{A_T^l}\right] = \frac{1}{E_t^T \left[Z_0^{*,T}\right]} \times E_t^T \left[Z_0^{*,T} \frac{X_T A_t^l}{A_T^l}\right] \tag{11}$$

here

$$Z_0^{*,T} = \frac{dP^*}{dP^T} = \frac{A_T^l B_{0,T}^l}{A_0^l B_{T,T}^l} \tag{12}$$

is a P^T-martingale. Inserting (11) into (10) yields

$$V_t = \frac{A_0^l B_{t,T}^l}{A_t^l B_{0,T}^l} \times E_t^T \left[\frac{A_T^l B_{0,T}^l}{A_0^l B_{T,T}^l} \frac{X_T A_t^l}{A_T^l} \right] \tag{13}$$

which is identical to (8). Having outlined the main pricing tools to be used, I briefly add a result which will facilitate the valuation of continuously resettled claims, the proof of which is adapted from Duffie and Singleton (1995).

Proposition 4.1 The price of a continuously resettled claim H_t which pays a continuous dividend of y_t and by contract specification is set to a value $H_T = X$ at maturity is given by

$$H_t = E_t^* \left[H_T + \int_t^T y_s ds \right] \tag{14}$$

Proof:

Under the spot risk measure the trading gains process associated with a long position in this contract must be given by

$$dV_u^H = r_u^l V_u^H du + \varphi_u^H \left(dH_u + y_u du \right) \tag{15}$$

where V_u^H is the value of the margin account and

$$\varphi_u^H = \exp \int_t^u r_s^l ds \tag{16}$$

is the process indicating the size of the position. Noting that

$$d\left(\frac{V_u^H A_t^H}{A_u^H} \right) = \frac{A_t^H}{A_u^H} \varphi_u^H \left(dH_u + y_u du \right) \tag{17}$$

(18) can be integrated to yield.

$$\frac{V_T^H A_t^H}{A_T^H} = V_t^H + H_T - H_t + \int_t^T y_s ds \tag{18}$$

Setting $V_t^H = H_t$ and $\varphi_u^H = A_u^H / A_t^H$ and noting that the normalized trading gains process must be a P^*-martingale establishes (14).

This completes the description of general pricing operators to be applied in the following section. I proceed to the pricing of specific contracts.

4.2. Forward Contracts

The current section is devoted to the valuation of forward contracts on zero coupon bond prices and zero coupon bond yields. Regarding the structure of contracts priced, the following is relevant. Forward prices or yields carry the subscript F: The forward price at time t for a contract maturing at time T is denoted by $B^F_{t,T,q}$, where $M = T+q$ is the maturity of the underlying instrument. The corresponding expression for the forward yield is given by $B^F_{t,T,q}$. In both cases, additional superscripts will indicate the type of underlying instrument.
Of particular interest will be the appearance of convexity adjustments arising from the presence of credit risk when considering contracts the payoffs of which are non linear functions of credit risky zero coupon bond prices.

4.2.1. Forward Contracts on Prices

The current section is devoted to the valuation of forward contracts on zero coupon bond prices. I consider the four different underlying assets in turn.

Proposition 4.2 The forward price $B^{g,F}_{t,T,q}$ for a zero value contract on the treasury bond is given by

$$B^{g,F}_{t,T,q} = \frac{B^g_{t,M}}{B^l_{t,T}} \times \exp\left(-\int_t^T y_s ds\right) \tag{19}$$

Proof:
Using (8) and the contract's payoff definition, the value of this contract is given by

$$V_t = B^l_{t,T} \times E^T_t\left[B^g_{T,M} - B^{g,F}_{t,T,q}\right] \tag{20}$$

Furthermore, under P^T, the following restriction must hold:

$$\begin{aligned} \frac{B^g_{T,M}}{B^l_{T,T}} &= \frac{B^g_{t,M}}{V^l_{t,T}} \times \exp\left(-\int_t^T y_s ds\right) \\ &\times \exp\left(\sum_{i=1}^3 \int_t^T \left(q^{gi}_s a^g(s,M) - q^{li}_s a^l(s,T)\right) dM^{i,c,T}_s\right) \\ &\times \exp\left(-\frac{1}{2}\sum_{i=1}^3 \int_t^T \left(q^{gi}_s a^g(s,M) - q^{li}_s a^l(s,T)\right)^2 ds\right) \end{aligned} \tag{21}$$

Inserting (22) into (21) and setting the contract value to zero yields (20).□

(20) shows the forward price for the treasury bond to be given by the standard cash-and-carry term, multiplied by a component involving the cumulated convenience yield. Note that $B^{g,F}_{t,T,q} < B^{g}_{t,M}/B^{l}_{t,T}$, reflecting the fact that by taking a long position in the forward contract instead of buying in the spot market, investors forego the convenience yield accruing to investors with long positions in the instrument. While (20) does not appear in the literature, a similar result could be established in the framework proposed by Grinblatt (1993).

Proposition 4.3 The forward price $B^{g,F}_{t,T,q}$ for a zero value contract on the Libor bond is given by

$$B^{l,F}_{t,T,q} = \frac{B^{l}_{t,M}}{B^{l}_{t,T}} \tag{22}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$V_t = B^{l}_{t,T} \times E^{T}_{t}\left[B^{l}_{T,M} - B^{l,F}_{t,T,q}\right] \tag{23}$$

Setting the contract value to zero yields (23).□

(23) is the standard cash-and-carry textbook result (see Fitzgerald (1993) or Hull (1993)). I turn to the valuation of forward contracts on credit risky zero coupon bonds.

Proposition 4.4 The forward price $B^{d,F}_{t,T,q}$ for a zero value contract on the corporate bond is given by

$$B^{d,F}_{t,T,q} = \frac{B^{d}_{t,M}}{B^{l}_{t,T}} \tag{24}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$V_t = B^{l}_{t,T} \times E^{T}_{t}\left[V^{d}_{T,M} - B^{l,F}_{t,T,q}\right] \tag{25}$$

Under P^T, the following restriction must hold:

$$\begin{aligned}
\frac{V^{d}_{T,M}}{V^{l}_{T,T}} &= \frac{V^{d}_{t,M}}{V^{l}_{t,T}} \times \exp\left(-\int_{t}^{T} w_s \lambda^{T}_{s} ds - k\int_{t}^{T} h_s dN_s\right) \\
&\quad \times \exp\left(\sum_{i=1}^{3}\int_{t}^{T}\left(q^{li}_{s}(a^{l}(s,M) - a^{l}(s,T)) + w_s q^{\lambda i}_{s} a^{\lambda}(s,M)\right) dM^{i,c,T}_{s}\right) \\
&\quad \times \exp\left(-\frac{1}{2}\sum_{i=1}^{3}\int_{t}^{T}\left(q^{li}_{s}(a^{l}(s,M) - a^{l}(s,T)) + w_s q^{\lambda i}_{s} a^{\lambda}(s,M)\right)^{2} ds\right)
\end{aligned} \tag{26}$$

Inserting (27) into (26) and setting the contract value to zero yields (25).□

(25) demonstrates that the standard cash-and-carry result holds for forward prices on credit risky zero coupon bonds. With regard to the valuation of forward contracts on the refreshed corporate zero coupon bond, the following proposition can be proved:

Proposition 4.5 The forward price $B^{r,F}_{t,T,q}$ for a zero value contract on the refreshed corporate bond is given by

$$B^{r,F}_{t,T,q} = \frac{B^d_{t,M}}{B^l_{t,T}} \times E^T_t \left[\exp\left(-\int_t^T w_s \lambda^T_s ds \right) \right] \tag{27}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$V_t = B^l_{t,T} \times E^T_t \left[V^d_{T,M} \cdot I_{\{T<\tau\}} - B^{l,F}_{t,T,q} \right] \tag{28}$$

Under P^T, the following restriction must hold:

$$\begin{aligned} \frac{V^d_{T,M} \cdot I_{\{T<\tau\}}}{V^l_{T,T}} &= \frac{V^d_{t,M} \cdot I_{\{T<\tau\}}}{V^l_{t,T}} \times \exp\left(-\int_t^T w_s \lambda^T_s ds \right) \\ & \times \exp\left(\sum_{i=1}^3 \int_t^T \left(q^{li}_s (a^l(s,M) - a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M) \right) dM^{i,c,T}_s \right) \\ & \times \exp\left(-\frac{1}{2} \sum_{i=1}^3 \int_t^T \left(q^{li}_s (a^l(s,M) - a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M) \right)^2 ds \right) \end{aligned} \tag{29}$$

Inserting (30) into (29) and setting the contract value to zero yields (28).□

Note that (28) differs from (25) by a term involving the integral over the instantaneous credit spread. Given that the spread is specified to be stochastic, the integral is embraced by a conditional expectation. This adjustment reflects the contractual specification that the corporate bond to be delivered into the forward contract has its credit parameters refreshed at the maturity date of the forward contract. Between time t and maturity M, this artificial instrument accrues at an instantaneous return differential, relative to the Libor asset, of $(1-e^{-kh_t})\lambda^T_t$. Results (28) and (25) quantify the impact of credit risk in the underlying asset on forward prices. Results of this kind do not appear in the literature. I now turn to the valuation of forward contracts of yields, the payoffs to which are non linear functions of zero coupon bond prices. Relative to the pricing results established in the present section, one should expect convexity adjustments due to nonlinearity effects.

4.2.2. Forward Contracts on Yields

The current section focuses on the valuation of contracts on yields. Relative to contracts written on prices, it will be of interest to establish the precise form of convexity adjustment arising from the presence of credit risk. In general, the contracts priced in the current section mature at time T and are written on money market yields on reference instruments maturing at time M.

Consider the valuation of a forward contract on the zero coupon yield on the treasury instrument.

Proposition 4.6 The forward yield $L^{g,F}_{t,T,q}$ for a zero value contract on the treasury bond yield is given by

$$\begin{aligned} 1+\delta L^{g,F}_{t,T,q} &= \frac{B^l_{t,T}}{B^g_{t,M}} \times \exp\left(-\int_t^T y_s ds\right) \\ & \times \exp\left(-\int_t^T \left(a^g(s,M)^2 - \rho^{gl}_s a^g(s,M)a^l(s,T) + a^l(s,T)^2\right) ds\right) \end{aligned} \tag{30}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= B^l_{t,T} \times E^T_t \left[L^g_{T,M} - L^{g,F}_{t,T,q}\right] \\ &= B^l_{t,T} \times E^T_t \left[\frac{1}{B^g_{T,M}} - (1+\delta L^{g,F}_{t,T,q})\right] \end{aligned} \tag{31}$$

Under P^T, the following restriction must hold:

$$\begin{aligned} \frac{B^l_{T,T}}{B^g_{T,M}} &= \frac{B^l_{t,T}}{B^g_{t,M}} \times \exp\left(\int_t^T y_s ds\right) \\ & \times \exp\left(-\sum_{i=1}^{3}\int_t^T \left(q^{gi}_s a^g(s,M) - q^{li}_s a^l(s,T)\right) dM^{i,c,T}_s\right) \\ & \times \exp\left(\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(q^{gi}_s a^g(s,M) - q^{li}_s a^l(s,T)\right)^2 ds\right) \end{aligned} \tag{32}$$

Inserting (33) into (32) and setting the contract value to zero yields (31).□

Regarding the pricing of a forward contract on the zero coupon yield on the Libor deposit, the following result can be established.

Proposition 4.7 The forward yield $L^{l,F}_{t,T,q}$ for a zero value contract on the Libor bond yield is given by

$$1 + \delta L^{l,F}_{t,T,q} = \frac{B^l_{t,T}}{B^l_{t,M}} \times \exp\left(\int\limits_t^T \left(a^l(s,M) - a^l(s,T)\right)^2 ds \right) \tag{33}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= B^l_{t,T} \times E^T_t \left[L^l_{T,M} - L^{l,F}_{t,T,q} \right] && (34) \\ &= B^l_{t,T} \times E^T_t \left[\frac{1}{B^l_{T,M}} - (1 + \delta L^{l,F}_{t,T,q}) \right] \end{aligned}$$

Under P^T, the following restriction must hold:

$$\begin{aligned} \frac{B^l_{T,T}}{B^l_{T,M}} &= \frac{B^l_{t,T}}{B^l_{t,M}} \times \exp\left(-\sum_{i=1}^{3} \int\limits_t^T q^{l_i}_s \left(a^l(s,M) - a^l(s,T)\right) dM^{i,c,T}_s \right) && (35) \\ &\quad \times \exp\left(\frac{1}{2} \sum_{i=1}^{3} \int\limits_t^T \left(a^l(s,M) - a^l(s,T)\right)^2 ds \right) \end{aligned}$$

Inserting (36) into (35) and setting the contract value to zero yields (34).□

(31) and (34) illustrate the impact of payoffs being non linear functions of prices for the case of interest rate risk. Results of this type are standard and can be found in, among others, Hanweck (1995). Results relating to convexity adjustments arising from uncertainty in credit risk have so far not been published. The following two propositions demonstrate that such adjustments are readily identifiable within the present framework.

Proposition 4.8 The forward yield $L^{d,F}_{t,T,q}$ for a zero value contract on the corporate bond yield is given by

$$\begin{aligned} 1 + \delta L^{d,F}_{t,T,q} &= \frac{B^l_{t,T}}{B^d_{t,M}} \times E^T_t \left[\exp\left(\int\limits_t^T \left(e^{-kh_s} - 1 + e^{kh_s} - 1\right) \lambda^T_s ds \right) \right] && (36) \\ &\quad \times \exp\left(\int\limits_t^T \left(q^{l_i}_s (a^l(s,M) - a^l(s,T)) + w_s q^{\lambda_i}_s a^\lambda(s,M) \right)^2 ds \right) \end{aligned}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= B^l_{t,T} \times E^T_t \left[L^d_{T,M} - L^{d,F}_{t,T,q} \right] && (37) \\ &= B^l_{t,T} \times E^T_t \left[\frac{1}{V^d_{T,M}} - (1 + \delta L^{d,F}_{t,T,q}) \right] \end{aligned}$$

Under P^T, the following restriction must hold:

$$\begin{aligned}\frac{B^l_{T,T}}{V^d_{T,M}} &= \frac{B^d_{t,T}}{V^l_{t,M}} \times \exp\left(\int_t^T w_s\lambda^T_s ds + k\int_t^T h_s dN_s\right) \\ &\times \exp\left(-\sum_{i=1}^{3}\int_t^T \left(q^{li}_s(a^l(s,M)-a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M)\right) dM^{i,c,T}_s\right) \\ &\times \exp\left(\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(q^{li}_s(a^l(s,M)-a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M)\right)^2 ds\right)\end{aligned} \tag{38}$$

Inserting (39) into (38) and setting the contract value to zero yields (37).□

Proposition 4.9 The forward yield $L^{r,F}_{t,T,q}$ for a zero value contract on the refreshed corporate bond yield is given by

$$1+\delta L^{r,F}_{t,T,q} = \frac{B^l_{t,T}}{B^d_{t,M}} \times E^T_t\left[\exp\left(\int_t^T w_s\lambda^T_s ds\right)\right] \tag{39}$$

$$\times \exp\left(\int_t^T \left(q^{li}_s(a^l(s,M)-a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M)\right)^2 ds\right)$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= B^l_{t,T} \times E^T_t\left[L^d_{T,M}\cdot I_{\{T<\tau\}} - L^{r,F}_{t,T,q}\right] \\ &= B^l_{t,T} \times E^T_t\left[\frac{1}{V^d_{T,M}\cdot I_{\{T<\tau\}}} - (1+\delta L^{r,F}_{t,T,q})\right]\end{aligned} \tag{40}$$

Under P^T, the following restriction must hold:

$$\begin{aligned}\frac{B^l_{T,T}}{V^d_{T,M}\cdot I_{\{T<\tau\}}} &= \frac{B^d_{t,T}}{V^l_{t,M}\cdot I_{\{T<\tau\}}} \times \exp\left(\int_t^T w_s\lambda^T_s ds\right) \\ &\times \exp\left(-\sum_{i=1}^{3}\int_t^T \left(q^{li}_s(a^l(s,M)-a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M)\right) dM^{i,c,T}_s\right) \\ &\times \exp\left(\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(q^{li}_s(a^l(s,M)-a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M)\right)^2 ds\right)\end{aligned} \tag{41}$$

Inserting (42) into (41) and setting the contract value to zero yields (40).□

(55) into (40) demonstrate that within the present framework explicit convexity adjustments arising from the presence of credit risk can be calculated. Furthermore,

$$L^{d,F}_{t,T,q} \geq L^{r,F}_{t,T,q} \tag{42}$$

Given that the refreshed corporate bond is default free between t and M, if it was available for trading it would have to sell at a higher price, or lower yield, than the credit risky corporate bond.

It should be noted that the convexity adjustments identified above are similar to terms arising when effecting a change of probability measure. It thus seems likely that probability measures can be identified under which contracts are priced as if no convexity existed. Application of the change of measure methodology in order to eliminate convexity terms seems an interesting area for future research.

I next turn to the pricing of forward contracts on yields with settlements in arrears.

4.2.3. Forward Contracts on Yields with Settlement in Arrears

Contracts on yields examined in the previous section were assumed to settle at time T. It is possible that contracts trade which stipulate for payoffs, determined at time T, to be settled at time M, where $T < M$. Individual payments in a swap agreement provide one example of such contracts. The following section examines pricing of contracts written on yields, determined at time T, that settle in arrears at time M.

Proposition 4.10 The forward yield $L^{l,F}_{t,M,q}$ for a zero value contract on the Libor yield with settlement in arrears is given by

$$1 + \delta L^{l,F}_{t,M,q} = \frac{B^l_{t,T}}{B^l_{t,M}} \tag{43}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= B^l_{t,M} \times E^M_t \left[L^l_{T,M} - L^{l,F}_{t,M,q} \right] \\ &= B^l_{t,M} \times E^M_t \left[\frac{1}{B^l_{T,M}} - (1 + \delta L^{l,F}_{t,M,q}) \right] \\ &= B^l_{t,M} \times E^T_t \left[\frac{B^l_{T,M} B^l_{t,T}}{B^l_{t,M} B^l_{T,T}} \left(\frac{1}{B^l_{T,M}} - (1 + \delta L^{l,F}_{t,M,q}) \right) \right] \\ &= B^l_{t,T} \times E^T_t \left[\frac{B^l_{T,T}}{B^l_{T,T}} - \frac{B^l_{T,M}}{B^l_{T,T}} (1 + \delta L^{l,F}_{t,M,q}) \right] \end{aligned} \tag{44}$$

Using (47) and setting the contract value in (49) to zero yields (48).

(48) demonstrates that the reciprocal of the forward price for a forward contract on yields with settlement in arrears at time M is identical to the forward price for a forward contract on prices for settlement at time T. This equivalence can only be

generated when the underlying asset in the forward contract is identical with the numeraire asset. In the case of the treasury bond, examination of (44) demonstrates that the above equivalence does not hold. It must further be noted that this result hinges on a change of probability measure from P^M to P^T; without this step the equivalence does not hold.

Consider the pricing of forward contracts on yields on the straight and the refreshed corporate zero coupon bond with settlement in arrears.

Proposition 4.11 The forward yield $L^{d,F}_{t,M,q}$ for a zero value contract on the corporate bond yield with settlement in arrears is given by

$$\begin{aligned} 1+\delta L^{d,F}_{t,M,q} &= \frac{B^l_{t,T}}{B^d_{t,M}} \times E^T_t \left[\exp\left(\int_t^T \left(e^{-kh_s} - 1 + e^{kh_s} - 1\right) \lambda^T_s ds \right) \right] \\ &\times \exp\left(\int_t^T \left(a^\lambda(s,M)^2 - \rho^{\lambda l}_s a^l(s,M)(a^l(s,T) - a^l(s,M))\right) ds \right) \end{aligned} \tag{45}$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= B^l_{t,M} \times E^M_t \left[L^d_{T,M} - L^{d,F}_{t,M,q} \right] \\ &= B^l_{t,M} \times E^M_t \left[\frac{1}{V^d_{T,M}} - (1+\delta L^{d,F}_{t,M,q}) \right] \\ &= B^l_{t,M} \times E^T_t \left[\frac{B^l_{T,M} B^l_{t,T}}{B^l_{t,M} B^l_{T,T}} \left(\frac{1}{V^d_{T,M}} - (1+\delta L^{d,F}_{t,M,q}) \right) \right] \\ &= B^l_{t,T} \times E^T_t \left[\frac{B^l_{T,M}}{V^d_{T,M}} - \frac{B^l_{T,M}}{B^l_{T,T}} (1+\delta L^{d,F}_{t,M,q}) \right] \end{aligned} \tag{46}$$

Under the P^T measure, the following restrictions must hold

$$\begin{aligned} \frac{B^l_{T,M}}{V^d_{T,M}} &= \frac{B^l_{t,M}}{V^d_{t,M}} \times \exp\left(\int_t^T w_s \lambda^T_s ds + k \int_t^T h_s dN_s \right) \\ &\times \exp\left(-\sum_{i=1}^3 \int_t^T q^{\lambda i}_s a^\lambda(s,M) dM^{i,c,T}_s + \frac{1}{2} \sum_{i=1}^3 \int_t^T a^\lambda(s,M)^2 ds \right) \\ &\times \exp\left(-\sum_{i=1}^3 \int_t^T q^{li}_s q^{\lambda i}_s a\lambda(s,M)(a^l(s,T) - a^l(s,M)) ds \right) \end{aligned} \tag{47}$$

Using (47) and setting the contract value in (49) to zero yields (48).

Proposition 4.12 The forward yield $L^{r,F}_{t,M,q}$ for a zero value contract on the refreshed corporate bond yield with settlement in arrears is given by

$$1+\delta L^{r,F}_{t,M,q}=\frac{B^l_{t,T}}{B^d_{t,M}}\times E^T_t\left[\exp\left(\int_t^T w_s\lambda^T_s ds\right)\right] \tag{48}$$

$$\times\exp\left(\int_t^T\left(a^\lambda(s,M)^2-\rho^{\lambda l}_s a^l(s,M)(a^l(s,T)-a^l(s,M))\right)ds\right)$$

Proof:

Using (8) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned}
V_t &= B^l_{t,M}\times E^M_t\left[L^d_{T,M}\cdot I_{\{T<\tau\}}-L^{d,F}_{t,M,q}\right] \qquad (49)\\
&= B^l_{t,M}\times E^M_t\left[\frac{1}{V^d_{T,M}\cdot I_{\{T<\tau\}}}-(1+\delta L^{d,F}_{t,M,q})\right]\\
&= B^l_{t,M}\times E^T_t\left[\frac{B^l_{T,M}B^l_{t,T}}{B^l_{t,M}B^l_{T,T}}\left(\frac{1}{V^d_{T,M}\cdot I_{\{T<\tau\}}}-(1+\delta L^{d,F}_{t,M,q})\right)\right]\\
&= B^l_{t,T}\times E^T_t\left[\frac{B^l_{T,M}}{V^d_{T,M}\cdot I_{\{T<\tau\}}}-\frac{B^l_{T,M}}{B^l_{T,T}}(1+\delta L^{d,F}_{t,M,q})\right]
\end{aligned}$$

Under the P^T measure, the following restrictions must hold

$$\begin{aligned}
\frac{B^l_{T,M}}{V^d_{T,M}\cdot I_{\{T<\tau\}}} &= \frac{B^l_{t,M}}{V^d_{t,M}}\times\exp\left(\int_t^T w_s\lambda^T_s ds\right) \qquad (50)\\
&\times\exp\left(-\sum_{i=1}^3\int_t^T q^{\lambda_i}_s a^\lambda(s,M)dM^{i,c,T}_s+\frac{1}{2}\sum_{i=1}^3\int_t^T a^\lambda(s,M)^2ds\right)\\
&\times\exp\left(-\sum_{i=1}^3\int_t^T q^{l_i}_s q^{\lambda_i}_s a\lambda(s,M)(a^l(s,T)-a^l(s,M))ds\right)
\end{aligned}$$

Using (55) and setting the contract value in (54) to zero yields (53).□

The above results demonstrate that the 'settlement in arrears at time M' feature generates additional convexity adjustments relative to the pricing of contracts with settlement at time T. Only for the forward contract on Libor yields is it the case that the settlement in arrears feature neutralizes the convexity adjustment in forward contracts on yields with settlement at time T. I proceed to consider the pricing of continuously resettled contracts.

4.3. Futures Contracts

The current section is devoted to the analysis of the continuously resettled counterparts of the contracts in the preceding section. The superscript H indicates futures prices or yields.

4.3.1. Futures Contracts on Prices

Turning to the valuation of futures contracts on the corporate assets the following two propositions can be established.

Proposition 4.13 The futures price $B^{d,H}_{t,T,q}$ for a continuously resettled zero value contract on the corporate bond is given by

$$B^{d,H}_{t,T,q} = \frac{B^d_{t,M}}{B^l_{t,T}} \times \exp\left(\int_t^T \left(a^l(s,T)^2 - a^l(s,T)a^l(s,M)/2 + \rho^{\lambda l}_s a^\lambda(s,M)(a^l(s,M) - a^l(s,T))\right) ds\right) \tag{51}$$

Proof:

Using (14) and the contract's payoff definition, the value of this contract is given by

$$V_t = E^*_t\left[V^d_{T,M} - B^{d,H}_{t,T,q}\right] \tag{52}$$

Under P^*, the following restriction must hold:

$$\begin{aligned} \frac{V^d_{T,M}}{B^l_{T,T}} &= \frac{V^d_{t,M}}{B^l_{t,T}} \times \exp\left(-\int_t^{T\wedge\tau} w_s\lambda^*_s ds - k\int_t^T h_s dN_s\right) \\ &\times \exp\left(\sum_{i=1}^3 \int_t^T \left(q^{li}_s(a^l(s,M) - a^l(s,T)) + w_s q^{\lambda i}_s a^\lambda(s,M)\right) dM^{i,c,*}_s\right) \\ &\times \exp\left(-\frac{1}{2}\sum_{i=1}^3 \int_t^T \left(a^l(s,M)^2 - a^l(s,T)^2 + w_s^2 a^\lambda(s,M)^2 - \rho^{\lambda l}_s a^\lambda(s,M)a^l(s,M)\right) ds\right) \end{aligned} \tag{53}$$

Using (64) and setting the contract value in (63) to zero yields (62).□

Proposition 4.14 The futures price $B^{r,H}_{t,T,q}$ for a continuously resettled zero value contract on the refreshed corporate bond is given by

$$B^{r,H}_{t,T,q} = \frac{B^d_{t,M}}{B^l_{t,T}} \times E^*_t\left[\exp\left(-\int_t^T w_s\lambda^*_s ds\right)\right] \tag{54}$$

$$\times \exp\left(\int_t^T \left(a^l(s,T)^2 - a^l(s,T)a^l(s,M)/2 + \rho_s^{\lambda l} a^\lambda(s,M)(a^l(s,M) - a^l(s,T))\right) ds\right)$$

Proof:

Using (14) and the contract's payoff definition, the value of this contract is given by

$$V_t = E_t^* \left[V_{T,M}^d \cdot I_{\{T<\tau\}} - B_{t,T,q}^{r,H}\right] \tag{55}$$

Under P^*, the following restriction must hold:

$$\begin{aligned} \frac{V_{T,M}^d \cdot I_{\{T<\tau\}}}{B_{T,T}^l} &= \frac{B_{t,M}^d}{B_{t,T}^l} \times \exp\left(-\int_t^{T\wedge\tau} w_s \lambda_s^* ds\right) \\ &\times \exp\left(\sum_{i=1}^3 \int_t^T \left(q_s^{li}(a^l(s,M) - a^l(s,T)) + w_s q_s^{\lambda i} a^\lambda(s,M)\right) dM_s^{i,c,*}\right) \\ &\times \exp\left(-\frac{1}{2}\sum_{i=1}^3 \int_t^T \left(a^l(s,M)^2 - a^l(s,T)^2 + w_s^2 a^\lambda(s,M)^2 - \rho_s^{\lambda l} a^\lambda(s,M) a^l(s,M)\right) ds\right) \end{aligned} \tag{56}$$

Using (67) and setting the contract value in (66) to zero yields (65).□

(62) and (65) provide pricing formulae for futures contracts on zero coupon bonds subject to credit risk. I am not aware of any results of this type in the literature.

4.3.2. Futures Contracts on Yields

The present section considers valuation of futures contracts on yields.

Proposition 4.15 The futures yield $L_{t,T,q}^{g,H}$ for a continuously resettled zero value contract on the treasury bond yield is given by

$$1 + \delta L_{t,T,q}^{g,H} = \frac{B_{t,T}^l}{B_{t,M}^g} \times \exp\left(\int_t^T a^g(s,M)\left(a^g(s,M) - \rho_s^{gl} a^l(s,T)\right) ds + \int_t^T y_s ds\right) \tag{57}$$

Proof:

Using (14) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= E_t^* \left[\left(L_{T,M}^g - L_{t,T,q}^{g,H}\right)\delta\right] \\ &= E_t^* \left[\frac{1}{B_{T,M}^g} - \left(1 + \delta L_{t,T,q}^{g,H}\right)\right] \end{aligned} \tag{58}$$

Under P^*, the following restriction must hold:

$$\frac{B_{T,T}^l}{B_{T,M}^g} = \frac{B_{t,T}^l}{B_{t,M}^g} \times \exp\left(\int_t^T y_s ds\right) \tag{59}$$

$$\times \exp\left(-\sum_{i=1}^{3}\int_t^T \left(q_s^{gi}a^g(s,M) - q_s^{li}a^l(s,T)\right) dM_s^{i,c,*}\right)$$

$$\times \exp\left(\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(a^g(s,M)^2 - a^l(s,T)^2\right) ds\right)$$

Using (59) and setting the contract value in (58) to zero yields (69).□

Proposition 4.16 The futures yield $L_{t,T,q}^{l,H}$ for a continuously resettled zero value contract on the Libor yield is given by

$$1 + \delta L_{t,T,q}^{l,H} = \frac{B_{t,T}^l}{B_{t,M}^l} \times \exp\left(\int_t^T a^l(s,M)\left(a^l(s,M) - a^l(s,T)\right) ds\right) \tag{60}$$

Proof:

Using (14) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= E_t^*\left[\left(L_{T,M}^l - L_{t,T,q}^{l,H}\right)\delta\right] \qquad (61)\\ &= E_t^*\left[\frac{1}{B_{T,M}^l} - \left(1 + \delta L_{t,T,q}^{l,H}\right)\right] \end{aligned}$$

Under P^*, the following restriction must hold:

$$\begin{aligned} \frac{B_{T,T}^l}{B_{T,M}^l} = \frac{B_{t,T}^l}{B_{t,M}^l} \quad &\times \exp\left(-\sum_{i=1}^{3}\int_t^T q_s^{li}\left(a^l(s,M) - a^l(s,T)\right) dM_s^{i,c,*}\right) \qquad (62)\\ &\times \exp\left(\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(a^l(s,M)^2 - a^l(s,T)^2\right) ds\right) \end{aligned}$$

Using (62) and setting the contract value in (61) to zero yields (60).□

Remark 4.1 (60) can be shown to be the relevant rate for the pricing of Eurodollar futures contracts. Defining $H_{t,T,q}^{ED}$ as the Eurodollar price, the value of the Eurodollar contract maturing at time T is given by

$$\begin{aligned} V_t &= E_t^*\left[(1 - \delta L_{T,M}^l) - H_{t,T,q}^{ED}\right] \qquad (63)\\ &= E_t^*\left[(1 + \delta L_{t,T,q}^{ED}) - \frac{1}{B_{T,M}^l}\right] \end{aligned}$$

where $H_{t,T,q}^{ED} = 1 - \delta L_{t,T,q}^{ED}$ and $L_{t,T,q}^{ED}$ is the Eurodollar yield. from (60), it then follows that

$$1 + \delta L_{t,T,q}^{ED} = \frac{B_{t,T}^l}{B_{t,M}^l} \times \exp\left(\int_t^T a^l(s,M)\left(a^l(s,M) - a^l(s,T)\right) ds\right) \tag{64}$$

and using the definition of money market yields, the "bond equivalent" of the Eurodollar price $H^{ED}_{t,T,q}$ is given by

$$B^{ED}_{t,T,q} = \frac{1}{1+\delta L^{ED}_{t,T,q}} = \frac{B^l_{t,M}}{B^l_{t,T}} \times \exp\left(-\int_t^T a^l(s,M)\left(a^l(s,M) - a^l(s,T)\right) ds\right) \quad (65)$$

which includes the expected adjustment term for the continuous settlement feature.

Proposition 4.17 The futures yield $L^{d,H}_{t,T,q}$ for a continuously resettled zero value contract on the corporate bond yield is given by

$$\begin{aligned} 1+\delta L^{d,H}_{t,T,q} &= \frac{B^l_{t,T}}{B^d_{t,M}} \times E^*_t\left[\exp\left(\int_t^T \left(e^{-kh_s} - 1 + e^{kh_s} - 1\right)\lambda^*_s ds\right)\right] && (66)\\ &\quad \times \exp\left(\int_t^T \left(a^l(s,M)^2 + a^\lambda(s,M)^2 - \rho^{\lambda l}_s a^\lambda(s,M)a^l(s,T) - a^l(s,M)a^l(s,T)\right) ds\right) \end{aligned}$$

Proof:

Using (14) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= E^*_t\left[\left(L^d_{T,M} - L^{d,H}_{t,T,q}\right)\delta\right] && (67)\\ &= E^*_t\left[\frac{1}{B^d_{T,M}} - \left(1+\delta L^{d,H}_{t,T,q}\right)\right] \end{aligned}$$

Under P^*, the following restriction must hold:

$$\begin{aligned} \frac{B^l_{T,T}}{B^d_{T,M}} &= \frac{B^l_{t,T}}{B^d_{t,M}} \times \exp\left(\int_t^{T\wedge\tau} w_s\lambda^*_s ds + k\int_t^T h_s dN_s\right) && (68)\\ &\quad \times \exp\left(-\sum_{i=1}^{3}\int_t^T \left(q^{li}_s\left(a^l(s,M) - a^l(s,T)\right) + w_s q^{\lambda i}_s a^\lambda(s,M)\right) dM^{i,c,*}_s\right)\\ &\quad \times \exp\left(\frac{1}{2}\sum_{i=1}^{3}\int_t^T \left(a^l(s,M)^2 - a^l(s,T)^2 + w_s^2 a^\lambda(s,M)^2 - \rho^{\lambda l}_s a^\lambda(s,M)a^l(s,M)\right) ds\right) \end{aligned}$$

Using (68) and setting the contract value in (67) to zero yields (66).□

Proposition 4.18 The futures yield $L^{r,H}_{t,T,q}$ for a continuously resettled zero value contract on the refreshed corporate bond yield is given by

$$\begin{aligned} 1+\delta L^{d,H}_{t,T,q} &= \frac{B^l_{t,T}}{B^d_{t,M}} \times E^*_t\left[\exp\left(\int_t^T w_s\lambda^*_s ds\right)\right] && (69)\\ &\quad \times \exp\left(\int_t^T \left(a^l(s,M)^2 + a^\lambda(s,M)^2 - \rho^{\lambda l}_s a^\lambda(s,M)a^l(s,T) - a^l(s,M)a^l(s,T)\right) ds\right) \end{aligned}$$

Proof:

Using (14) and the contract's payoff definition, the value of this contract is given by

$$\begin{aligned} V_t &= E_t^*\left[\left(L_{T,M}^d \cdot I_{\{T<\tau\}} - L_{t,T,q}^{\tau,H}\right)\delta\right] \qquad (70)\\ &= E_t^*\left[\frac{1}{B_{T,M}^d \cdot I_{\{T<\tau\}}} - \left(1+\delta L_{t,T,q}^{\tau,H}\right)\right] \end{aligned}$$

Under P^*, the following restriction must hold:

$$\begin{aligned} \frac{B_{T,T}^l}{B_{T,M}^d \cdot I_{\{T<\tau\}}} &= \frac{B_{t,T}^l}{B_{t,M}^d \cdot I_{\{T<\tau\}}} \times \exp\left(\int_t^T w_s \lambda_s^* ds\right) \qquad (71)\\ &\times \exp\left(-\sum_{i=1}^3 \int_t^T \left(q_s^{li}(a^l(s,M)-a^l(s,T)) + w_s q_s^{\lambda i} a^\lambda(s,M)\right) dM_s^{i,c,*}\right)\\ &\times \exp\left(\frac{1}{2}\sum_{i=1}^3 \int_t^T \left(a^l(s,M)^2 - a^l(s,T)^2 + w_s^2 a^\lambda(s,M)^2 - \rho_s^{\lambda l} a^\lambda(s,M) a^l(s,M)\right) ds\right) \end{aligned}$$

Using (71) and setting the contract value in (70) to zero yields (69).□

Note that (66) and (69) provide pricing formulae for futures contracts on credit risky yields and incorporate adjustments for convexity and the continuous settlement feature in the presence of credit risk.

4.4. Credit Spread Options

The appearance of options on credit spreads is a fairly new phenomenon. The following proposition demonstrates that valuation of such contracts, though tedious, can be handled within the framework developed in chapter 2.

Specifically, I consider the pricing of options on the $M-T$ period credit spread.

Proposition 4.19 At time t, the value of a call option, maturing at time T, on the $M-T$ period credit spread is given by

$$C_t(K) = K_1 N(\eta) - K_2 N(\eta - \zeta) \qquad (72)$$

The value of the corresponding put is given by

$$P_t(K) = K_2 N(\zeta - \eta) - K_1 N(-\eta) \qquad (73)$$

where

$$\zeta = var(X) = \int_t^T w_s^2 \left(a^\lambda(s,M) - w_s(a^\lambda(u,T) - a^\lambda(u,t))\right)^2 ds \qquad (74)$$

$$\eta = \frac{1}{\zeta}\ln\left(\frac{K_1}{K_2}\right) + \frac{\zeta}{2} \qquad (75)$$

$$X = \sum_{i=1}^{3} \int_t^T w_s q_s^{\lambda i} \left(a^\lambda(s,M) - w_s(a^\lambda(u,T) - a^\lambda(u,t))\right) dM_s^{i,c,T} \tag{76}$$

$$\begin{aligned}
K_1 &= \frac{B^d_{t,M} B^l_{t,T}}{B^l_{t,M}} \times \exp\left(\frac{\zeta}{2} - \frac{1}{2}\int_t^T w_s^2 a^\lambda(s,M)^2 ds\right) \qquad (77)\\
&\times \exp\left(\sum_{i=1}^{3}\int_t^T w_s q_s^{\lambda i} q_s^{li} a^\lambda(s,M)(a^l(s,T) - a^l(s,M))ds\right)\\
&\times \exp\left(-\sum_{i=1}^{3}\int_0^t w_s^2 a^\lambda(u,t)^2 dM_u^{i,c,T}\right)\\
&\times \exp\left(-\int_t^T w_s \lambda^T_{0,s} ds - \int_t^T\int_0^s w_s \sigma^\lambda(u,s) a^\lambda(u,s) du ds\right)\\
&\times \exp\left(-\int_t^T\int_0^s w_s \rho_u^{\lambda l}\left(a^l(u,s)\sigma^\lambda(u,s) + \sigma^l(u,s)a^\lambda(u,s) - a^l(u,T)\sigma^\lambda(u,s)\right) du ds\right)\\
&\times \exp\left(\sum_{i=1}^{3}\int_t^T\int_0^s w_s q_s^{\lambda i}\sigma^\lambda(u,s)(a^l(u,T) - a^l(u,M)) du ds\right)
\end{aligned}$$

$$K_2 = K B^l_{t,T} \tag{78}$$

Proof:

Using (8) and the contract's payoff definition, the value of the call and put, respectively, is given by

$$C_t(K) = B^l_{t,T} \times E_t^T\left[\left(\frac{V^d_{T,M}\cdot I_{\{T<\tau\}}}{B^l_{T,M}} - K\right)^+\right] \tag{79}$$

$$P_t(K) = B^l_{t,T} \times E_t^T\left[\left(K - \frac{V^d_{T,M}\cdot I_{\{T<\tau\}}}{B^l_{T,M}}\right)^+\right] \tag{80}$$

Under P^M, the following restriction must hold

$$\frac{V^d_{T,M}\cdot I_{\{T<\tau\}}}{B^l_{T,M}} = \frac{B^d_{t,M}}{B^l_{T,M}} \exp\left(-\int_t^T w_s \lambda_s^M ds + \sum_{i=1}^{3}\int_t^T w_s q_s^{\lambda i} a^\lambda(s,M) dM_s^{i,c,M} - \frac{1}{2}\int_t^T w_s^2 a^\lambda(s,M)^2 ds\right) \tag{81}$$

Under P^T, (81) is equivalent to

$$\begin{aligned}
\frac{V^d_{T,M}\cdot I_{\{T<\tau\}}}{B^l_{T,M}} &= \frac{B^d_{t,M}}{B^l_{T,M}} \times \exp\left(\sum_{i=1}^{3}\int_t^T w_s q_s^{\lambda i} q_s^{li} a^\lambda(s,M)(a^l(s,T) - a^l(s,M))ds\right) \qquad (82)\\
&\times \exp\left(-\int_t^T w_s \lambda_s^M ds + \sum_{i=1}^{3}\int_t^T w_s q_s^{\lambda i} a^\lambda(s,M) dM_s^{i,c,T} - \frac{1}{2}\int_t^T w_s^2 a^\lambda(s,M)^2 ds\right)
\end{aligned}$$

Furthermore, under P^M:

$$\int_t^T w_s \lambda_s^M ds = \int_t^T w_s \lambda_{0,s}^M ds - \int_t^T \int_0^s w_s \sigma^\lambda(u,s) a^\lambda(u,s) du ds \tag{83}$$
$$+ \sum_{i=1}^3 \int_t^T \int_0^s w_s q_s^{\lambda i} \sigma\lambda(u,s) dM_u^{i,c,M} ds$$
$$- \int_t^T \int_0^s w_s \rho_u^{\lambda l} \left(a^l(u,s)\sigma^\lambda(u,s) + \sigma^l(u,s)a^\lambda(u,s) - a^l(u,T)\sigma^\lambda(u,s)\right) du ds$$

whereas under P^T:

$$\int_t^T w_s \lambda_s^M ds = \int_t^T w_s \lambda_{0,s}^T ds - \int_t^T \int_0^s w_s \sigma^\lambda(u,s) a^\lambda(u,s) du ds \tag{84}$$
$$+ \sum_{i=1}^3 \int_t^T \int_0^s w_s q_s^{\lambda i} \sigma^\lambda(u,s) dM_u^{i,c,T} ds$$
$$- \int_t^T \int_0^s w_s \rho_u^{\lambda l} \left(a^l(u,s)\sigma^\lambda(u,s) + \sigma^l(u,s)a^\lambda(u,s) - a^l(u,T)\sigma^\lambda(u,s)\right) du ds$$
$$+ \sum_{i=1}^3 \int_t^T \int_0^s w_s q_s^{\lambda i} \sigma^\lambda(u,s)(a^l(u,T) - a^l(u,M)) du ds$$
$$= \int_t^T w_s \lambda_{0,s}^T ds - \int_t^T \int_0^s w_s \sigma^\lambda(u,s) a^\lambda(u,s) du ds$$
$$- \int_t^T \int_0^s w_s \rho_u^{\lambda l} \left(a^l(u,s)\sigma^\lambda(u,s) + \sigma^l(u,s)a^\lambda(u,s) - a^l(u,T)\sigma^\lambda(u,s)\right) du ds$$
$$+ \sum_{i=1}^3 \int_t^T \int_0^s w_s q_s^{\lambda i} \sigma^\lambda(u,s)(a^l(u,T) - a^l(u,M)) du ds$$
$$+ \sum_{i=1}^3 \int_0^t w_s^2 a^\lambda(u,t)^2 dM_u^{i,c,T} + \sum_{i=1}^3 \int_t^T w_s^2 q_s^{\lambda i}(a^\lambda(u,T) - a^\lambda(u,t)) dM_u^{i,c,T} ds$$

Application of standard results[23] now yields (72) and (73).□

[23] Note that if $X \sim N(0,\sigma_x^2)$ and $Y \sim N(0,\sigma_y^2)$, then the following hold:

$$E\left[\left(K_1 e^{X-\sigma_x^2} - K_2 e^{Y-\sigma_y^2}\right)^+\right] = K_1 N(\eta) - K_2 N(\eta - \zeta)$$
$$E\left[\left(K_2 e^{Y-\sigma_y^2} - K_1 e^{X-\sigma_x^2}\right)^+\right] = K_2 N(\zeta - \eta) - K_1 N(-\eta)$$

where

$$\eta = \frac{1}{\zeta} \ln\left(\frac{K_1}{K_2}\right) + \frac{\zeta}{2}$$
$$\zeta = var(X - Y)$$

(72) and (73) provide closed form solutions for the pricing of calls and puts on credit spreads. Similar results are derived by Litterman and Scheinkman (1991). However, their approach is based on an exogenous spread specification, whereas the present results are derived within a reduced form model where the spread is composed of terms relating to the instantaneous probability of default and the recovery rate.

4.5. Structured Products

A number of structured products incorporating credit components have appeared recently in the markets (particular structures are described in practitioner's articles such as Parsley (1996), Smithson (1995), Smithson (1996) and van Dyun (1995)). These structures aim to incorporate credit elements into structured notes or swaps. After a brief exposition of the structure of conventional swap contracts, a number of generic credit contract components are considered before finally some emerging credit structures are discussed.

4.5.1. Swap Contracts and Swaptions

Consider the value at time t of a payer swap[24] SW_t for n exchanges, at respective payment dates $T_1 ... T_n$ with $0 < t < T_1 < T_n < 1$, of Libor yields against payment of the fixed swap yield c_t. The value of such an agreement is given by

$$\begin{aligned} SW_t &= \sum_{i=1}^{n} B^l_{t,T_i} \times E_t^{T_i} \left[\delta(L_{T_{i-1},T_i} - c_t)\right] \\ &= \sum_{i=1}^{n} B^l_{t,T_i} \times E_t^{T_i} \left[\frac{1}{B^l_{T_{i-1},T_i}} - (1+\delta c_t)\right] \end{aligned} \tag{85}$$

where the principal on which respective yields are being paid is normalized to unity. Effecting a change of numeraire, valuation proceeds under the $P^{T_{i-1}}$ measure:

$$\begin{aligned} SW_t &= \sum_{i=1}^{n} B^l_{t,T_i} \times E_t^{T_{i-1}} \left[\frac{B^l_{T_{i-1},T_i} \cdot B^l_{t,T_{i-1}}}{B^l_{T_{i-1},T_{i-1}} \cdot B^l_{t,T_i}} \left(\frac{1}{B^l_{T_{i-1},T_i}} - (1+\delta c_t)\right)\right] \\ &= \sum_{i=1}^{n} \left(B^l_{t,T_{i-1}} - B^l_{t,T_i}(1+\delta c_t)\right) \end{aligned} \tag{86}$$

[24] In the present framework, the terms "payer" or "receiver" in association with swaps relate to the fixed side of the transaction; thus, a counterparty which has taken a long position in a payer swap swap has contracted to make, on each payment date, a fixed payment in return for a floating rate payment.

$$= B^l_{t,T_0} - B^l_{t,T_n} - \sum_{i=1}^{n} B^l_{t,T_i} c_t$$

Setting the value of the contract to zero yields the fixed swap rate

$$c_t = \frac{B^l_{t,T_0} - B^l_{t,T_n}}{\sum_{i=1}^{n} B^l_{t,T_i}} \tag{87}$$

In the present framework, the value of a payer swaption SW^P_t is given by:

$$\begin{aligned} SW^P_t &= B^l_{t,T_0} \times E^{T_0}_t \left[\sum_{i=1}^{n} B^l_{T_0,T_i} \times \delta \left(c_{T_0} - K^P_t \right)^+ \right] \\ &= B^l_{t,T_0} \times E^{T_0}_t \left[1 - B^l_{T_0,T_n} - K^P_t \left(\sum_{i=1}^{n} B^l_{T_0,T_i} \right) \right] \end{aligned} \tag{88}$$

where K^P_t is the strike rate and c_{T_0} is the future swap spread at time T_0. The value of a receiver swaption SW^R_t is given by:

$$\begin{aligned} SW^R_t &= B^l_{t,T_0} \times E^{T_0}_t \left[\sum_{i=1}^{n} B^l_{T_0,T_i} \times \delta \left(K^P_t - c_{T_0} \right)^+ \right] \\ &= B^l_{t,T_0} \times E^{T_0}_t \left[B^l_{T_0,T_n} + K^P_t \left(\sum_{i=1}^{n} B^l_{T_0,T_i} \right) - 1 \right] \end{aligned} \tag{89}$$

Having outlined the pricing of some standard swap related contracts, I proceed to the pricing of structures involving credit components.

4.5.2. Generic Credit Contingent Structures

Many credit contingent structures contain typical elements. Components relate either to payoffs contingent on the event of default or to payoffs linked to recovery rates. Consider the following credit contract components to be used as building blocks in more complex credit structures.

Proposition 4.20 The value of a contract to make a fixed payment of P at time T, subject to default on a reference asset not having occurred, is given by

$$V_t = E^*_t \left[P \exp \left(- \int_t^T \left(r^l_s + \lambda^*_s \right) ds \right) \right] \tag{90}$$

Proof:

Using (9) and the contract's payoff specification $X_T = PI_{\{T<\tau\}}$, the value of the above contract is

$$V_t = E_t^* \left[X_T \exp\left(-\int_t^T r_s^l ds \right) \right] \tag{91}$$

Noting that

$$E_t^* \left[I_{\{T<\tau\}} \right] = \exp\left(-\int_t^T \lambda_s^* ds \right) \tag{92}$$

and inserting (92) into (91) yields (90).□

Proposition 4.21 The value of a contract paying a continuous dividend y_s up to time T, subject to default on a reference asset not having occurred, is given by

$$V_t = E_t^* \left[\int_t^T y_s \exp\left(-\int_t^T \left(r_s^l + \lambda_s^* \right) ds \right) ds \right] \tag{93}$$

Proof:

Using (9) and the contract's payoff specification $c_s = y_s I_{\{s<\tau\}}$, the value of the above contract is

$$V_t = E_t^* \left[\int_t^T c_s \exp\left(-\int_t^s r_u^l du \right) ds \right] \tag{94}$$

Noting that

$$E_t^* \left[I_{\{s<\tau\}} \right] = \exp\left(-\int_t^s \lambda_s^* ds \right) \tag{95}$$

inserting (95) into (94) yields (93).□

Proposition 4.22 The value of a contract to make a fixed payment of P at the time of default of a specified reference asset is given by

$$V_t = P \times E_t^* \left[\int_t^T \lambda_s^* \exp\left(-\int_t^T \left(r_u^l + \lambda_u^* \right) du \right) ds \right] \tag{96}$$

Proof:

Using (9) and the contract's payoff specification $X_s = P I_{\{s=\tau\}}$, the value of the above contract is

$$V_t = E_t^* \left[\int_t^T X_s \exp\left(-\int_t^s r_u^l du \right) ds \right] \tag{97}$$

Noting that

$$\begin{aligned} E_t^* \left[I_{\{s=\tau\}} \right] &= P_t^* \left[s = \tau \right] \\ &= \frac{\partial}{\partial s} P_t^* \left[\tau \le s \right] \\ &= -\frac{\partial}{\partial s} P_t^* \left[\tau > s \right] \end{aligned} \tag{98}$$

$$= \lambda_s^* \exp\left(-\int_t^s \lambda_s^* du\right)$$

inserting (98) into (97) yields (96).□

While contracts the payoffs on which are linked to recovery rates are discussed below, the above results illustrate how components the payoffs on which are contingent on the incidence of default are valued within the present framework.

4.5.3. Specific Credit Contingent Structures

Financial institutions have started to issue structured notes containing credit elements. The following set of structures is by no means exhaustive but illustrates the fact that the market has stared to develop transfer mechanisms for components of credit risk. With regard to the following pricing results, the principal values are normalized to unity.

The principal repayment on so-called recovery notes is tied to the recovery rate of a given credit risky reference asset: If the reference asset has defaulted by the time the note matures, investors receive the realized recovery value on the asset; otherwise, the principal is returned. The value of such a contract is given by

$$V_t = E_t^* \left[\exp\left(-\int_t^T r_s^l ds\right) D_T^{d,j}\right] \tag{99}$$

where $D_T^{d,j}$ is the default process associated with the issuer of the reference asset j. In the framework of chapter 2, (99) is equivalent to

$$\begin{aligned} V_t &= E_t^* \left[R_T^j \exp\left(-\int_t^T r_s^l ds\right)\right] \\ &\quad - E_t^* \left[\left(R_T^j - 1\right) \exp\left(-\int_t^T \left(r_u^l + \lambda_u^{j,*}\right) ds\right)\right] \end{aligned} \tag{100}$$

A variation of the above structure, the zero one note, returns the principal in the event of default not having occurred and nothing otherwise. The value of this contract is

$$V_t = E_t^* \left[\exp\left(-\int_t^T r_s^l ds\right) I_{\{T<\tau^j\}}\right] \tag{101}$$

which, in terms of our credit risk specification, is equivalent to

$$V_t = E_t^* \left[\exp\left(- \int_t^T \left(r_u^l + \lambda_u^{j,*} \right) ds \right) \right] \tag{102}$$

Range notes tied to credit spreads have made their appearance. Payoffs on these notes are contingent on credit spreads staying within specific ranges:

$$V_t = E_t^* \left[\exp\left(- \int_t^T r_s^l ds \right) X_T I_{\{S_l \leq \frac{B^d}{B^l} \leq S_h\}} \right] \tag{103}$$

Finally, credit linked notes are structured such that the protection seller pays the protection buyer a principal amount in return for a series of fixed coupon payments over the life of the note. At the maturity date, if the reference asset has not defaulted, the protection buyer returns to the protection seller the principal amount. If the reference assets has defaulted, the protection seller receives the amount recovered. The value of this note to the protection seller is given as follows:

$$\begin{aligned} V_t &= c_t \sum_{i=1}^{n} E_t^* \left[I_{\{T_i < \tau\}} \exp\left(- \int_t^{T_i} r_s^l ds \right) \right] - 1 \\ &\quad + E_t^* \left[\exp\left(- \int_t^{s} r_u^l du \right) D_T^{d,j} \right] \end{aligned} \tag{104}$$

Using results from the preceding section, this expression is equivalent to:

$$\begin{aligned} V_t &= c_t \sum_{i=1}^{n} E_t^* \left[I_{\{T_i < \tau\}} \exp\left(- \int_t^{T_i} \left(r_u^l + \lambda_u^{j,*} \right) ds \right) \right] - 1 + \\ &\quad E_t^* \left[R_T^j \exp\left(- \int_t^{T} r_s^l ds \right) \right] - \\ &\quad E_t^* \left[\left(R_T^j - 1 \right) \exp\left(- \int_t^{T} \left(r_u^l + \lambda_u^{j,*} \right) ds \right) \right] \end{aligned} \tag{105}$$

The merit of transferring the principal value of the contract at inception lies in the fact that the protection buyer's exposure to the credit risk of the protection seller is thereby eliminated.

Credit components have been explicitly embedded in swap structures. The most common contract, the credit default option (also termed default swap), stipulates for an exchange of a fixed fee c_t, up to the time of default, in return for a payment, at the time of default, of the realized loss on a given credit risky reference asset. The value of this contract is given by

$$V_t = c_t \sum_{i=1}^{n} E_t^* \left[I_{\{T_i < \tau\}} \exp\left(- \int_t^{T_i} r_s^l ds \right) \right] \tag{106}$$

$$+E_t^*\left[\int_t^T w_s I_{\{s=\tau\}} \exp\left(-\int_t^s r_u^l du\right) ds\right]$$

which is identical to

$$\begin{aligned} V_t &= c_t \sum_{i=1}^n E_t^*\left[\exp\left(-\int_t^{T_i} \left(r_u^l + \lambda_u^*\right) du\right)\right] \\ &\quad + E_t^*\left[\int_t^T w_s \lambda_s^* \exp\left(-\int_t^T \left(r_u^l + \lambda_u^*\right) du\right) ds\right] \end{aligned} \tag{107}$$

Finally, the total return swap stipulates for the exchange of a fixed rate c_t in return for the total capital gains on a credit risky reference asset, independently of whether default occurs or not. The value of this contract is given by

$$\begin{aligned} V_t &= c_t \sum_{i=1}^n E_t^*\left[I_{\{T_i<\tau\}} \exp\left(-\int_t^{T_i} r_s^l ds\right)\right] \\ &\quad - E_t^*\left[\int_t^T c_s I_{\{s<\tau\}} \exp\left(-\int_t^s r_u^l du\right) ds\right] \\ &\quad - E_t^*\left[\exp\left(-\int_t^T r_s^l ds\right)\left(1-\exp\left(-k\int_t^T h_s dN_s\right)\right)\right] \end{aligned} \tag{108}$$

which is identical to

$$\begin{aligned} V_t &= c_t \sum_{i=1}^n E_t^*\left[\exp\left(-\int_t^{T_i} \left(r_u^l + \lambda_u^*\right) ds\right)\right] \\ &\quad - E_t^*\left[\int_t^T c_s \exp\left(-\int_t^s \left(r_u^l + \lambda_u^*\right) du\right) ds\right] \\ &\quad - E_t^*\left[\exp\left(-\int_t^T r_s^l ds\right)\left(\exp\left(-\int_t^T w_s \lambda_s^* ds\right) - 1\right)\right] \end{aligned} \tag{109}$$

This concludes the discussion of pricing of specific credit contingent structures. The above results demonstrate that the credit risk specification developed in this thesis is capable of accommodating the task of pricing contracts the value of which depend in complicated ways on various components of credit risk.

Conclusion

Given the framework developed in the preceding chapter, the current chapter has been concerned with the pricing of specific credit contingent claims. It has been demonstrated that certain convexity adjustment arising from the presence of credit risk are required for the purpose of pricing of contracts the payoffs to which are non linear functions of credit risky zero coupon bonds. Explicit results of this nature have not been reported in the literature. Furthermore, the impact of continuous settlement features on the value of credit contingent claims has been examined. Finally, the pricing of a number of emerging credit contingent structures, such as credit spread options, credit linked notes and credit swaps, has been considered. The present results demonstrate the capability of the credit risk specification in the current thesis to address pricing issues relating to these contracts. Given that such contracts potentially contain elements relating to isolated credit risk components like default probability or recovery rate, it is apparent that the modelling choice in this thesis (both recovery rates and probability of default are allowed to be stochastic) is particularly suited to address the pricing of such contracts. This may not be the case for models which assume that either the probability of default or recovery rates are deterministic, or which fail to take account of the correlation structure between credit risk and market risk.

References

Broadie, M and J. Detemple (1995): "American capped call options on dividend paying assets," *The Review of Financial Studies*, 8/1, 161-191.

Carr, P. (1993): "Deriving derivatives of derivative assets," *Working Paper.*

Carr, P. (1995): "Two extensions to barrier option valuation", *Applied Mathematical Finance*, 2, 173-209.

Carr, P. and R. Jarrow (1990): "The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic value and time value," *The Review of Financial Studies*, 3/3, 469-492.

Carr, P., R. Jarrow and R. Myneni (1992): "Alternative characterizations of American put options," *Mathematical Finance*, 2/2, 87-106.

Conze, A. and R. Viswanathan (1991): "Path dependent options: The case of lookback options," *Journal of Finance*, 46/5, 1893-1907.

Das, S. (1995): "Credit risk derivatives," *Journal of Derivatives* (Spring), 7-23.

Dravid, A., M. Richardson and T. Sun (1994): "The pricing of dollar denominated YEN/DM warrants," *Journal of International Money and Finance*, 13, 517-536.

Duffie, D. and R. Stanton (1992): "Pricing continously resettled contingent claims," *Journal of Economic Dynamics and Control*, 16, 561-573.

Fitzgerald, D. (1993): *Financial Futures* Euromoney Publications.

Flesaker, B., L. Hughston, L. Schreiber and L. Sprung (1994): "Taking all the credit," *Risk* (September), 7/9.

Grinblatt, M. (1993): "An analytic solution for interest rate swap spreads," *Working Paper.*

Grinblatt, M. and N. Jegadeesh (1996): "Relative pricing of Eurodollar futures and forward contracts", *Journal of Finance* (September), 51/4, 1499-1522.

Hanweck, G. (1995): "An asset pricing formula for interest rate derivative assets with applications to Eurodollar futures and options," *Working Paper.*

Hull, J. (1993): *Options, Futures, and other Derivative Securities* Prentice-Hall International.

Jamshidian, F. (1990): "Commodity option valuation in the Gaussian futures term structure model," *Review of Futures Markets*, 10, 324-346.

Jamshidian, F. (1991): "Bond and option valuation in the Gaussian interest rate model," *Research in Finance*, 9, 131-170.

Jamshidian, F. (1993): "Option and futures evaluation with deterministic volatilities," *Mathematical Finance*, 3/2, 149-159.

Kim, I. and G. Yu (1996): "A simplified approach to the valuation of American options and its applications," *Review of Derivatives Research*, 61-85.

Kunitomo, N. and M. Ikeda (1992): "Pricing options with curved boundaries," *Mathematical Finance*, 2/4, 275-298.

Longstaff, F. and E. Schwartz (1995): "Valuing credit derivatives," *The Journal of Fixed Income* (June), 6-12.

Myneni, R. (1992): "The pricing of american options," *The Annals of Applied Probability*, 2, 1-23.

Parsley, M. (1996): "Credit derivatives get cracking," *Euromoney* (March), 28-34.

Smithson, C. (1995): "Credit derivatives (1)," *Risk* (December), 8/12.

Smithson, C. (1996): "Credit derivatives (2)," *Risk* (June), 9/6.

van Dyun, A. (1995): "Credit risk for sale. Any buyers ?," *Euromoney* (April), 41-43.

Chapter 5: Quantifying and Managing Counterparty Credit Risk

Introduction

Credit risk is among the most relevant sources of uncertainty[25] faced by banks. Financial institutions are facing large amounts of credit risk and increased competition in traditional business areas is forcing banks manage this risk more efficiently. This affects bank's activities in a number of areas, in particular the pricing of credit risk, performance measurement of business activities generating credit risk as well as the formulation of performance based compensation schemes.

Regulatory agencies, responsible for ascertaining that market participants hold adequate capital cushions commensurate with the risks incurred, have indicated their willingness to reward sophisticated risk management practices (see BCBS (1993), BCBS (1995a) and BCBS (1995b)). While this trend sofar relates primarily to market risk, there are indications that similar regimes will come into place regarding credit risk. A statistic which has gained widespread acceptance among both regulators and practitioners for the purpose of market risk measurement is VaR (Value at Risk). This statistic indicates the maximum loss, subject to a given confidence level and liquidation horizon, which a bank is liable to experience on a given portfolio due to adverse movements in market variables. Methods of calculating such a statistic have reached a considerable level of sophistication. With regard to credit risk, no similar method has emerged. The ability to calculate Value at Risk for credit risk would be desirable for banks as well as regulators, and I demonstrate below that the credit risk specification in the present thesis is able to generate such statistic.

The preceding chapter provided a number of pricing results for contracts written on credit variables and illustrated how the presence of credit risk in the underlying instrument affects valuation. Credit risk arising from the possibility of any of the parties to a financial contract being subject to default was not considered. Arguably it is the existence of the latter risk which prompted the development of markets in contracts explicitly written on credit variables.

The present chapter considers issues relating to quantification and management of counterparty risk within the framework constructed in chapter 2 and is structured as follows. Section 1 is devoted to an analysis of measures to quantify credit risk in conventional contracts. Specifically, different exposure and loss statistics are examined. Section 2 contains an analysis of different mechanisms devised to manage this

[25] Other relevant risks include market risk, relating to uncertainty due to changes in market variables such as interest rates, equity prices and exchange rates; operational risk, associated with failure to adequately structure, implement and control operational procedures; liquidity risk, relating to the risk of loss of value due to failure to fund illiquid assets within given time limits; and legal risk, arising from the possibility that contractual features may be invalidated by changes in the legal environment.

risk. Credit rationing and credit mitigation techniques are discussed. Finally, credit management through pricing is discussed; specifically, I illustrate that the theoretical framework developed above can be employed to determine equilibrium swap rates in swap agreements subject to bilateral counterparty risk.

5.1. Quantification of Credit Risk

Quantification of credit risk is a fundamental activity for financial institutions. While academics have focused their efforts on developing methods to price credit risk, practitioners have tended to view credit risk in terms of exposure and loss statistics. I consider each of these in turn.

5.1.1. Exposure

The standard measure of credit risk is credit exposure, measuring the maximal amount to be lost, at a point in time or over a given time interval, due to a counterparty failing to fulfill its contractual obligation under a given set of financial contracts. Credit exposure has long been the standard statistic for measuring credit risk for both financial institutions (see Derivatives Policy Group (1995), Group of 30 (1993) , Heldring (1995), Wisener (1994), Rowe (1995), Matten (1996), Mark (1995), ?, Lawrence (1995)) and regulators (see BCBS (1988), BCBS (1995b), BCBS (1996c) , European Commission (1994b), European Commission (1994e)), and different approaches of credit risk measurement differ not in that other concepts than exposure are devised, but rather in that methods of varying levels of sophistication are being employed to calculate exposure; recently academics have taken an interest in this concept (see Smithson and Smith (1995), Jamshidian and Zhu (1996), Iben and Brotherton-Ratcliffe (1994) , Hansen and Scheinkman (1995)) and it has been demonstrated that financial theory is able to contribute in a significant way to further developing this concept.4

Conventional contracts or balance sheet items constitute either an asset or a liability for a given company. If the contract is an asset, exposure is given by the market value of that asset. If the contract is a liability, exposure is zero because no money can be lost to the counterparty.

Innovative contracts and some off balance sheet items potentially constitute either an assets or a liability at different times of their lives. Over periods of the contract during which the instrument constitutes an asset, exposure is given by it's market value. Over periods of the contract during which the instrument constitutes a liability, exposure is zero.

These considerations can be formalized by defining current exposure at time t to counterparty j, X_t^j, as

$$X_t^j = \sum_{i=1}^{n(j)} \left(V_t^{i,j}(\omega), 0\right)^+ \tag{1}$$

where $n(j)$ is the number of outstanding contracts with this counterparty and $V_t^{i,j}(\omega)$ is the current state-dependent market value of the i-th contract. This market value is a function of both market and credit state variables. The summation across outstanding contracts outside the bracket formalizes the conservative assumption that in the event of default, bankrupt counterparties might engage in the practice of 'cherry picking', choosing to honor those contract under which they stand to gain and refusing payments under contracts under which they owe money. While such practice is not allowed in most jurisdictions, there is considerable uncertainty as to whether such rules can and will effectively be enforced. By making allowance for the possibility of the defaulting counterparty to maximize its potential to choose which contracts to honor the contracts of its choice, the exposure statistic embodies maximum conservatism by assuming a worst case scenario. To the extent that relative to a given set of contracts absolute certainty exist such that 'cherry picking' can be avoided, the relevant measure of credit risk is net exposure, expanded on below. Current exposure is the maximum amount which can be lost due to default if counterparty j defaults at the current time. Given that market values change over time, so does exposure. A measure of exposure at some future date $T > t$ is given by the future expected exposure $X_{t,T}^{e,j}$:

$$X_{t,T}^{e,j} = E_t \left[\sum_{i=1}^{n(j)} \left(V_T^{i,j}(\omega), 0\right)^+ \right] \tag{2}$$

where the expectation is taken over the distribution of state variables, stochastic variations in which drive the uncertainty in market values $V_T^{i,j}(\omega)$.

While (2) gives an estimate of the average exposure at time T, a more conservative measure is given by the future maximal exposure $X_{t,T}^{m,j}$:

$$X_{t,T}^{m,j} = \max_{\omega \in \Omega} \sum_{i=1}^{n(j)} \left(V_T^{i,j}(\omega), 0\right)^+ \tag{3}$$

This statistic determines the maximal exposure to counterparty j at time T subject to a specified confidence level on the distribution of exposures at time T.

As an illustration of the exposure concept[26], consider the valuation of a standard swap agreement to exchange a set of n Libor payments in returns for the same number of fixed payments of c_t, subject to both counterparties being subject to default.

Suppose counterparty X has contracted to receive the fixed rate and pay the floating rate. The contractual components of X's position is equivalent to a portfolio of a long position in a floor (n floorlets):

$$V_t = E_t\left[\sum_{j=1}^{n}\frac{A_t^l}{A_{T_j}^l}\left(c_t - L_{t_{j-1},t_j}\right)^+ \delta P\right] \tag{4}$$

and a short position in a cap (n caplets):

$$V_t = E_t\left[\sum_{j=1}^{n}\frac{A_t^l}{A_{T_j}^l}\left(L_{t_{j-1},t_j} - c_t\right)^+ \delta P\right] \tag{5}$$

where

$$A_t^l/A_{T_j}^l = \exp\int_0^t r_s^l ds \tag{6}$$

is the discount factor associated with the i-th payment. Given the assumption that both counterparties are subject to default, non contractual positions must be included. The fact that X itself is liable to default means that X holds a long position in an American payer swaption:

$$SW_t^P = \max_{i\in[1,n]} E_t\left[\sum_{j=1}^{n}\frac{A_t^l}{A_{T_j}^l}\left(c_{T_i} - c_t\right)^+ \delta P\right] \tag{7}$$

whereas the default risk of counterparty Y results in X having a short position in an American receiver swaption.

$$SW_t^R = \max_{i\in[1,n]} E_t\left[\sum_{j=1}^{n}\frac{A_t^l}{A_{T_j}^l}\left(c_t - c_{t_i}\right)^+ \delta P\right] \tag{8}$$

X's cumulative position is thus composed of contractual components (4) and (5); the presence of default risk generates the non contractual components (7) and (8).

Counterparty X has contracted to pay the fixed rate and receive the floating rate. Its contractual position under the swap agreement, taking account of credit risk, is equivalent to a long position in a cap (n caplets):

$$V_t = E_t\left[\sum_{j=1}^{n}\frac{A_t^l}{A_{T_j}^l}\left(L_{t_{j-1},t_j} - c_t\right)^+ \delta P\right] \tag{9}$$

[26] This example is a formalization of arguments proposed in Sorensen and Bollier (1994).

and a short position in a floor (n floorlets):

$$V_t = E_t \left[\sum_{j=1}^{n} \frac{A_t^l}{A_{T_j}^l} \left(c_t - L_{t_{j-1}, t_j} \right)^+ \delta P \right] \tag{10}$$

Due to the presence of default risk, Y's position includes non contractual components in the form of a long position in an American receiver swap

$$SW_t^R = \max_{i \in [1,n]} E_t \left[\sum_{j=1}^{n} \frac{A_t^l}{A_{T_j}^l} \left(c_t - c_{t_i} \right)^+ \delta P \right] \tag{11}$$

arising from its own credit risk, and a short position in an American payer swap

$$SW_t^P = \max_{i \in [1,n]} E_t \left[\sum_{j=1}^{n} \frac{A_t^l}{A_{T_j}^l} \left(c_{T_i} - c_t \right)^+ \delta P \right] \tag{12}$$

arising from the risk of default associated with X. In this example, (7), (8), (11) and (13) constitute measures of exposure in that the respective values subsume the maximum loss of value due to counterparty default. Ultimately, financial institutions will want to incorporate the credit risk into the pricing of the swap. The preceding example demonstrates that the exposure concept is insufficient for this purpose. There is no role for the counterparties' respective default probabilities or recovery rates and settlement features specific to the swap market are not incorporated. Incorporation of other credit risk elements should facilitate formulation of more sophisticated measures of credit risk.

Nevertheless, establishment of exposure statistics relative to a given counterparty is a necessary first step in the process of quantification of credit risk. These measures provide a conservative estimate of the maximal loss to be incurred due to the default of a given counterparty and provide useful information for the purposes of monitoring how changes in the credit quality of a given counterparty could affect the banks profits.

5.1.2. Loss

While measures of exposure generate estimates of maximal losses due to the presence of default risk, estimates of credit losses can be made more precise by incorporating elements relating to recovery rates and the probability of default.

Future expected loss at time T, conditional on information at time t, $L_{t,T}^{e,j}$, is given by

$$L_{t,T}^{e,j} = E_t \left[I_{\{T=\tau\}} \times \left(\sum_{i=1}^{n(j)} (1 - R_T^{i,j}(\omega))(V_T^{i,j}(\omega), 0)^+ \right) \right] \tag{13}$$

where $R^{i,j}_T(\omega)$ is the state-dependent recovery rate on contract i with counterparty j and $V^{i,j}_T(\omega)$ is the value of contract i with counterparty j. In the framework of chapter 2, (14) is equivalent to

$$L^{e,j}_{t,T} = \qquad (14)$$

$$E_t \left[\lambda^j_T(\omega) \exp\left(- \int_t^T \lambda^j_s(\omega) ds \right) \times \left(\sum_{i=1}^{n(j)} (1 - R^{i,j}_T(\omega))(V^{i,j}_T(\omega), 0)^+ \right) \right]$$

Note that

$$\lambda^j_T(\omega) \exp\left(- \int_t^T \lambda^j_s(\omega) ds \right) \qquad (15)$$

is the probability that no default occurs over the interval $[t, T]$ and default occurs at time T. As such , (17) is the marginal probability of default at time T. Alternatively, one could consider using the cumulative probability of default over the interval $[t, T]$ to derive the statistic

$$L^{e,j}_{t,T} = \qquad (16)$$

$$E_t \left[\left(1 - \exp\left(- \int_t^T \lambda^j_s(\omega) ds \right) \right) \times \left(\sum_{i=1}^{n(j)} (1 - R^{i,j}_T(\omega))(V^{i,j}_T(\omega), 0)^+ \right) \right]$$

This statistic can then be interpreted as indicating the expected loss due to counterparty default over the interval $[t, T]$, using exposure at time T . Consider the effect on expected loss due to a credit downgrading of a given counterparty. Two cases can be considered. First, $V^{i,j}_T(\omega)$ does not depend on the credit quality of the counterparty; assuming that a downgrading results in higher $\lambda^j_s(\omega)$ as well as lower expected recovery rates, $R^{i,j}_T(\omega)$, such a downgrading will raise the value of (16). Secondly, if $V^{i,j}_T(\omega)$ is a function of the counterparty's credit quality as well as other state variables, the rise in expected loss will be dampened due to a simultaneous fall in $V^{i,j}_T(\omega)$; in other words, part of the potential loss due to credit deterioration has been realized through the fall in $V^{i,j}_T(\omega)$, leaving less potential to lose in the future.

A more conservative measure of loss than expected loss is given by the future maximal loss to counterparty j at time T, $L^{m,j}_{t,T}$, given by

$$L^{m,j}_{t,T} = \qquad (17)$$

$$\max_{\omega \in \Omega} \lambda^j_T(\omega) \exp\left(- \int_t^T \lambda^j_s(\omega) ds \right) \times \left(\sum_{i=1}^{n(j)} (1 - R^{i,j}_T(\omega))(V^{i,j}_T(\omega), 0)^+ \right)$$

The maximum is taken subject to a given confidence level on the distribution of losses at time T. Again, instead of using the marginal probability of default, one can use the cumulative probability of default to derive a measure of maximum loss not at a point in time T, but over the interval $[t, T]$:

$$L_{t,T}^{m,j} = \max_{\omega\in\Omega}\left(1-\exp\left(-\int_t^T \lambda_s^j(\omega)ds\right)\right)\times\left(\sum_{i=1}^{n(j)}(1-R_T^{i,j}(\omega))(V_T^{i,j}(\omega),0)^+\right) \tag{18}$$

Note that (18) indicates the maximum loss, subject to a given confidence level and liquidation horizon T, which a financial institution is liable to experience due to changes in market variables and changes in the credit quality of a given counterparty. Calculation of (18), which is likely to involve numerical techniques, can therefore be likened to a Value at Risk figure for credit risk. This expression allows for stochastic changes in credit risk components (recovery rate and probability of default) as well as in the credit risk free components $V_T^{i,j}(\omega)$. Note that due to the explicit role of the correlation between the forward probability of default and risk free rates in our model, the size of (18) will depend on this parameter. This fact is of particular relevance in light of the fact that currently, banks are required to calculate risk capital for credit risk and market risk separately. The banks total capital requirements are determined as the sum of capital requirements for market risk and credit risk. The above expression demonstrates that such a calculation fails to take account of interaction between market risk and credit risk: Only by taking account of this interaction will capital requirements adequately reflect the risks to which a bank is exposed.
(18), like a Value at risk number for market risk, has the additional interpretation of indicating the amount of capital needed to support a business activity exposing the financial institution to credit risk. As such, this type of calculation will not only be relevant for calculating capital requirements but also for purposes of performance measurement and compensation policies.

5.2. Management of Credit Risk

Having established measures quantifying credit risk, I turn to the discussion of how this risk is managed. Financial institutions have devised a number of mechanisms to manage credit risk. The objective of the current section is not to provide an exhaustive analysis of such mechanisms, but rather to give a brief outline of established practice and highlight areas in which the credit risk specification introduced in chapter 2 should prove useful.

5.2.1. Rationing Credit Risk

The least sophisticated approach to managing credit risk is to ration exposure. Institutions adopting this approach typically set credit limits for specific counterparties. Typical limit structures are formulated in terms of notional amounts, type of underlying or contract maturity.

5.2.2. Mitigation of Credit Risk

Financial markets have responded to the increasing presence of credit risk with the creation of a number of mechanisms designed to mitigate this risk. These mechanisms facilitate participation in a wider set of business activities generating credit risk than would be possible under a simple rationing policy, given a level of credit risk an organization is willing to assume.

5.2.2.1. Netting Arrangements

Netting refers to the practice of allowing for obligations arising from individual contracts with a given counterparty to be subsumed into a new contract, respective obligations under which reflect the cumulative obligations arising from the individual contracts. In practice it is rare that all contracts with a given counterparty qualify for complete netting. More typical is a situation in which arrangements exist to net values across specific subsets of contracts. This fact necessitates introduction of an index of netting sets. Exposure measures can then be adapted to take account of such mechanism.. Thus, net current exposure, $X_t^{n,j}$, is defined to be

$$X_t^{n,j} = \sum_{k=1}^{ns(j)} \left(\sum_{i=1}^{n(j,k)} V_t^{j,i}(\omega), 0 \right)^+ \tag{19}$$

where $ns(j)$ is the number of netting sets in place with counterparty j, $n(j,k)$ is the number of contracts in netting set k and $V_t^{j,i}(\omega)$ is the value of contract i in netting set j. Net future expected exposure to counterparty j at time $T > t$, $X_{t,T}^{ne,j}$, is given by

$$X_{t,T}^{ne,j} = E_t \left[\sum_{k=1}^{ns(j)} \left(\sum_{i=1}^{n(j,k)} V_T^{j,i}(\omega), 0 \right)^+ \right] \tag{20}$$

while net maximal exposure at time $T > t$, $X^{nm,j}_{t,T}$, is given by

$$X^{nm,j}_{t,T} = \max_{\omega\in\Omega} \sum_{k=1}^{ns(j)} \left(\sum_{i=1}^{n(j,k)} V^{j,i}_T(\omega), 0 \right)^+ \tag{21}$$

(21) is understood to be calculated subject to a given confidence level on the exposure distribution at time T and makes use of the marginal probability of default at time T; a similar statistic involving the cumulative probability of default over the interval $[t, T]$ is easily constructed.

The measures of loss in (14) and (18) can be augmented to take account of the presence of netting arrangements. Net future expected losses to counterparty j at time $T > t$, $L^{ne,j}_{t,T}$, is given by

$$L^{ne,j}_{t,T} = E_t \left[\lambda^j_T(\omega) \exp\left(-\int_t^T \lambda^j_s(\omega) ds \right) \times \sum_{k=1}^{ns(j)} \left(\sum_{i=1}^{n(j,k)} (1 - R^{j,i}_T(\omega)) V^{j,i}_T(\omega), 0 \right)^+ \right] \tag{22}$$

while net future maximal losses to counterparty j at time $T > t$, $L^{nm,j}_{t,T}$, is given by

$$L^{nm,j}_{t,T} = \max_{\omega\in\Omega} \lambda^j_T(\omega) \exp\left(-\int_t^T \lambda^j_s(\omega) ds \right) \times \left(\sum_{i=1}^{n(j,k)} \left(1 - R^{j,i}_T(\omega)\right) V^{j,i}_T(\omega), 0 \right)^+ \tag{23}$$

Calculation of (23), using the cumulative probability of default over the interval $[t, T]$ and subject to a given confidence level, will indicate the maximum loss due to counterparty default in the presence of netting.

5.2.2.2. Collateral Arrangements

Another mechanism used to mitigate credit risk is the practice of requiring counterparties to post collateral in order to reduce exposure. Collateralized current exposure to counterparty j, $X^{c,j}_t$, is given by

$$X^{c,j}_t = \left(\sum_{i=1}^{n(j)} V^{j,i}_t(\omega) - C^j_t(\omega), 0 \right)^+ \tag{24}$$

where C^j_t is the collateral's market value, posted by counterparty j, at time t. The impact on future exposure of future changes in the collateral's market value can be identified by examining the collateralized future expected exposure, $X^{ce,j}_{t,T}$:

$$X^{ce,j}_{t,T} = E_t \left[\left(\sum_{i=1}^{n(j)} V^{j,i}_T(\omega) - C^j_T(\omega), 0 \right)^+ \right] \tag{25}$$

or the collateralized future maximal exposure, $X_{t,T}^{cm,j}$:

$$X_{t,T}^{cm,j} = \max_{\omega\in\Omega}\left(\sum_{i=1}^{n(j)} V_T^{j,i}(\omega) - C_T^j(\omega), 0\right)^+ \tag{26}$$

(26) is understood to be calculated subject to a given confidence level on the distribution of collateralized exposure at time T. The impact of collateral arrangements on loss measures can be measured by examining the collateralized future expected loss to counterparty j, $L_{t,T}^{ce,j}$:

$$L_{t,T}^{ce,j} = E_t\left[\lambda_T^j(\omega)\exp\left(-\int_t^T \lambda_s^j(\omega)ds\right) \times \left(\sum_{i=1}^{n(j,k)}(1-R_T^{j,i}(\omega))V_t^{j,i}(\omega) - C_t^j(\omega), 0\right)^+\right] \tag{27}$$

as well as the collateralized future maximal loss to the same counterparty, $L_{t,T}^{cm,j}$:

$$L_{t,T}^{cm,j} = \max_{\omega\in\Omega}\lambda_T^j(\omega)\exp\left(-\int_t^T \lambda_s^j(\omega)ds\right) \times \left(\sum_{i=1}^{n(j,k)}(1-R_T^{j,i}(\omega))V_t^{j,i}(\omega) - C_t^j(\omega), 0\right)^+ \tag{28}$$

Finally, existence of netting agreements and collateral arrangements can be included into exposure measures by defining the net collateralized current exposure to counterparty j, $X_t^{nc,j}$:

$$X_t^{nc,j} = \sum_{k=1}^{cs(j)}\left(\sum_{i=1}^{n(j,k)} V_t^{j,i}(\omega) - C_t^{j,k}(\omega), 0\right)^+ \tag{29}$$

where $C_t^{j,k}(\omega)$ is the collateral for netting set k, posted by counterparty j. Net collateralized expected future exposure, $X_{t,T}^{nce,j}$, is given by

$$X_{t,T}^{nce,j} = E_t\left[\sum_{k=1}^{cs(j)}\left(\sum_{i=1}^{n(j,k)} V_t^{j,i}(\omega) - C_t^{j,k}(\omega), 0\right)^+\right] \tag{30}$$

and net collateralized maximal future exposure, $X_{t,T}^{ncm,j}$, is given by

$$X_{t,T}^{ncm,j} = \max_{\omega\in\Omega}\sum_{k=1}^{cs(j)}\left(\sum_{i=1}^{n(j,k)} V_t^{j,i}(\omega) - C_t^{j,k}(\omega), 0\right)^+ \tag{31}$$

Net collateralized expected future loss, $L_{t,T}^{nce,j}$, is given by

$$L_{t,T}^{nce,j} = E_t\left[\lambda_T^j(\omega)\exp\left(-\int_t^T \lambda_s^j(\omega)ds\right) \times \sum_{k=1}^{cs(j)}\left(\sum_{i=1}^{n(j,k)}(1-R_T^{j,i}(\omega))V_t^{j,i}(\omega) - C_t^{j,k}(\omega), 0\right)^+\right] \tag{32}$$

and net collateralized maximal future loss, $L_{t,T}^{ncm,j}$, can be defined to be

$$L_{t,T}^{ncm,j} = \max_{\omega\in\Omega} \lambda_T^j(\omega) \exp\left(-\int_t^T \lambda_s^j(\omega) ds\right) \times \sum_{k=1}^{cs(j)} \left(\sum_{i=1}^{n(j,k)} (1 - R_T^{j,i}(\omega))V_t^{j,i}(\omega) - C_t^{j,k}(\omega), 0\right)^+ \tag{33}$$

The above statistics should facilitate calculation of Value at Risk for credit risk in the presence of netting and collateral arrangements. In the form given above, these formulae demonstrate that the credit risk specification in the current thesis is capable to address both pricing and risk management issues in a unified framework.

5.2.2.3. Special Purpose Vehicles

A number of institutions have set up highly rated derivative product companies (DPC) in order to mitigate their own credit risk and thus be able to capture greater market share in dealings with credit sensitive investors (see Bahar and Gold (1995) and Curry et al. (1995)). Derivative product companies enter into mirror transactions with their parent companies in order to off-load the market risk incurred through dealings with clients. Separate capitalization isolates the vehicle from the parent company's credit risk. Derivative product companies differ with respect to liquidation procedures in the event of the parent's default.
With regard to termination structures, in the event of default contracts held by the vehicle are marked to market and the vehicle is terminated. The statutory termination clause leads to potentially expensive closing of open positions at unfavorable market rates.
Continuation structures continue to trade after the parent company has defaulted until all positions are closed out. This structure has the merit of avoiding the forced shutdown at potentially unfavorable market rates but requires a more elaborate administrative structure to insure smooth operation without the parent company's support.

5.2.3. Pricing of Credit Risk

In contrast to rationing or mitigation techniques, only the ability to price credit risk will allow financial institutions to create the exact credit risk profile they desire. Application of results established in the preceding chapters will facilitate such endeavors.. We proceed by analyzing the pricing of a number of specific contracts subject to counterparty risk.

5.2.3.1. Forward Contracts and Futures Contracts on Yields

The present section is devoted to the valuation of forward and futures contracts subject to counterparty risk. The aim here is to demonstrate that, given the credit risk specification introduced in chapter 2, closed form solutions for the value of such contracts can be established. This contrasts with the quantification of counterparty risk in swap agreements, to be considered in the subsequent section.

Proposition 5.1 The value of a forward contract, arranged between two counterparties 1 and 2, on the $M - T = q$ maturity Libor yield with settlement in arrears is given by

$$V_t = K_{2,x} N(\eta_1) - K_{2,y} N(\eta_1 - \zeta_1) - K_{1,y} N(\eta_2) + K_{1,x} N(\eta_2 - \zeta_2) \tag{34}$$

where, for $j \in [1, 2]$:

$$Y_j = \sum_{i=1}^{3} \left(\int_t^T (w_u^j)^2 q_u^{\lambda i} a^{\lambda_j}(u, M) + q_s^{li}(a^l(s, M) - a^l(s, T)) + \int_T^M (w_u^j)^2 q_u^{\lambda_i} a^{\lambda_j}(u, M) \right) dM_u^{i,c,T} \tag{35}$$

$$X_j = \sum_{i=1}^{3} \int_t^M (w_u^j)^2 q_u^{\lambda i} a^{\lambda_j}(u, M) dM_u^{i,c,T} \tag{36}$$

$$\sigma_{j,y}^2 = \int_t^T (w_u^j)^2 a^{\lambda_j}(u, M)^2 du + \int_T^M \left((w_u^j)^2 q_u^{\lambda i} a^{\lambda_j}(u, M) + q_s^{li}((a^l(s, M) - a^l(s, T))\right)^2 du \tag{37}$$

$$\sigma_{j,x}^2 = \int_t^M (w_u^j)^2 a^{\lambda_j}(u, M)^2 du \tag{38}$$

$$\begin{aligned} K_{j,x} \quad = \quad & B_{t,T}^l \times \exp\left(\frac{\sigma_{j,x}^2}{2} + \int_t^M w_s^i \lambda_{0,s}^{i,T} ds - \int_t^M \int_0^s (w_u^i)^2 \sigma^{\lambda_i}(u, s) a^{\lambda_i}(u, s) du ds \right) \\ & \times \exp\left(- \int_t^M \int_0^s (w_u^i)^2 \rho_u^{\lambda l} \left(a^l(u, s)\sigma^{\lambda_i}(u, s) + \sigma^l(u, s)a^{\lambda_i}(u, s) - a^l(u, T)\sigma^{\lambda_i}(u, s)\right) du ds \right) \\ & \times \exp\left(\sum_{i=1}^{3} \int_0^t (w_u^i)^2 q_u^{\lambda i}(a^{\lambda_i}(u, M) - a^{\lambda_i}(u, t)) dM_u^{i,c,T} \right) \end{aligned} \tag{39}$$

$$K_{j,y} = (1+\delta L^{l,F}_{t,M,q})B^l_{t,M} \times \exp\left(\frac{\sigma^2_{j,y}}{2} - \frac{1}{2}\int_t^T \left(a^l(s,M) - a^l(s,T)\right) ds\right) \tag{40}$$
$$\times \exp\left(\int_t^M w^i_s \lambda^{i,T}_{0,s} ds - \int_t^M \int_0^s (w^i_u)^2 \sigma^{\lambda_i}(u,s) a^{\lambda_i}(u,s) du ds\right)$$
$$\times \exp\left(-\int_t^M \int_0^s (w^i_u)^2 \rho^{\lambda l}_u \left(a^l(u,s)\sigma^{\lambda_i}(u,s) + \sigma^l(u,s)a^{\lambda_i}(u,s) - a^l(u,T)\sigma^{\lambda_i}(u,s)\right) du ds\right)$$
$$\times \exp\left(\sum_{i=1}^3 \int_0^t (w^i_u)^2 q^{\lambda_i}_u (a^{\lambda_i}(u,M) - a^{\lambda_i}(u,t)) dM^{i,c,T}_u\right)$$

$$\zeta_j = var\left(X_j - Y_j\right) \tag{41}$$

$$\eta_j = \frac{1}{\zeta_j} \ln\left(\frac{K_{j,x}}{K_{j,y}}\right) + \frac{\zeta_j}{2} \tag{42}$$

Proof:

Given the contract's payoff definition, the value of the contract is given by

$$\begin{aligned} V_t &= B^l_{t,M} \times E^M_t\left[D^{2,d}_M \left(L_{T,M} - L^{l,F}_{t,M,q}\right)^+ \delta\right] - B^l_{t,M} \times E^M_t\left[D^{1,d}_M \left(L^{l,F}_{t,M,q} - L_{T,M}\right)^+ \delta\right] \\ &= B^l_{t,M} \times E^T_t\left[\frac{B^l_{T,M}B^l_{t,T}}{B^l_{t,M}B^l_{T,T}}\left(D^{2,d}_M\left(L_{T,M} - L^{l,F}_{t,M,q}\right)^+ \delta - D^{1,d}_M\left(L^{l,F}_{t,M,q} - L_{T,M}\right)^+ \delta\right)\right] \\ &= B^l_{t,M} \times E^T_t\left[\frac{B^l_{T,M}B^l_{t,T}}{B^l_{t,M}B^l_{T,T}}\left(D^{2,d}_M\left(\frac{1}{B^l_{T,M}} - (1+\delta L^{l,F}_{t,M,q})\right)^+ - D^{1,d}_M\left((1+\delta L^{l,F}_{t,M,q}) - \frac{1}{B^l_{T,M}}\right)^+\right)\delta\right] \\ &= B^l_{t,T} \times E^T_t\left[D^{2,d}_M\left(1 - B^l_{T,M}(1+\delta L^{l,F}_{t,M,q})\right)^+\right] - B^l_{t,T} \times E^T_t\left[D^{1,d}_M\left(B^l_{T,M}(1+\delta L^{l,F}_{t,M,q}) - 1\right)^+\right] \end{aligned} \tag{43}$$

Under the P^T-measure, the following restriction must hold:

$$\begin{aligned} \frac{B^l_{T,M}}{B^l_{T,T}} &= \frac{B^l_{t,M}}{B^l_{t,T}} \times \exp\left(\sum_{i=1}^3 \int_t^T q^{li}_s \left(a^l(s,M) - a^l(s,T)\right) dM^{i,c,T}_s\right) \\ &\quad \times \exp\left(-\frac{1}{2}\int_t^T \left(a^l(s,M) - a^l(s,T)\right) ds\right) \end{aligned} \tag{44}$$

Taking conditional expectation of the recovery rate process for both counterparties:

$$E^T_M\left[D^{i,d}_M\right] = \exp\left(\int_t^M w^i_s \lambda^{i,T}_s ds\right) \tag{45}$$

Furthermore,

$$\int_t^M w^i_s \lambda^{i,T}_s ds = \int_t^M w^i_s \lambda^{i,T}_{0,s} ds - \int_t^M \int_0^s (w^i_u)^2 \sigma^{\lambda_i}(u,s) a^{\lambda_i}(u,s) du ds \tag{46}$$

$$+\sum_{i=1}^{3}\int_{t}^{M}\int_{0}^{s}(w_u^i)^2 q_s^{\lambda i}\sigma^{\lambda_i}(u,s)dM_u^{i,c,T}ds$$

$$-\int_{t}^{M}\int_{0}^{s}(w_u^i)^2\rho_u^{\lambda l}\left(a^l(u,s)\sigma^{\lambda_i}(u,s)+\sigma^l(u,s)a^{\lambda_i}(u,s)-a^l(u,T)\sigma^{\lambda_i}(u,s)\right)duds$$

$$=\int_{t}^{M} w_s^i\lambda_{0,s}^{i,T}ds-\int_{t}^{M}\int_{0}^{s}(w_u^i)^2\sigma^{\lambda_i}(u,s)a^{\lambda_i}(u,s)duds-$$

$$\int_{t}^{M}\int_{0}^{s}(w_u^i)^2\rho_u^{\lambda l}\left(a^l(u,s)\sigma^{\lambda_i}(u,s)+\sigma^l(u,s)a^{\lambda_i}(u,s)-a^l(u,T)\sigma^{\lambda_i}(u,s)\right)duds+$$

$$\sum_{i=1}^{3}\int_{t}^{M}(w_u^i)^2 q_u^{\lambda i}a^{\lambda_i}(u,M)dM_u^{i,c,T}+\sum_{i=1}^{3}\int_{0}^{t}(w_u^i)^2 q_u^{\lambda i}(a^{\lambda_i}(u,M)-a^{\lambda_i}(u,t))dM_u^{i,c,T}$$

Application of standard techniques now yields (34).□

Note that setting the value of the contract to zero yields the forward yield $L_{t,M,q}^{l,F}$ for a forward contract subject to counterparty risk. (34) is admittedly tedious but illustrates how the presence of bilateral counterparty credit risk, even under the simplifying assumption of deterministic recovery rates, complicates the pricing of even relatively simple contracts.

Consider the continuously resettled equivalent of (34):

Proposition 5.2 The value of a futures contract, arranged between two counterparties 1 and 2, on the $M-T=q$ maturity Libor yield with settlement in arrears is given by

$$V_t = K_{2,x}N(\eta_1) - K_{2,y}N(\eta_1-\zeta_1) - K_{1,y}N(\eta_2) + K_{1,x}N(\eta_2-\zeta_2) \tag{47}$$

where

$$X_j=\sum_{i=1}^{3}\left(\int_{t}^{T}(w_u^j)^2 q_u^{\lambda i}a^{\lambda_j}(u,M)+q_s^{li}(a^l(s,M)-a^l(s,T))+\int_{T}^{M}(w_u^j)^2 q_u^{\lambda i}a^{\lambda_j}(u,M)\right)dM_u^{i,c,*} \tag{48}$$

$$Y_j=\sum_{i=1}^{3}\int_{t}^{M}(w_u^j)^2 q_u^{\lambda i}a^{\lambda_j}(u,M)dM_u^{i,c,*} \tag{49}$$

$$\sigma_{j,x}^2=\int_{t}^{T}(w_u^j)^2 a^{\lambda_j}(u,M)^2du+\int_{T}^{M}\left((w_u^j)^2 q_u^{\lambda i}a^{\lambda_j}(u,M)+q_u^{li}(a^l(u,M)+a^l(u,T))\right)^2du \tag{50}$$

$$\sigma_{j,y}^2=\int_{t}^{M}(w_u^j)^2 a^{\lambda_j}(u,M)^2du \tag{51}$$

$$K_{j,x} = B^l_{t,T} \times \exp\left(\frac{\sigma^2_{j,x}}{2} + \frac{1}{2}\int_t^T a^l(s,M)^2 ds\right) \tag{52}$$
$$\times \exp\left(\int_t^M w^i_s \lambda^{i,*}_{0,s} ds - \int_t^M \int_0^s (w^i_u)^2 \sigma^{\lambda_i}(u,s) a^{\lambda_i}(u,s) du ds\right)$$
$$\times \exp\left(-\int_t^M \int_0^s (w^i_u)^2 \rho^{\lambda l}_u \left(a^l(u,s)\sigma^{\lambda_i}(u,s) + \sigma^l(u,s) a^{\lambda_i}(u,s)\right) du ds\right)$$
$$\times \exp\left(\sum_{i=1}^3 \int_0^t (w^i_u)^2 q^{\lambda i}_u (a^{\lambda_i}(u,M) - a^{\lambda_i}(u,t)) dM^{i,c,T}_u\right)$$
$$\times \exp\left(-\int_t^T f^l_{0,s} ds + \int_t^T \int_0^s \sigma^l(u,s) a^l(u,s) du ds - \sum_{i=1}^3 \int_0^t q^{li}_u (a^l(u,T) - a^l(u,t)) dM^{i,c,*}_u\right)$$

$$K_{j,y} = (1 + \delta L^{l,F}_{t,M,q}) B^l_{t,M} \times \exp\left(\frac{\sigma^2_{j,y}}{2}\right) \tag{53}$$
$$\times \exp\left(\int_t^M w^i_s \lambda^{i,*}_{0,s} ds - \int_t^M \int_0^s (w^i_u)^2 \sigma^{\lambda_i}(u,s) a^{\lambda_i}(u,s) du ds\right)$$
$$\times \exp\left(-\int_t^M \int_0^s (w^i_u)^2 \rho^{\lambda l}_u \left(a^l(u,s)\sigma^{\lambda_i}(u,s) + \sigma^l(u,s) a^{\lambda_i}(u,s)\right) du ds\right)$$
$$\times \exp\left(\sum_{i=1}^3 \int_0^t (w^i_u)^2 q^{\lambda_i}_u (a^{\lambda_i}(u,M) - a^{\lambda_i}(u,t)) dM^{i,c,T}_u\right)$$

$$\zeta_j = var\left(X_j - Y_j\right) \tag{54}$$

$$\eta_j = \frac{1}{\zeta_j} \ln\left(\frac{K_{j,x}}{K_{j,y}}\right) + \frac{\zeta_j}{2} \tag{55}$$

Proof:

Given the payoff definition of the futures contract, the value of the contract is given by

$$V_t = E^*_t\left[D^{2,d}_M \left(L_{T,M} - L^{l,F}_{t,M,q}\right)^+ \delta\right] - E^*_t\left[D^{1,d}_M \left(L^{l,F}_{t,M,q} - L_{T,M}\right)^+ \delta\right] \tag{56}$$
$$= E^*_t\left[D^{2,d}_M \left(\frac{1}{B^l_{T,M}} - (1 + \delta L^{l,F}_{t,M,q})\right)^+\right] - E^*_t\left[D^{1,d}_M \left((1 + \delta L^{l,F}_{t,M,q}) - \frac{1}{B^l_{T,M}}\right)^+ \delta\right]$$

Under the P^* measure, the following restriction must hold

$$\frac{B^l_{t,M}}{B^l_{T,M}} = \exp\left(-\int_t^T r^l_s ds - \sum_{i=1}^3 \int_t^T q^{li}_s a^l(s,M) dM^{i,c,*}_s + \frac{1}{2}\int_t^T a^l(s,M)^2 ds\right) \tag{57}$$

$$E^*_M\left[D^{i,d}_M\right] = \exp\left(\int_t^M w^i_s \lambda^{i,*}_s ds\right) \tag{58}$$

$$\int_t^T r_s^l ds = \int_t^T f_{0,s}^l ds - \int_t^T \int_0^s \sigma^l(u,s)a^l(u,s)duds + \sum_{i=1}^{3} \int_t^T \int_0^s q_s^{li}\sigma^l(u,s)dM_u^{i,c,*}ds \quad (59)$$

$$= \int_t^T f_{0,s}^l ds - \int_t^T \int_0^s \sigma^l(u,s)a^l(u,s)duds + \sum_{i=1}^{3} \int_0^t q_u^{li}(a^l(u,T) - a^l(u,t))dM_u^{i,c,*}$$

$$+ \int_t^T q_u^{li} a^l(u,T)dM_u^{i,c,*}$$

$$\int_t^M w_s^i \lambda_s^{i,*} ds = \int_t^M w_s^i \lambda_{0,s}^{i,*} dsc - \int_t^M \int_0^s (w_u^i)^2 \sigma^{\lambda_i}(u,s)a^{\lambda_i}(u,s)duds \quad (60)$$

$$+ \sum_{i=1}^{3} \int_t^M \int_0^s (w_u^i)^2 q_s^{\lambda i} \sigma^{\lambda_i}(u,s)dM_u^{i,c,*}ds$$

$$- \int_t^M \int_0^s (w_u^i)^2 \rho_u^{\lambda l} \left(a^l(u,s)\sigma^{\lambda_i}(u,s) + \sigma^l(u,s)a^{\lambda_i}(u,s)\right) duds$$

$$= \int_t^M w_s^i \lambda_{0,s}^{i,*} ds - \int_t^M \int_0^s (w_u^i)^2 \sigma^{\lambda_i}(u,s)a^{\lambda_i}(u,s)duds -$$

$$\int_t^M \int_0^s (w_u^i)^2 \rho_u^{\lambda l} \left(a^l(u,s)\sigma^{\lambda_i}(u,s) + \sigma^l(u,s)a^{\lambda_i}(u,s)\right) duds +$$

$$\sum_{i=1}^{3} \int_t^M (w_u^i)^2 q_u^{\lambda_i} a^{\lambda_i}(u,M)dM_u^{i,c,*} + \sum_{i=1}^{3} \int_0^t (w_u^i)^2 q_u^{\lambda i} \left(a^{\lambda_i}(u,M) - a^{\lambda_i}(u,t)\right) dM_u^{i,c,*}$$

Application of standard techniques now yields (47).□

Note that (34) and (47) constitute closed form solution for the value of contracts subject to counterparty risk. Development of such formulae is possible only for very simple contracts. In the following section, it is demonstrated that in the case of swap contracts, incorporation of counterparty credit risk will require numerical techniques in order to price the contract.

5.2.3.2. Swap Contracts

Considering the pricing of swap contracts subject to counterparty risk, two facts must be taken into account. On the one hand, determination of the market value of the contract at a given payment date not only depends on the net payment at that particular date, but on the expected discounted value of the sequence of remaining payments up to the maturity date. With regard to the contracts analyzed in the previous section, it was precisely the fact that contract values were determined by payments at the single payment date which allowed for closed form solutions to be established. This will not be possible with regard to swap arrangements.

On the other hand, in contrast to forward contracts, the specific settlement procedures in the swap market necessitate a more articulate approach to the formalization of counterparty risk. Under the "one way" rule[27], the payment due to the non-defaulting party is the higher of the market value of its position and zero; the non defaulting party is not obliged to compensate the defaulting party if the value of the contract to the defaulting party is positive. The "two way" rule[28] stipulates that the party for which the value of the contract is negative must compensate the other party, regardless of the identity of the defaulting party; this rule is most frequently employed by market participants.

Proposition 5.3 The value of an interest rate swap SW_t between two credit risky counterparties is given by

$$SW_t = \tag{61}$$
$$\sum_{i=1}^{n} B^l_{t,T_i} \times E^{T_i}_t \left[\delta(L_{T_{i-1},T_i} - c_t) \times \exp\left(- \int_t^{T_i} (I_{\{SW_{T_i} \geq 0\}} s^2_s + I_{\{SW_{T_i} < 0\}} s^1_s) ds \right) \right]$$

or, equivalently,

$$SW_t = \tag{62}$$
$$\sum_{i=1}^{n} E^*_t \left[\delta(L_{T_{i-1},T_i} - c_t) \times \exp\left(- \int_t^{T_i} (r^l_s + I_{\{SW_{T_i} \geq 0\}} s^2_s + I_{\{SW_{T_i} < 0\}} s^1_s) ds \right) \right]$$

where

$$s^1_s = \left(1 - e^{-k_1}\right) \lambda^{1,T_i}_t + \left(1 - e^{-d_2}\right) \lambda^{2,T_i}_t \tag{63}$$

$$s^2_s = \left(1 - e^{-k_2}\right) \lambda^{2,T_i}_t + \left(1 - e^{-d_1}\right) \lambda^{1,T_i}_t \tag{64}$$

Proof:

The value of the contract is given by

$$SW_t = \sum_{i-1}^{n} B^l_{t,T_i} \times E^{T_i}_t \left[\delta(L_{T_{i-1},T_i} - c_t) \prod_{t < s \leq T_i} \left(1 + D^1_s\right) \left(1 + D^2_s\right) \right] \tag{65}$$

where the default processes, augmented for the settlement features specific to the swap market, are given by

$$D^1_s = \left(I_{\{SW_{T_i} < 0\}} \left(e^{-k_1} - 1\right) + I_{\{SW_{T_i} \geq 0\}} \left(e^{-d_2} - 1\right) \right) dN^1_s \tag{66}$$

$$D^2_s = \left(I_{\{SW_{T_i} \geq 0\}} \left(e^{-k_2} - 1\right) + I_{\{SW_{T_i} < 0\}} \left(e^{-d_1} - 1\right) \right) dN^2_s \tag{67}$$

---

27 ISDA master agreement (1992), section 6(e)(i)(1).

28 ISDA master agreement (1992), section 6(e)(i)(3).

The one way settlement rule corresponds to $d_1 = d_2 = \infty$, whereas the two way settlement rule corresponds to $d_1 = d_2 = 0$, reflecting the fact that counterparties for which the contract value is positive are required to compensate the defaulting party. Conditioning on information at time T_i inside the expectation in (65) yields the following expression:

$$E_{T_i}^{T_i}\left[\prod_{t<s\leq T_i}\left(1+D_s^1\right)\left(1+D_s^2\right)\right] = \tag{68}$$

$$\begin{aligned}
& E_{T_i}^{T_i}\left[\prod_{t<s\leq T_i}\left(1+I_{\{SW_{T_i}<0\}}\left(e^{-k_1}-1\right)+I_{\{SW_{T_i}\geq 0\}}\left(e^{-d_2}-1\right)\right)^{dN_s^1}\right] \\
& \times E_{T_i}^{T_i}\left[\prod_{t<s\leq T_i}\left(1+I_{\{SW_{T_i}\geq 0\}}\left(e^{-k_2}-1\right)+I_{\{SW_{T_i}<0\}}\left(e^{-d_1}-1\right)\right)^{dN_s^2}\right] \\
= \; & E_{T_i}^{T_i}\left[\exp\left(-\int_t^{T_i}(I_{\{SW_{T_i}\geq 0\}}k_2 + I_{\{SW_{T_i}<0\}}d_1)dN_s^2\right)\right] \\
& \times E_{T_i}^{T_i}\left[\exp\left(-\int_t^{T_i}(I_{\{SW_{T_i}\geq 0\}}k_1 + I_{\{SW_{T_i}<0\}}d_2)dN_s^1\right)\right]
\end{aligned}$$

Evaluating the latter expectation yields:

$$E_{T_i}^{T_i}\left[\cdot\prod_{t<s\leq T_i}\left(1+D_s^1\right)\left(1+D_s^2\right)\right] = \exp\left(-\int_t^{T_i}(I_{\{SW_{T_i}\geq 0\}}s_s^2 + I_{\{SW_{T_i}<0\}}s_s^1)ds\right) \tag{69}$$

Inserting (69) into (65) yields (61).□

As long as the value of the contract to counterparty 1 is positive, counterparty 1 is effectively advancing a loan to counterparty 2; the rate on this loan si given by the risk free rate plus the credit spread of counterparty 2. When the value of the contract to counterparty 1 is negative, counterparty 1 is effectively borrowing from counterparty 2; the rate at which counterparty 1 can borrow from counterparty 2 is given by the risk free rate plus the credit spread of counterparty 1. Setting the value of the contract to zero yields the fixed swap rate c_t; such an exercise will involve searches over possible interest rate and credit spread paths, determining intervals over which the contract has positive or negative value to each of the counterparties involved, discounting those values at the corresponding spread and finally searching for the value c_t for which $SW_t = 0$. Expression similar to (61) and (62) have been derived by ??. The above proposition demonstrates that such results are consistent with and can be generated within the more elaborate credit risk framework of the present thesis.

Conclusion

Among other risks, banks face model risk, capturing the possibility that the methods and models implemented to capture uncertainty in real life phenomena represent an incomplete or even wrong mapping of such uncertainties. Any formal risk model is subject to this kind of risk, and the current credit risk specification is no exception. That said, the following conclusion is in order. For internal purposes, financial institutions require establishment of measures of counterparty credit risk. At the same time, banks are facing increasingly comprehensive regulatory requirements with respect to credit risk reporting and control. In order to respond to these requirements in a timely fashion, market participants need to implement measures quantifying the counterparty credit risk they are exposed to. The analysis in the preceding sections has demonstrated that the credit risk specification proposed in the current thesis, based on a stochastic probability of default, stochastic recovery rates and an explicit correlation structure between market risk and credit risk, is well suited to be applied in a risk management context: A number of exposure and loss measures, based on the credit risk specification developed in chapter 2, can be formulated and quantification of credit risk in conventional contracts is shown to be feasible.. The fact that one methodology can be consistently applied to both the pricing of credit contingent claims, as demonstrated in chapter 3, as well as for the purposes of counterparty credit risk management, as illustrated in the present chapter, will be appealing to both financial institutions and regulatory agencies.

A risk measure which has gained increased popularity is Value at Risk, measuring the potential loss in portfolio value, within a given confidence interval and over a specified time horizon, due to changes in market variables. Regulatory authorities have conceded considerable freedom to financial institutions in devising models generating such market risk measures, subject to the specification of benchmark parameters. The amount of risk thus established determines, among other things, the amount of capital the financial institution must set aside for its exposure to market risk. With regard to credit risk, however, implementation and acceptance by regulators of sophisticated measures of credit risk for the purposes of determining capital requirements is far less developed. In addition, while capital requirements for market risk take account of interactions of risk across asset classes, no account is taken of interaction of credit risk and market risk or of comovements of credit risks associated with different borrowers for the purpose of calculating capital requirements.

Existing credit risk specifications are unable to address such issues because of their inability to generate sufficient realism by stipulating either fixed recovery rates or deterministic default probabilities. The failure to model explicitly correlation parameters relating to interaction of credit risk and market risk components means that existing specifications cannot address issues relating to potential loss scenarios associated with combined credit and market risk exposure profiles.

In contrast to existing modelling approaches, the present credit risk specification opens up the possibility to examine these issues in a formal way. Specification of stochastic recovery rates and of an uncertain default probability, in conjunction with specific probabilistic assumptions relating to the uncertainty in these credit risk components, will facilitate simulation of realistic credit scenarios, opening up the possibility of formalizing Value at Risk measures for credit risk. The explicit role for a correlation structure between credit risk and market risk should facilitate formulation of capital requirements taking account of comovements in risk factors not just within risk classes, but across risk classes. Establishment of models taking account of such features will be desirable from a regulatory point of view if existence of comovements across risk classes exaggerates overall risk profiles. In turn, if incorporation of correlation features linking credit and market risk variables reduces overall risk and thus entails lower capital requirements, banks will have incentives to implement models that permit such features to be captured. While the empirical evidence relating to the size and direction of such effects is still limited, market participants and regulators will need to address these issues in due time and the provision of the present credit risk specification should facilitate this process.

References

Alexander, C. (1994): "History debunked," *Risk*, 7/12.

Alexander, C. (1996): *The Handbook of Risk Management and Analysis* Wiley

Allen, M. (1994): "Building a role model," *Risk*, 7/9.

Bahar, R. and M. Gold (1995): "Structuring derivative product companies: Risks and safeguards," in: *Derivative Credit Risk*, Risk Publications.

Bank of England (1996): "Developing a supervisory approach to credit derivatives," *Discussion Paper.*

Basle Committee on Banking Supervision (1988): "International convergence of capital measurement and capital standards," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1990): "Report of the committee on interbank netting schemes of the central banks of the group of ten countries," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1993): "The supervisory treatment of market risks," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1995a): "Planned supplement to the capital accord to incorporate market risks," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1995b): "An internal model-based approach to market risk capital requirements," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1996a): "Interpretation of the capital accord for the multilateral netting of forward value foreign exchange transaction," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1996b): "Supervisory framework for the use of 'backtesting' in conjunction with the internal models approach to market risk capital requirements," *Bank for International Settlements.*

Basle Committee on Banking Supervision (1996c): "Treatment of potential exposure for off-balance-sheet items," *Bank for International Settlements.*

Beder, T. (1995): "VAR: Seductive but Dangerous ," *Financial Analysts Journal*, (September-October), 12-24.

Boudoukh, J., M. Richardson and R. Whitelaw (1995): "Expect the worst," *Risk*, 8/9.

Bralver, C. and A. Kuritzkes (1993): "Risk adjusted performance measurement in the trading room," *Journal of Applied Corporate Finance*, 6/3, 109-131.

Chew, L. (1994a): "Protect and survive," *Risk*, 7/3.

Chew, L. (1994b): "Conscious efforts," *Risk*, 7/3.

Chew, L. (1994c): "Shock treatment," *Risk*, 7/9.

Cumby, R., S. Figlewski and J. Hasbrouk (1993): "Forecasting volatilities and correletions with EGARCH models," *The Journal of Derivatives* (Winter), 51-63.

Curry, D., J. Gluck, W. May and A. Backman (1995): "Evaluating derivative product companies", in: *Derivative Credit Risk*, Risk Publications.

Derivatives Policy Group (1995): "Framework for voluntary oversight."

Dimson, E. and P. Marsh (1995): "Capital requirements for securities firms," *The*

Journal of Finance, 50/3, 821-851.

Dimson, E. and P. Marsh (1995): "Stress tests of capital requirements," *Working Paper.*

Duffee, G. (1995): "On measuring credit risks of derivative instruments," *Journal of Banking and Finance*, 20, 805-833.

Duffee, G. and C. Zhou (1996): "Banks and credit derivatives: Is it always good to have more risk management tools?," *Working Paper.*

Duffie, D. and M. Huang (1996): "Swap rates and credit quality," *The Journal of Finance*, 51/3, 921-949.

Elderfield, M. (1995a): "Capital countdown," *Risk*, 8/2.

Elderfield, M. (1995b): "Capital incentives," *Risk*, 8/9.

Engle, R. (1982): "Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation," *Econometrica*, 50/4, 987-1007.

Engle, R. and J. Mezrich (1995): "Grappling with GARCH," *Risk*, 8/9.

European Commission (1993): "Concil directive on the capital adequacy of investment firms and credit institutions," *Official Journal of the European Communities*, L 141, 1-26.

European Commission (1994a): "Proposal for a European Parliament and Council Directive amending Concil Directive 89/647/EEC with respect to the supervisory recognition of contracts for novation and netting agreements," *Official Journal of the European Communities*, L 142, 8-11.

European Commission (1994b): "Council Directive on a solvency ratio for credit institutions," *Official Journal of the European Communities*, L 386, 14-22.

European Commission (1994c): "Council Directive for the own funds of credit institutions," *Official Journal of the European Communities*, L 124, 16-20.

European Commission (1994d): "Council Directive on the supervision of credit institutions on a consolidated basis," *Official Journal of the European Communities*, L 110, 52-58.

European Commission (1994e): "Council Directive on the monitoring and control of large exposures of credit institutions," *Official Journal of the European Communities*, L 29, 1-8.

European Commission (1994f): "Council Directive on investment services in the securities field," *Official Journal of the European Communities*, L 110, 52-58.

European Commission (1994g): "Council Directive on investment services in the securities field," *Official Journal of the European Communities*, L 141, 27-46.

European Commission (1995): "Amended Proposal for a European Parliament and Council Directive amending Concil Directive 89/647/EEC with respect to the supervisory recognition of contracts for novation and netting agreements," *Official Journal of the European Communities*, C 165, 6-8.

Fallon W. (1996): "Calculating Value at Risk," *Working Paper.*

Falloon, W. (1995): "2020 visions," *Risk*, 8/10.

Garman, M. (1996): "Improving on VAR," *Risk*, 9/5.

Group of 30 (1993): "Derivatives: Practices and principles."

Gray, S. (1996): "Modellling the conditional distribution of interest rates as a regime switching process," *Journal of Financial Economics*, 42, 27-62.

Guldimann, T. (1995): "RiskMetrics Technical document," 3rd ed., Morgan

Guaranty Trust Company, New York.

Gumerlock, R. (1993a): "When the snoozing had to stop," *Risk*, 6/9.

Gumerlock, R. (1993b): "Double trouble," *Risk*, 6/9.les".

Heldring, O. (1995): "Alpha plus," *Risk*, 8/1.

Hendricks, D. (1996): "Evaluation of Value-at-Risk Models using historical data", *FRBNY Economic Policy Review* (April), 39-68.

Hentschel, L. and C. Smith (1995): "Controlling risks in derivative markets," *The Journal of Financial Engineering*, 4/2, 101-125.

Heynen, R. and H. Kat (1994): "Volatility prediction: A comparison of the stochastic volatility, GARCH(1,1), and EGARCH(1,1) models," *The Journal of Derivatives* (Winter), 50-65.

Heynen, R., H. Kat and T. Vorst (1994): "Analysis of the term structure of implied volatilities," *Journal of Financial and Quantitative Analysis*, 29/1, 31-56.

Hull, J. (1989): "Assessing credit risk in a financial institution's off-balance sheet commitments," *Journal of Financial and Quantitative Analysis*, 24/4, 489-501.

Iben, B. and R. Brotherton-Ratcliffe (1994): "Credit loss distributions and required capital for derivatives portfolios," *The Journal of Fixed Income* (June), 6-14.

Irving, R. (1996): "Credit derivatives come good," *Risk*, 9/7.

Jackson, P., D. Maude and W. Perraudin (1996): "Capital requirements and value-at-risk analysis," *Working Paper*.

Jamshidian, F. and Yu Zhu (1996): "Scenario simulation: Theory and methodology," *Working Paper*.

Jorion, P. (1996): "Risk^2: Measuring the risk in value at risk," *Financial Analysts Journal* (November-December), 47-56.

Kupiec, P. (1995): "Techniques for verifying the accuracy of risk measurement models," *The Journal of Derivatives* (Winter), 73-84.

Kupiec, P. and J. O'Brian (1995a): "Model alternative," *Risk*, 8/6.

Kupiec, P. and J. O'Brian (1995b): "Internal affairs," *Risk*, 8/5.

Lam, J. (1995): "Integrated risk measurement," in: *Derivative Credit Risk*, Risk Publications.

Lawrence, D. (1995): "Aggregating credit exposures: The simulation approach," in: *Derivative Credit Risk*, Risk Publications.

Lawrence, C. and G. Robinson (1995a): "Liquid measures," *Risk*, 8/7.

Lawrence, C. and G. Robinson (1995b): "How safe is riskmetrics ?," *Risk*, 8/1.

Longerstaey, J. and P. Zangari, (1995): "A transparent tool," *Risk*, 8/1.

Lopez, J. (1996): "Regulatory evaluation of VaR models," *Working Paper*.

Lucas, D. (1995): "Measuring credit risk and required capital," in: *Derivative Credit Risk*, Risk Publications.

Lucas, D. (1995a): "The effectiveness of downgrade provisions in reducing counterparty credit risk," *The Journal of Fixed Income* (June), 32-41.

Lucas, D. (1995b): "Default correlation and credit analysis," *The Journal of Fixed Income* (March), 76-87.

Mahoney, J. (1996): "Empirical based versus model based approaches to Value-at-Risk: An examination of foreign exchange and global equity portfolios", *Working Paper.*

Mark, R. (1995): "Integrated credit risk measurement," in: *Derivative Credit Risk*, Risk Publications.

Matten, C. (1996): *Managing Bank Capital* Wiley

Merton, R. (1995a): "Financial innovation and the management and regulation of financial institutions," *Journal of Banking and Finance*, 19, 461-481.

Merton, R. (1995b): "A functional perspective of financial intermediation," Financial Management (Summer), 24/2, 23-41.

Nelson, D. (1990): "ARCH models as diffusion approximations," *Journal of Econometrics*, 45, 7-38.

Phelan, M. (1995): "Probability and statistics applied to the practice of financial risk management: The case of JP Morgan's RiskMetrics," *Working Paper.*

Pritsker, M. (1996): "Evaluating value at risk methodologies: Accuracy versus computational time", *Working Paper.*

Rowe, D. (1995): "Aggregating credit exposures: The primary risk source approach", in: *Derivative Credit Risk*, Risk Publications.

Schaefer, S. (1993): "Financial regulation: The contribution of the theory of finance," *Working Paper.*

Shimko, D. (1995): "What is VAR ?," *Risk*, 8/12.

Smithson, C. (1996): "Value-at-risk," *Risk*, 9/1.

Smithson, C. and C. Smith (1995): *Managing Financial Risk* Irwin.

Sorensen, E. and T. Bollier (1994): "Pricing swap default risk," *Financial Analysts Journal* (May-June), 23-33.

Taylor, S. (1994): "Modeling stochastic volatility: A review and comparative study," *Mathematical Finance*, 4/2, 183-204.

Wilson, D. (1994): "In good company," *Risk* (August), 7/8, 35-39.

Wilson, D. (1995a): "VAR in operation," *Risk*, 8/12.

Wilson, D. (1995b): "Marriage of ideals," *Risk*, 8/7.

Wilson, T. (1994a): "Debunking the myths," *Risk*, 7/4, .

Wilson, T. (1994b): "Plugging the gap," *Risk*, 7/10.

Wisener, P. (1994): "Net errors," *Risk*, 7/5.

Chapter 6: The Term Structure of Credit Spreads and the Valuation of Credit Sensitive Claims: An Empirical Investigation

Introduction

Despite the considerable attention the analysis of credit risk has received from both academics and practitioners, little is known about the empirical properties of credit variables and credit contingent claims. Identification of such properties is relevant to the quantification and management of counterparty credit risk. At the same time, development of transfer mechanisms for credit risk, for instance in the form of credit derivatives, require establishment of the nature of uncertainty associated with specific credit risk components, a task which cannot be resolved by theoretical means. The fact that empirical analysis of credit risk has lagged theoretical developments can be attributed to three considerations. Relative to market risk data, data relating to credit risk is subject to a number of limitations. Contrary to market risk, credit risk contains elements specific to a particular issuer. While the credit quality of any two companies will be linked through association with a particular industry and exposure to macroeconomic trends, it must be assumed that market participants having incurred credit risk exposure, be it through exposure to counterparty risk in conventional contracts or through participation in contractual arrangements to transfer credit risk elements, will ultimately bear a cost for failure to take account of idiosyncratic risk components. Making proper allowance for the fact that credit risk incorporates significant firm specific components means that less data is available. In addition to quantity limitations due to data being firm specific, given that the occurrence of default is a rare event, establishment of credit properties from past observations is difficult because of the rarity of actual default; while this problem is less relevant for speculative grade instruments, the use of historical default rates relating to investment grade issues could be misleading. Finally, because the market for credit contingent claims is at an early stage in its development cycle, trading volume is low and contract variety is limited; in conjunction with the fact that the contracts which are being traded are all traded over the counter this means that researchers face severe data constraints.

At the same time, estimation techniques relating to credit risk free instruments are subject to ongoing development. In the context of conventional term structure models, which form the point of departure for most credit risk models, early attempts sought to estimate parameters using either cross sectional information on traded instruments (e.g. Brown and Dybvig (1986) or Brown and Schaefer (1994)) or time series information relating to a selection of state variable proxies (e.g. Chan et al. (1992) , Litterman and Scheinkman (1991), Longstaff and Schwartz (1993)). Subsequently, a number of researchers pointed out that particular model specifications generated joint cross sectional and time series restrictions; application of any one of these approaches without taking account of the additional constraints would be inconsistent and lead to loss of information. Techniques consistent with these considerations were introduced by Gibbons and Ramaswamy (1993) and subsequently extended by Chen and Scott (1993), Pearson and Sun (1994), Duan (1994) and Singh (1995). While these approaches still rely one the assumption that state variables can be observed without error, most recently researchers have proposed the use of filtering methods for the purpose of parameter estimation and forecasting of relevant state variables, where contrary to earlier approaches allowance is made for state variables to be observed with measurement error (e.g. Pennachi (1991), Geyer and Pichler (1995), Chen and Scott (1992) and Duan and Simonato (1995)). Despite the apparent progress, empirical work based on the above techniques leads to regular rejection of broad classes of models, indicating that both model specification as well as estimation techniques require further refinement.

Finally, no conclusive consensus has yet emerged as to how credit risk should be modelled. As a consequence, problems relating to data quality and estimation strategy are compounded by model uncertainty.

6.1. Existing evidence

Empirical evidence based on specifications consistent with the capital structure approach has generated mixed results. The widely cited study by Jones, Mason and Rosenfeld (1984) examined the ability of Merton's to price credit risky debt. Restricting analysis to firms whose capital structure consists of equity and multiple issues of callable nonconvertible debt, the authors find that the capital structure models offers no improvement relative to a model which neglects credit risk. Nevertheless, it is noted that "introducing stochastic interest rates as well as taxes would improve the model's improvement", and the study's inconclusive results may attest to the difficulty of implementing empirical work based on this specification rather than the validity of the specification itself. Titman and Torous (1989) provide an application of the capital structure approach to the valuation of commercial mortgages. The authors restrict their analysis to mortgages which are free of prepayment risk and thereby similar to credit risky bonds. With this restriction, default is defined as the moment at which the value of the mortgaged property falls below the value of the mortgage. The authors report that "the magnitude of the observed default premia for a sample of nonprepayable fixed rate bullet mortgages can be explained by the contingent claims approach".

While these results are encouraging, it must be noted that the instruments being analyzed are contracts contingent on the value of commercial properties, not on firm value. Given that it is precisely the difficulty of observing the latter which has prevented comprehensive implementation, successful application in a context in which underlying value is readily observable permits only limited inferences with regard to the validity of the capital structure specification.

Studies based on direct specifications of credit spreads are more numerous. In a study by Ramaswamy and Sundaresan (1986), the authors report that a model based on a square root specification for the credit spread is unable to explain observed default premia on certificates of deposits. Longstaff and Schwartz (1995a) use data on Moody's bond yield indices to establish properties of credit spreads; they report significant negative correlation between interest rates and credit spreads. Duffee (1996) argues that a number of bond indices on which empirical work is based include yields on callable bonds; this and the fact that often credit quality is kept constant for the purpose of construction of some indices means that inference of corporate credit characteristics based on index data is flawed. Using data that takes account of these considerations, he reports that credit spreads are negatively correlated with interest rates and notes that such behaviour could be consistent with business cycle explanations. Duffie and Singleton (1995) employ a multifactor square root specification in analyzing swap spreads.

More recently, a study by Duffee (1996b) examines the pricing of straight corporate bonds based on a reduced form model with a fixed recovery rate and a square root specification for the instantaneous probability of default. Assuming that one bond issued by a specific firm can be observed without measurement error, Duffee uses both cross sectional and time series information to parameterize his specification. His results are sensitive to the choice of the bond which is chosen to be measured without error and concludes that "while the estimated models are moderately successful, there is strong evidence of misspecification".

6.2. Methodology

The current analysis differs from earlier studies both in objective and methodology. Independently of the mechanism generating the contingency of default, every credit risk model generates a credit spread. The second chapter was devoted to the development of a no-arbitrage framework for the pricing of credit contingent claims. The modelling approach chosen there can be characterized as belonging to the category

of reduced form models: Credit spreads are composed of elements relating to the probability of default and recovery rates. Nevertheless, my specification differs from other models using this approach. More precisely, I argue that uncertainty should be attached to the probability of default and make explicit allowance for credit risk and interest rate to be correlated. More fundamentally, it is demonstrated that the absence of any interaction between market risk and credit risk, while admittedly generating convenient separation properties for pricing applications, is not an inherent property of reduced form models, an alleged characterization often adduced in support of such specifications. Specifications belonging to either the capital structure approach or the reduced form approach are thus capable of generating such interaction, as well as uncertainty relating to the evolution of credit risk. The question of whether such properties merit explicit modelling is thus more fundamental than the question of which modelling approach to chose. In addition, while no one would deny that future credit quality is uncertain, there is no consensus as to the merit of formalizing this uncertainty. Duffee (1996b)reports that his "results support the idea that models of corporate bond pricing do not need to incorporate correlations between default free rates and instantaneous default probabilities". Hull and White (1995b) conclude that "a simplifying assumption that is likely to have a great deal of appeal is that the variables concerned with defaults are independent of the variables underlying the value of the derivative security in a no-default world". The objective of the current analysis is not to determine whether the capital structure approach is more adequate than the reduced form approach; of interest here is whether explicit specification of uncertainty in credit risk and its correlation with market risk warrants the sacrifice of parsimony and tractability that such a practice entails.
In contrast to other studies, our methodology places only limited weight on statistical measures of model validity. The metric I propose measures model validity relative to pricing errors generated by different credit risk specifications. While the market for credit contingent claims is still in its infancy and pricing data for such instruments is not available, pricing data for a large class of credit sensitive securities in the form of corporate bonds has long been available. In particular, data is available on both straight and callable coupon debt. While a straight corporate bonds might display limited sensitivity to the specification of credit risk, failure of the call component implicit in callable bonds to respond to more elaborate credit risk specification relative to a simple benchmark model would facilitate more decisive rejection of the need for sophisticated credit modelling.

6.2.1. Specification

Examination of the interaction of credit risk and market risk necessitates inclusion of stochastic credit free rates, for which a one factor Vasicek specification is selected:

$$dx_{g,t} = \kappa_g(\theta_g - x_{g,t})dt + \sigma_g dW_t^g \tag{1}$$

In (1), κ_g is the speed of mean reversion, θ_g is the long rum mean and σ_g is the instantaneous volatility of the risk free factor. No arbitrage arguments in conjunction with (1) lead to the well known bond pricing formula:

$$B_{t,T}^g = \exp\left(a^g(T-t) + b^g(T-t)x_{g,t}\right) \tag{2}$$

where

$$\begin{aligned} a^g(T-t) &= -\theta_g(T-t) - \frac{1}{\kappa_g}e^{-\kappa_g(T-t)}\theta_g + \frac{1}{2\kappa_g^2}\sigma_g^2(T-t) \\ &\quad + \frac{1}{\kappa_g^3}e^{-\kappa_g(T-t)}\sigma_g^2 - \frac{1}{4\kappa_g^3}e^{-2\kappa_g(T-t)}\sigma_g^2 - \frac{1}{4}\frac{3\sigma_g^2 - 4\theta_g\kappa_g^2}{\kappa_g^3} \end{aligned} \tag{3}$$

and

$$b^g(T-t) = -\frac{1}{\kappa_g} + \frac{e^{-\kappa_g(T-t)}}{\kappa_g} \tag{4}$$

My approach to valuing the credit risky corporate bonds is based on the spread approach. I consider three nested specifications.

The benchmark specification is based on the assumption of the credit spread being fixed; the price of a credit risky corporate zero coupon bond is given by:

$$B_{t,T}^d = \exp\left(a^g(T-t) + b^g(T-t)x_{g,t} + (T-t)\overline{S}\right) \tag{5}$$

where $\overline{S}$ is the fixed yield spread arising from the presence of credit risk. This specification will be referred to as model 1.

The second specification introduces uncertainty attached to credit risk by assuming that the instantaneous spread $x_{d,t}$ evolves according to

$$dx_{d,t} = \kappa_d(\theta_d - x_{d,t})dt + \sigma_d dW_t^d \tag{6}$$

The two state variables $x_{g,t}$ and $x_{d,t}$ are assumed to be uncorrelated:

$$d\left\langle W^d, W^g\right\rangle_t = 0 \tag{7}$$

The corresponding credit risky corporate zero coupon bond price is given by:

$$B_{t,T}^d = \prod_{j=g,d} \exp\left(a^j(T-t) + b^j(T-t)x_{j,t}\right) \tag{8}$$

where for $j \in \{g, d\}$

$$\begin{aligned} a^j(T-t) &= -\theta_j(T-t) - \frac{1}{\kappa_j}e^{-\kappa_j(T-t)}\theta_j + \frac{1}{2\kappa_j^2}\sigma_j^2(T-t) \\ &\quad + \frac{1}{\kappa_j^3}e^{-\kappa_j(T-t)}\sigma_j^2 - \frac{1}{4\kappa_j^3}e^{-2\kappa_j(T-t)}\sigma_j^2 - \frac{1}{4}\frac{3\sigma_j^2 - 4\theta_j\kappa_j^2}{\kappa_j^3} \end{aligned} \tag{9}$$

$$b^j(T-t) = -\frac{1}{\kappa_j} + \frac{e^{-\kappa_j(T-t)}}{\kappa_j} \tag{10}$$

The third specification is equivalent to (6) but relaxes (7) to make allowance for the possibility that credit risk is correlated to market risk:

$$d\left\langle W^d, W^g\right\rangle_t = \rho_{dg}dt \tag{11}$$

While closed form expressions for bond prices similar to (8) are available for this specification, I chose to estimate the correlation parameters using the time series properties of the estimated state variables as explained below. Model 2 refers to the stochastic spread specification based on zero correlation between the spread and interest rates, while the equivalent specification without the zero correlation constraint will be referred to as model 3.

In the context of the above specifications, the price on day t of the i-th coupon bond maturing at T with coupon c and face value P is given by

$$P_{t,T}^{d,i} = \sum_{j=1}^{nc(i)} c^j B_{t,T_j}^d \tag{12}$$

where $nc(i)$ is the number of remaining coupon payments on bond i, $c^j = c/2$ for $j = 1...nc(i)-1$ and $c^{nc(i)} = c/2+P$ and B_{t,T_j}^d is given by (5) or (8). The programming code implementing the above specifications can be found in the 'Model Code' section in the code section following this chapter.

Elsewhere in this thesis it was demonstrated that in a framework consistent with the reduced form approach the price of a credit risky zero coupon bond is identical to the price of credit risk free zero coupon bond, the instantaneous rate of return is given by $r_s^l + (1 - e^{-kh_s})\lambda_t^*$ (see chapter 3) :

$$B_{t,T}^{d,c} = E_t^*\left[\exp\left(-\int_t^T (r_s^l + (1 - e^{-kh_s})\lambda_t^*)ds\right)\right] \tag{13}$$

where e^{-kh_s} is the possibly stochastic recovery rate, λ_t^* is the instantaneous probability of default under the risk neutral measure and r_s^l is the risk free rate. Introducing the notation $x_{g,t}$ for the risk free rate and $x_{d,t}$ for the instantaneous spread

$$x_{d,t} = (1 - e^{-kh_s})\lambda_t^* \tag{14}$$

and attaching specifications (1) and (6) to both these variables demonstrates that our specification is consistent with the reduced form approach, the link being provided by (14). Accordingly, the current specification is also consistent with the capital structure approach and an expression similar to (14) relating to the defining components in such a framework could be derived. Ultimately, interest is in establishment of the properties of the components e^{-kh_s} and λ_t^*. Nevertheless, given the multiplicative form of (14) in conjunction with the fact that prices for contract written on only one of those components are not available, it is not possible to estimate both e^{-kh_s} and λ_t^* separately. One possible approach to this problem is to fix one of these components. The analysis in Duffee (1996b) is based on the assumption that historical data can be used to fix e^{-kh_s} and proceeds to use an expression similar to (13) to identify the properties λ_t^*. It must be pointed out that there is no reason why recovery rates should be deterministic while the probability of default is uncertain. Accordingly, one might proceed by assuming that the probability of default is deterministic and use a closed form specification based on (13) to identify the properties of e^{-kh_s}. Given the decision to model credit risk through specification of e^{-kh_s} and λ_t^*, apart from providing welcome simplifications for the purpose of facilitating implementation it is not clear that such restrictions generate better understanding of the uncertainty associated with those components.. As long as no additional information is available, the merit of approaches which do not take account of the possibility that both recovery rates and probability of default are uncertain and vary over time remains unclear. For this reason the present analysis is based on specifications for the credit spread instead of separate specifications for the probability of default process and the recovery rate process.

6.2.2. Data

Data for risk free as well as credit risky instruments is obtained from datastream. For the purpose of parameterizing (1), I employ the datastream coupon strip list.
The need to preserve some homogeneity with respect to credit risk must be balanced with data requirements for the purposes of estimation. It was noted above that the credit risk associated with a credit risky security is likely to contain firm specific as well market related elements, and pooling all available data for purposes of establishing the properties of credit risk is liable to preclude identification of issuer specific variation of these properties. To take account of this variation, albeit not at the level of the issuer, I use both industry membership as well as the credit rating as categorization

variables. A list of corporate bonds issued by financial companies is available from datastream. While such data is available for other industries, establishment of the credit properties of this particular industry is of particular interest in light of the fact that establishment of such properties will facilitate quantification of credit risk in OTC contracts, a substantial portion of which is contracted between financial institutions. The use of a credit rating serves as an additional screening device employed to assure relative homogeneity of credit risk. The imposition of both the credit rating (single A) and the industry (financial institutions) filter significantly reduce the number of data points available.
Both the coupon strip list as well as the financial corporate bond list are available on the following five dates: 05/01/96, 08/01/96, 09/01/96, 10/01/96 and 11/01/96. The corporate list includes the universe of US financial corporate bonds and for the purpose of the current analysis a number of observation must be deselected. Given that I employ the bond's rating as an indicator of credit homogeneity, bonds without a rating are deselected. While both Moody's and Standard & Poor's ratings are available, I rely on the latter's credit classification. Furthermore, any bonds with attached derivative features such as convertibles, equity or bond warrants are deselected. With regard to interest payment type, only bonds which make fixed interest payments are retained; bonds making floating, variable graduated or index-linked payments are thus deselected. Finally, instruments with optional redemption features other than being callable, such as being subject to early redemption or extendible at the bond holder's option, are deselected. This selection results in a distribution of straight and callable rated bonds on the five days in my sample, an example of which for AAA to A rated issues is reported in the following table. This distribution is representative

Rating	Callable	Straight
AAA	0	2
AA	2	1
A	6	10

Table 1 : Number of AAA, AA and A rated corporate financial bonds on 05/01/96.

for the bond list constituency on the remaining days in my sample and illustrates the trade-off between data requirements and the desire to make adequate allowance for firm specific variations in credit risk. Categorizing bonds according to industry membership and rating results in only two AAA rated bonds and one A rated bond to be available for estimation purposes; this is not sufficient. Data for speculative grade issues is equally scarce. On the other hand, the list provides price data for about ten straight A rated bonds on each day in the sample. This information, while limited, should make estimation possible and for the remainder of the analysis we use only data on A rated bonds. According to Standard & Poor's rating definition, "bonds rated A have a strong capacity to pay interest and repay principal, although they are somewhat more susceptible to the adverse effects of changes in circumstances and economic conditions than bonds in higher rated categories". Regarding the restricted set of instruments I have selected, the datastream list provides information on price, duration, redemption yield, coupon rate, coupon dates and redemption dates; for the callable bonds information on call schedules is also available.

6.2.3. Estimation Results

Specification of state variable dynamics impose time series as well as cross sectional restrictions on the behavior of bond prices. Estimation techniques taking account of only a subset of these specifications cannot be considered as taking account of all available information and resulting parameter estimates can be expected to differ from estimates based on techniques taking account of all available information.
The most sophisticated approach to parameter estimation and state variable forecasting are based on filtering methods. This technique is most general in that it allows for all bond prices to be observed with measurement errors, the distribution of which must be specified in order to make the method operational. While such techniques are generating increasing interest, existing evidence is scarce. Implementation for higher-dimensional state variable vectors is non trivial and must be judged as incompatible with the valuation focus (as opposed to estimation focus) of the current analysis.
The second approach taking account of the complete set of restrictions generated by a given state variable specification is the what is frequently referred to as the inversion technique. Specification of state variable dynamics generates time series restrictions through the corresponding conditional transition density of the state variables; cross sectional restriction are generated by the theoretical bond prices based on a given specification for the state variables. In contrast to filtering methods, the inversion technique requires that a number of bonds, corresponding to the number of state variables stipulated, is measured without measurement error. This set of bonds is then inverted, based on the theoretical bond price formula generated by a particular specification and possibly using numerical methods, to yield a set of state variables. While

the technique allows in principle for state variables to be unobservable, in practice the requirement that a set of traded instruments must be observed without measurement error is equivalent to state variables being observable. The inverted state variables are subsequently reinjected into the remaining bonds, assumed to be measured with measurement error, and estimation proceeds based on the minimization of some metric relating to the distance of theoretical and market prices for the set of instruments subject to measurement error. Relative to an estimation strategy based on minimization of distance measures relating to functionals of prices or yields incorporating only cross sectional information, a method taking account of time series restrictions based on the transition density of state variables will result in higher price/yield errors at the resulting parameter estimates. Such error can be further reduced by including a larger number of state variables. In this context it must be noted that imposition of a valuation based metric in evaluating the merit of alternative credit risk specifications requires subsequent implementation of the hypothesized specification. Considerations of hardware limitations limit the number of state variables to two. Given this constraint and the fact that for models 2 and 3, one state variable is reserved for the credit spread, means that only one factor is available to model uncertainty in the risk free term structure. Parameters for the benchmark factor were estimated using a single factor implementation of the inversion technique, based on the dynamics specified in (1). However, pricing errors were unacceptably large and this technique was not further applied.

Instead, parameter estimation for (1), (5) and (6) is based on the respective minimization of sum of squared error criterion using only cross sectional information. Parameterization of model 1, assuming a fixed credit spread, is based on minimization of the objective function

$$\min_{\boldsymbol{\theta}_1} \sum_{i=1}^{ng(t)} \left[B_{t,T}^{m,i} - B_{t,T}^{d,i}(\boldsymbol{\theta}_1)\right]^2 \tag{15}$$

where $\boldsymbol{\theta}_1 = \{x_{g,t}, \kappa_g, \theta_g, \sigma_g, \overline{S}\}$ is the relevant parameter vector, $B_{t,T}^{m,i}$ is the market price for the i-th A-rated straight coupon bond on date t and $B_{t,T}^{d,i}(\boldsymbol{\theta}_1)$ is the model price of the same instrument, based on specification (5). Parameterization of models 2 and 3, assuming a stochastic credit spread, is based on minimization of the objective function

$$\min_{\boldsymbol{\theta}_2} \sum_{i=1}^{ng(t)} \left[B_{t,T}^{m,i} - B_{t,T}^{d,i}(\boldsymbol{\theta}_2)\right]^2 \tag{16}$$

where $\boldsymbol{\theta}_2 = \{x_{g,t}, \kappa_g, \theta_g, \sigma_g, x_{d,t}, \kappa_d, \theta_d, \sigma_d\}$ is the relevant parameter vector, $B_{t,T}^{m,i}$ is the market price for the i-th A-rated straight coupon bond on date t and $B_{t,T}^{d,i}(\boldsymbol{\theta}_2)$ is the model price of the same instrument, based on specification (8). The state variables $x_{g,t}$ and $x_{d,t}$ are thus treated as additional parameters to be estimated.

Given the non-linearity of the theoretical bond prices in the parameters, estimation must proceed using numerical methods. Parameter estimation for the above specifications relied both on one- as well as multidimensional optimization methods.

All one dimensional methods require initial execution of a bracketing routine which determines a triplet of abscissa values bracketing the minimum; this triplet is supplied as an input to a number of minimization routines. The first method employed was the golden section search; this method is based on an efficient partitioning algorithm, slicing up the original bracketing interval until a minimum is found within a given tolerance. The number of steps required to identify a given minimum with this method can be reduced if the objective function is parabolic near the minimum. In this case, inverse parabolic interpolation should identify the minimum with a smaller number of steps. A method combining the robustness of the golden section search with the speed of inverse parabolic interpolation is Brent's method: In regions where the objective function is not well behaved (based on a set of specified criteria), the method applies the golden section method, otherwise the method switches to the faster inverse interpolation scheme. A variant of this methods uses derivative information in determining which half of a current bracketing interval the next test point should be taken from.

The golden section method as well as variants of Brent's method are applicable only to one-dimensional minimization problems. A variety of methods for multidimensional optimization problems have been introduced. All of these methods are based on specification of directions along with minimization algorithms along those directions. So-called direction set methods aim to construct a set of mutually orthogonal directions; if such a set can be identified, n-dimensional optimization reduces to n one dimensional optimizations. Powell's method is based on initializing the directions to unit vectors in each dimension and prescribes an algorithm to orthogonalize the direction set; minimization proceeds by successive minimization in each dimension along a given direction, subject to the requirement that this direction be orthogonal to all other directions associated with dimensions in which minimization has already taken place. Direction set methods only require the objective function values to be calculated, not its derivatives. If the gradient vectors of the objective function is available, such information can be used to initialize the direction set and proceed to create a set of orthogonal directions such that n-dimensional optimization again reduces to n one dimensional optimizations. Methods based on this approach are called conjugate gradient or steepest descent methods. The particular method implemented for the current analysis is the Fletcher-Reeves-Polak-Ribiere (FRPR) algorithm as described in Press et al. (1992). A second class of optimization techniques making use of derivative information is referred to as variable metric or quasi Newton methods. However, whereas steepest descent methods use derivative information to create a set of orthogonal directions, variable metric methods use this information to build up an approximation to the inverse of the objective function's Hessian matrix. In regions where the objective function is well approximated by a quadratic approximation, knowledge of the gradient vector and the Hessian matrix should permit identification of the function minimum in a single step. Using an approximation can be shown to be advantageous relative to the use of the true Hessian, assuming it could be calculated. For the purpose of the current analysis, the Broyden-Fletcher-Goldfarb-Shanno (BFGS) version of this approach was implemented.

Both the conjugate gradient approach as well as variable metric methods assume that second order derivative information for the objective function is not available and thus do not take account of such information. One method which does take account of such information is the Levenberg-Marquardt method, which is based on minimization of

$$\chi^2 = \sum_{i=1}^{ng(t)} \frac{1}{\sigma_i} \left[B_{t,T}^{m,i} - B_{t,T}^{g,i} \right]^2 \tag{17}$$

where $ng(t)$ is the number of coupon strips available on day t, $B_{t,T}^{g,i}$ is the theoretical zero coupon bond price, $B_{t,T}^{m,i}$ is the corresponding market price and σ_i is a parameter reflecting the measurement error in $B_{t,T}^{m,i}$. Note that if σ_i, the volatility of the measurement error of bond i, (or an estimate of it) was known, if the measurement errors were normally distributed and if $B_{t,T}^{m,i}$ was linear in the parameters to be estimated, χ^2 will follow a chi-square distribution with $ng(t) - npar$ degrees of freedom, where $npar$ is the number of parameters to be estimated. In this case, the numerical value of (17) could be used to calculate a goodness-of-fit measure for each of the three models. The corresponding statistics were calculated and, apart for the benchmark model fitted to the treasury coupon strips, indicated rejections of the specifications on a statistical basis. However, none of the three conditions on which the goodness-of-fit tests are based are likely to be met in the present situation. Therefore, only limited relevance is attributed to statistical measures of model fit and the relevant statistics are not reported here. For the purpose of the current analysis, no information relating to the size of measurement errors is available and σ_i is set to unity for all i. Due to the specific form of the objective function, second derivative information is easily calculated. Far from the minimum, the Levenberg-Marquardt method relies on the steepest descent method to get closer to the function minimum; in regions close to the minimum, the method switches to inverse Hessian method (which is similar to variable metric methods but in the Levenberg-Marquardt implementation does not rely on an approximation to the Hessian matrix but uses the true Hessian). The method has the additional advantage of providing standard errors of the parameters as part of its output.

All of the above routines require parameter starting values from which the optimization algorithms can depart. In order to provide such values, three approaches were taken. First, starting values were drawn at random from specified parameter ranges and the algorithm was started at the parameter vector corresponding to the lowest function value. Next, a parameter grid was specified and the objective function was evaluated at the grid nodes; the algorithm was started at the grid node with the lowest function value. Finally, the routines were started from parameter values which seemed economically reasonable. Due to the shape of the objective functions and the characteristics of the data, the first two approaches were not successful in establishing good starting values and it turned out that subjective starting values for all parameters led to the best optimization results; the corresponding starting value used are given in the tables under the respective parameter variables.

Parameters relating to specification (1) were estimated first. Both the Fletcher-Reeves-Polak-Ribiere (FRPR) algorithm, the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method and the Levenberg-Marquardt method were implemented. However, it turned out that the latter method was most robust with respect to variations in starting values and in its ability to find the objective function's minimum; therefore, results for the first two methods are not reported. The relevant programming code can be found in the 'Estimation Code' section in the code section following this chapter. Point estimates for the parameters relating to the risk free factor are given in the following table, with standard errors[29] reported in brackets.

Next, estimation of the parameters relating to different spread specifications was implemented. Again, all of the above methods were implemented but only results relating to the Levenberg-Marquardt method are reported. Estimation of parameters in (5) and (6) is based on minimization of

$$\chi^2 = \sum_{i=1}^{nas(t)} \frac{1}{\sigma_i} \left[P_{t,T}^{m,i} - \sum_{j=1}^{nc(i)} c^j B_{t,T_j}^d \right]^2 \tag{18}$$

where $nas(t)$ is the number of straight A rated coupon bonds on date t, σ_i is set to unity, $nc(i)$ is the number of remaining coupon payments on the particular bond, B_{t,T_j}^d is given by either (5) or (8) and $P_{t,T}^{m,i}$ is the market price of the respective instrument.

29 Standard errors were calculated by taking the inverse of the Hessian of the objective function at the optimal parameter values (for details see Press et al. (1992)).

Date	κ_g 0.2	θ_g 0.06	σ_g 0.0004	x_g 0.05
05/01/96	0.201987 (1.37e−5)	0.069905 (2.8e−4)	0.000251 (2.04e−6)	0.043724 (7.64e−5)
08/01/96	0.20192 (1.37e−5)	0.069923 (2.8e−4)	0.00026 (2.04e−6)	0.043736 (7.64e−5)
09/01/96	0.201495 (1.34e−5)	0.070295 (2.8e−4)	0.000268 (2.02e−6)	0.043635 (7.61e−5)
10/01/96	0.200583 (1.25e−5)	0.071565 (2.8e−4)	0.000266 (2.00e−6)	0.044731 (7.58e−5)
11/01/96	0.200583 (1.25e−5)	0.071659 (2.8e−4)	0.00026 (2.00e−6)	0.043873 (7.53e−5)
Average	0.201366	0.070669	0.000261	0.04394

Table 2 : Cross sectional point estimates of parameters relating to the risk free factor. κ_g is the speed of mean reversion, θ_g is the long rum mean and σ_g is the instantaneous volatility of the risk free factor. Standard errors are reported in brackets.

Estimates of the parameters relating to the risk free factor were injected into (18) and the routine was used to estimate the respective credit parameters; the alternative of estimating both parameter set simultaneously was also tested but estimates were so similar that only results relating to the former approach are reported.

Given estimates of the parameters relating to the risk free component, estimation of $\overline{S}_A$ via minimization of (18) is an exercise in one dimensional optimization. In addition to Fletcher-Reeves-Polak-Ribiere (FRPR), Broyden-Fletcher-Goldfarb-Shanno (BFGS) and Levenberg-Marquardt, the golden section method and Brent's method (with and without the use of derivative information) were implemented. Parameter estimates for all methods were basically identical and only the estimates of $\overline{S}_A$, based on the specification in (5), resulting from application of the Levenberg-Marquardt method are reported in the following table.

table. At around 200 basis points, my estimates of fixed credit spreads on straight A rated corporate financial bonds are high but not unreasonable. Finally, estimates of the spread parameters based on (6) are reported in the following table. Again, all of the optimization methods described above were implemented and results are reported for the Levenberg-Marquardt method. Estimates of the speed of mean reversion are slightly higher than the corresponding numbers for the benchmark factor. Although the long term mean of the spread factor is very high, these numbers need not be unreasonable. On the one hand, they might incorporate the risk of further credit deterioration prior of maturities of the relevant bonds. In addition, while the credit factor $x_{d,t}$ characterizes credit risk over an infinitesimally short time period, high estimates of the long term mean of this variable might capture a cumulative effect over

Date	$\overline{S}_A$ 0.01
05/01/96	0.020115 (2e−8)
08/01/96	0.020305 (2e−8)
09/01/96	0.020025 (2e−8)
10/01/96	0.019229 (2e−8)
11/01/96	0.019573 (2e−8)
Average	0.019849

Table 3 : Cross sectional point estimates of parameters relating to spread factor consistent with the fixed credit spread specification. Standard errors are reported in brackets.

the remaining life of the instruments involved. The negative estimates for the instantaneous credit spread are disturbing. Nevertheless, I believe they are the consequence of quantity constraints on available data and the specified spread dynamics; examination of robustness issues relating to these problems is outside the scope of the current analysis.

Finally, while the correlation parameter could be estimated on the basis of analytical bond price equations[30], I identify a correlation measure using the time series properties of the factor estimates; this estimate is given as $\rho_{dg} = -0.93033162$. Due to the small number of observations (note that state variable estimates on five consecutive days are available), statistical tests of the significance of this value are almost meaningless and such results are not reported here. The sign of the estimate is nevertheless consistent with estimates reported by Titman and Torous (1989), Duffee (1996) and Duffee (1996b), who report significant negative measures of comovement between credit spreads and interest rate levels.

6.2.4. Valuation Results

Parameterization of the credit risk specification is only the first step in the present analysis. I am interested in how different specifications of credit risk affect the valuation of credit sensitive instruments. While valuation of both straight and callable bonds will be affected by how credit risk is modeled, the component which is likely

[30] See the appendix.

Date	κ_d 0.25	θ_d 0.07	σ_d 0.01	x_d 0.03
05/01/96	0.281484 (4.43e−4)	0.05275 (5.04e−4)	0.002086 (1.61e−4)	−0.01797 (9.33e−4)
08/01/96	0.280903 (4.33e−4)	0.05301 (5.03e−3)	0.002081 (1.59e−4)	−0.0176 (9.28e−4)
09/01/96	0.280974 (4.27e−4)	0.053071 (5.02e−3)	0.002083 (1.59e−4)	−0.01773 (9.20e−4)
10/01/96	0.282938 (4.49e−4)	0.052055 (4.98e−3)	0.002021 (1.62e−4)	−0.01957 (9.17e−4)
11/01/96	0.282268 (4.40e−4)	0.052241 (4 98e−3)	0.002078 (1.61e−4)	−0.018663 (9.13e−4)
Average	0.281713	0.052625	0.00207	-0.01831

Table 4 : Cross sectional point estimates of parameters relating to the spread factor consistent with the stochastic credit spread specification. κ_d is the speed of mean reversion, θ_d is the long rum mean and σ_d is the instantaneous volatility of the spread factor. Standard errors are reported in brackets.

to display the highest sensitivity to credit risk specification is the call option embedded in the callable bonds. I will rank different specification according to their ability to price these embedded options. Given that price quotes on callable bonds are relate to the callable component of the contract and separate prices for the embedded straight bond and the embedded call option are not given, I derive a measure of the market price of the embedded call option.

A measure of the market price for the embedded call option is based on the following algorithm:

1.) Using the two risk free zero coupon yields whose maturities bracket the duration of the specific callable bond under consideration, I interpolate the zero coupon rate at the maturity corresponding to that duration. Before interpolation is conducted, I use the `sort2` routine from Press et al. (1992), p.334, to line up the bonds in increasing order of maturity. Subsequently, polynomial interpolation of different degrees was tested; given that results were very similar for different orders of interpolation, results from linear interpolation were employed for the remaining analysis.

2.) Using the two redemption yields on straight single A bonds whose durations bracket the duration of the specific callable bond under consideration, I interpolate the redemption yield at the maturity corresponding to that duration.

3.) The interpolated zero coupon yield is deducted from the interpolated redemption yield to arrive at a market based redemption yield spread corresponding to the duration of the corresponding callable corporate bond.

4.) Using the two risk free zero coupon yields whose maturities bracket the maturity of each coupon payment on a given callable bond, I interpolate the zero coupon rate at the maturity corresponding to that coupon payment. Adding to the zero coupon rate the market based spread as calculated above I generate a market based term structure of A rated zero coupon yields.

5.) A market based valuation of the straight component of the callable bond is given as the sum of coupons and principal on this instrument, discounted at the market based term structure of A rated zero coupon yields derived above.

6.) The market based valuation of the embedded call option is given as the difference between the derived market price of the straight component and the quoted market price of the callable component.

This market price will then be compared with model prices consistent with the three credit spread specifications. Model prices were calculated as follows:

1.) For a given spread model, the callable bond, the straight bond and the embedded call were set to their respective boundary condition at the maturity date.

2.) Iterating backwards through the tree, at each node it is checked whether the current date is an exercise date or not. If the date is not an exercise date, the value of the securities is set to their discounted value at the preceding node. Otherwise, the callable bond is set to the minimum of its market price or the exercise price and the value of the call is set to the difference between the value of the straight bond and the value of the callable bond.

The above procedure assumes that the issuer seeks to minimize his outstanding liabilities and that this is achieved by calling the whole issue when the market price falls below the exercise price; while the issuer's liability may be minimized by a strategy under which not all outstanding bonds are called at the same time, the current analysis abstracts from such issues.

Implementation of the valuation procedures can be found in the 'Pricing Code' section in the code section following this chapter.

Generation of model prices necessitates implementation of the three specifications for the credit spread using the parameter estimates as identified above. While other approaches are possible[31], I choose to implement the models using trinomial trees; this technique facilitates inclusion of mean reversion and correlation structure. In simple terms, the method is based on the discretization of the stipulated state variable dy-

[31] If an asset has a unique price, it can be shown that the following are equivalent (see Karatzas and Shreve (1989)):
-The price of the assets satifies a certain partial differential equation.
-The price of the asset is given by the expectation, under a certain measure, of its discounted payoffs. Implementations based on the former approach focus on finite difference schemes for the solutions of partial differential equations, whereas the latter approach is based on Monte Carlo simulation or lattice (i.e. tree) methods.

namics; given step sizes for time and state space increments, the method generates transition probabilities based on restrictions equating the moments of the continuous time specification and its discrete equivalent. Allowing for nonstandard branching, these restrictions, based on variable dynamics consistent with (1) and (6) are summarized as follows:

$$\mathbf{\Pi}_1^l \mathbf{\Pi}_2^l = \mathbf{\Pi}_3^l \tag{19}$$

$$\mathbf{\Pi}_1^l = \begin{pmatrix} 1 & 1 & 1 \\ (k+1-j)\triangle x_l & (k-j)\triangle x_l & (k-1-j)\triangle x_l \\ (k+1-j)^2\triangle x_l^2 & (k-j)^2\triangle x_l^2 & (k-1-j)^2\triangle x_l^2 \end{pmatrix}$$

$$\mathbf{\Pi}_2^l = \begin{pmatrix} pu_l(i,j) \\ pm_l(i,j) \\ pd_l(i,j) \end{pmatrix}$$

$$\mathbf{\Pi}_3^l = \begin{pmatrix} 1 \\ \kappa_l(\theta_l - x_l(i,j))\triangle t \\ \sigma_l^2\triangle t + \kappa_l^2(\theta_l - x_l(i,j))^2\triangle t^2 \end{pmatrix}$$

where j is the current node index, k is the middle node at the next time step and $l \in \{g, d\}$. Allowing for the possibility that $j \neq k$ is necessary in order to accommodate mean reversion. $x_l(i,j)$ is the value of $x_l, l \in \{g,d\}$, at the grid point (i,j): $x_l(i,j) = x_l(0,0) + j\triangle x$. Given the presence of mean reversion, no additional restrictions need to be placed on $x_l(i,j)$; in the absence of mean reversion, boundaries for the state variables and their respective behavior at those boundaries (e.g. reflection) should be specified. The first equation in (19) embodies the condition that probabilities must sum to unity; the second and third equation equate the first and second moments of the continuous specification and its discrete equivalent. Solving the above system yields the transition probabilities as functions of model parameters and time and state space increments:

$$\mathbf{\Pi}_2^l = \begin{pmatrix} \frac{((j-k)\triangle x + \kappa_l(\theta_l - x_l(i,j))\triangle t)^2 + \sigma_l^2\triangle t^2}{2\triangle x^2} + \frac{(j-k)\triangle x + \kappa_l(\theta_l - x_l(i,j))\triangle t}{2\triangle x} \\ \frac{((j-k)\triangle x + \kappa_l(\theta_l - x_l(i,j))\triangle t)^2 + \sigma_l^2\triangle t^2}{2\triangle x^2} - \frac{(j-k)\triangle x + \kappa_l(\theta_l - x_l(i,j))\triangle t}{2\triangle x} \\ \frac{\triangle x - ((j-k)\triangle x + \kappa_l(\theta_l - x_l(i,j))\triangle t)^2 - \sigma_l^2\triangle t^2}{\triangle x^2} \end{pmatrix} \tag{20}$$

where $l \in \{g, d\}$. At nodes where the branching is standard, $j = k$ and the transition probabilities are given by the following set of equations:

$$pu_l(i,j) = \frac{(\kappa_l(\theta_l - x_l(i,j))\triangle t)^2 + \sigma_l^2\triangle t^2}{2\triangle x^2} + \frac{\kappa_l(\theta_l - x_l(i,j))\triangle t}{2\triangle x} \tag{21}$$

$$pm_l(i,j) = \frac{(\kappa_l(\theta_l - x_l(i,j))\triangle t)^2 + \sigma_l^2\triangle t^2}{2\triangle x^2} - \frac{\kappa_l(\theta_l - x_l(i,j))\triangle t}{2\triangle x} \tag{22}$$

$$pd_l(i,j) = \frac{\Delta x - (\kappa_l(\theta_l - x_l(i,j))\Delta t)^2 - \sigma_l^2 \Delta t^2}{\Delta x^2} \tag{23}$$

For purposes of implementation, $\Delta x_l = 2\sigma\sqrt{\Delta t/3}$[32] and $\Delta t = 0.5$, that is, time increments are semiannual; time and state space increments are assumed constant across nodes and are therefore not indexed. Based on this implementation of state variable dynamics I generate three sets of model prices for the embedded call options. The benchmark specification assumes that the credit spread is fixed. Using parameter estimates for the interest rate process and the fixed credit spread $\overline{S}_A$ in conjunction with (20), I generate a tree for credit risky discount rates. Implementation of this procedure can be found in the 'Tree Code' section in the code section following this chapter. Using (3), (4), (5) and the available data on call schedules, backward recursion through the tree permits valuation of the embedded option component. The programming code relating to this step can be found in the 'Pricing Code' section in the code section following this chapter.

The second specification is based on the assumption that credit risk is stochastic but uncorrelated with interest rates; independence between the two processes permits implementation using the relevant parameter estimates and (20) for each process separately. Potentially each node in the combined tree generates nine new nodes; probabilities of reaching these nodes are given as products of relevant probabilities in the constituent trees:

$$puu(i,j) = pu_g(i,j) \times pu_d(i,j)$$

$$pum(i,j) = pu_g(i,j) \times pm_d(i,j)$$

$$pud(i,j) = pu_g(i,j) \times pd_d(i,j)$$

$$pmu(i,j) = pm_g(i,j) \times pu_d(i,j)$$

$$pmm(i,j) = pm_g(i,j) \times pm_d(i,j)$$

$$pmd(i,j) = pm_g(i,j) \times pd_d(i,j)$$

$$pdu(i,j) = pd_g(i,j) \times pu_d(i,j)$$

$$pdm(i,j) = pd_g(i,j) \times pm_d(i,j)$$

$$pdd(i,j) = pd_g(i,j) \times pd_d(i,j)$$

Using (3), (4), (8), (9) and (10), I generate model values for the embedded call option associated with a stochastic credit spread specification.

[32] The requirement that transition probabilities remain positive and bounded above by unity generates the following constraint for the state variable invrements: $2\sigma_l\sqrt{\Delta t/3} \leq \Delta x_l \leq 2\sigma_l\sqrt{\Delta t}$. Subsequent valuation results are robust to varying Δx_l as long as the above constraint is satisfied.

Finally, I generate the model prices corresponding to specification of a stochastic credit spread and non-zero correlation between credit spreads and interest rates. Implementation with nonzero correlation between the constituent trees requires adjustments as derived in Hull and White (1994a)and Hull and White (1994b)

$$puu(i,j) = pu_g(i,j) \times pu_d(i,j) + I_{\{\rho_{gd}>0\}} 5\varepsilon - I_{\{\rho_{gd}<0\}} \varepsilon$$

$$pum(i,j) = pu_g(i,j) \times pm_d(i,j) - 4\varepsilon$$

$$pud(i,j) = pu_g(i,j) \times pd_d(i,j) - I_{\{\rho_{gd}>0\}} \varepsilon + I_{\{\rho_{gd}<0\}} 5\varepsilon$$

$$pmu(i,j) = pm_g(i,j) \times pu_d(i,j) - 4\varepsilon$$

$$pmm(i,j) = pm_g(i,j) \times pm_d(i,j) + 8\varepsilon$$

$$pmd(i,j) = pm_g(i,j) \times pd_d(i,j) - 4\varepsilon$$

$$pdu(i,j) = pd_g(i,j) \times pu_d(i,j) - I_{\{\rho_{gd}>0\}} \varepsilon + I_{\{\rho_{gd}<0\}} 5\varepsilon$$

$$pdm(i,j) = pd_g(i,j) \times pm_d(i,j) - 4\varepsilon$$

$$pdd(i,j) = pd_g(i,j) \times pd_d(i,j) + I_{\{\rho_{gd}>0\}} 5\varepsilon - I_{\{\rho_{gd}<0\}} \varepsilon$$

where $\varepsilon = \rho_{gd}/36$. The above adjustments do not change the mean and standard deviation of x_g and x_d but allow for the correlation ρ_{gd} to be incorporated into state transitions. The programming code relating to the construction of the trees and the pricing of the calls, respectively, can be found in the 'Tree Code' and 'Pricing Code' section in the code section following this chapter.

The valuation metric I have proposed judges different credit risk specifications according to their ability to price call options embedded in callable bonds. The following table reports the mean absolute pricing errors for the embedded options, defined as the difference of the market based and model based call prices, across bonds and trading dates. Mean absolute pricing errors are large but not unreasonable, given

e^a_{sc}	e^a_{ss}	e^a_{cor}
17.412	10.335	5.566

Table 5 : Mean absolute pricing errors for embedded call options. Subscripts refer to credit risk specification; *sc* denotes the constant spread model, *ss* denotes the stochastic spread model with zero correlation and *cor* denotes the stochastic spread model with non zero correlation.

the complexity of the instrument under consideration. More significant is the sensitivity of the pricing errors to the introduction of sophisticated credit risk modelling features. Relative to a fixed credit spread specification, introduction of uncertainty reduces the mean absolute pricing errors by around 40%. Incorporation of the correlation between the credit spread and risk free rates reduces this error by another 45%.

A very similar pattern emerges for the mean relative pricing errors, defined as the mean of the individual absolute pricing errors normalized by the market based call prices, summarized in the following table. . Relative to a fixed credit spread specification, introduction of uncertainty reduces relative pricing errors by around 70%. Incorporation of correlation between the credit spread and risk free rates leads to a further decrease of mean relative pricing error of 23%.

e^r_{sc}	e^r_{ss}	e^r_{cor}
1.38	0.694	0.458

Table 6 : Mean relative pricing errors for embedded call options. Subscripts refer to credit risk specification; *sc* denotes the constant spread model, *ss* denotes the stochastic spread model with zero correlation and *cor* denotes the stochastic spread model with non zero correlation.

Conclusion

The above analysis is subject to a number of problems. Identification problems preclude characterization of individual credit components. Data constraints are severe. The estimation technique used is simple. Generation of market based valuations for the embedded call options is imprecise. Pricing errors are large. I conjecture that application of more sophisticated estimation techniques might lead to some improvements of these shortcomings. Availability of bigger datasets is certain to generate more precise insights than the ones delivered here. However, it is precisely due to some of these problems that researcher have shied away from empirical analysis relating to credit risk. Indeed, most papers proposing new credit risk models contain statements to the effect of declaring that "subject to reasonable parameter values, the new model is capable of replicating empirical spreads." While such statements are of interest in their own right, they must not be mistaken for being based on any real empirical analysis.

Very little work has been done in terms of econometric analysis of credit models. As far as has been done, particular models are rejected on the basis of statistical measures of fit. The present approach is not directed at examining empirical features of credit risk models from a purely econometric or statistical perspective. Given the data constraints and the lack of consensus as to which model is the correct one, I adopt a valuation based metric in order to identify technical features in the credit risk specification which improve the pricing of specific credit contingent claims. As such, the current analysis is unique in the recent credit literature (Litterman and Iben (1991) is a slight exception).

Despite the fact that a number of simplifying assumptions has been made and despite the fact that the econometric technique is simple, the above analysis provides empirical evidence for a refutable proposition. Uncertainty in credit risk is important not only from a conceptual point of view. Correlation between market risk and credit risk matters. Failure to take account of these complexities will prove costly for financial institutions having incurred credit risk exposure because they will misprice the risks they are facing.

Appendix

PDE's for coefficient functions in two factor bond price specification

Assume that credit risky bond prices are functions of e risk free factor $x_{g,t}$ and a credit factor $x_{g,t}$, whose evolution is given by

$$dx_{g,t} = \kappa_g(\theta_g - x_{g,t})dt + \sigma_g dW_t^g \tag{24}$$

$$dx_{d,t} = \kappa_d(\theta_d - x_{d,t})dt + \sigma_d dW_t^d \tag{25}$$

and allowance is made for both processes to be correlated:

$$d\left\langle W^d, W^g \right\rangle_t = \rho_{dg} dt \tag{26}$$

The assumption that bond prices are linear exponential in the state variables:

$$B_{t,T}^g = \exp\left(a(T-t) + b^g(T-t)x_{g,t} + b^d(T-t)x_{d,t}\right) \tag{27}$$

coupled with hypothesis that the expected return on assets is given by the risk free rate $x_{g,t}$

$$E\left[\frac{dB_{t,T}^g}{B_{t,T}^g}\right] = x_{g,t}dt \tag{28}$$

yields the following set of equations:

$$\begin{gathered} 1 + \frac{\partial b^j(u)}{\partial u} + \kappa b^j(u) = 0 \\ b^j(0) = 0 \end{gathered} \tag{29}$$

$$\begin{gathered} 0 = -\frac{\partial a^j(u)}{\partial u} + \left(\frac{e^{-\kappa_g x}}{\kappa_g} - \frac{1}{\kappa_g}\right)\kappa_g\theta_g + \left(\frac{e^{\ \kappa_g x}}{\kappa_g} - \frac{1}{\kappa_d}\right)\kappa_d\theta_d + \\ \frac{1}{2}\left(\sigma_g^2\left(\frac{e^{\ \kappa_g x}}{\kappa_g} - \frac{1}{\kappa_g}\right)^2 + 2\sigma_g\sigma_d\rho_{dg}\left(\frac{e^{\ \kappa_g x}}{\kappa_d} - \frac{1}{\kappa_g}\right)\left(\frac{e^{\ \kappa_d x}}{\kappa_d} - \frac{1}{\kappa_d}\right) + \sigma_d^2\left(\frac{e^{\ \kappa_d x}}{\kappa_d} - \frac{1}{\kappa_d}\right)^2\right) \\ a(0) = 0 \end{gathered} \tag{30}$$

Solutions to (29) and (30) yield closed form expression for bond prices.

References

Altman, E. (1985): "The success of business failure prediction models," *Journal of Banking and Finance* , 8, 171-198.

Altman, E. (1989): "Measuring corporate bond mortality and performance," *Journal of Finance* (September), 64/4, 909-922.

Altman, E. (1990): "Setting the record straight on junk bonds: A review of the research on default rates and returns," *Journal of Applied Corporate Finance* , 3, 82-95.

Altman, E. (1990): "Measuring corporate bond mortality and performance," *Journal of Finance* (September), 64/4, 909-922.

Altman, E. (1992): "Revisiting the high yield bond market," *Financial Management* , 21/2, 78-92.

Altman, E. (1992): "The implications of corporate bond rating drift," *Financial Analysts Journal* (May-June), 78-92.

Altman, R. and V.Kishore (1996): "Almost everything you wanted to know about recoveries on defaulted bonds," *Financial Analysts Journal* (November-December), 57-64.

Amin, K. (1991): "On the computation of continuous time option prices using discrete time approximations," *Journal of Financial and Quantitative Analysis,* 26, 477-496.

Amin, K. and J. Bodurtha (1995): "Discrete-time valuation of American options with stochastic interest rates," *The Review of Financial Studies,* 8/1, 193-234.

Amin, K. and A. Morton (1994): "Implied volatility functions in arbitrage-free term structure models," *Journal of Financial Economics,* 35, 141-180.

Asquith, P., D. Mullins and E.Wolff (1989): "Original issue high yield bonds: Ageing analysis of defaults, exchanges and calls," *Journal of Finance,* 923-947.

Balduzzi, P., S. Das, S. Foresi and R. Sundaresan (1996): "A simple approach to three factor affine term structure models," *The Journal of Fixed Income,* 43-53.

Barraquand, J. (1995): "Numerical valuation of high dimensional multivariate European securities," *Management Science* (December), 41/12, 1882-1891.

Barraquand, J and D. Martineau (1995): "Numerical valuation of high dimensional multivariate American securities," *Journal of Financial and Quantitative Analysis,* 30/3, 383-405.

Barraquand, J and T. Pudet (1996): "Pricing of American path-dependent contingent claims," *Mathematical Finance,* 6/1, 17-51.

Black, F. (1995): "Interest rates as options," *The Journal of Finance,* 50/7, 1371-1376.

Black, F. and P. Karasinski (1991): "Bond and option pricing when short rates are lognormal," *Financial Analysts Journal* (Juli-August), 52-59.

Boyle, P. (1977): "Options: A Monte Carlo approach," *Journal of Financial Economics,* 4, 323-338.

Boyle, P. (1988): "A lattice framework for option pricing with two state variable," *Journal of Financial and Quantitative Analysis,* 23/1, 1-12.

Boyle, P., J Evnine and J. Gibbs (1989): "Numerical evaluation of multivariate contingent claims," *The Review of Financial Studies,* 2/2, 241-250.

Brotherton-Ratcliffe, R. (1994): "Monte Carlo motoring," *Risk*, 7/12.

Brown, S. and P. Dybvig (1986): "The empirical implications of the Cox-Ingersoll-Ross theory of the term structure of interest rates," *Journal of Finance*, 41/3, 617-632.

Brown, R. and S. Schaefer (1994): "The term structure of real interest rates and the Cox-Ingersoll-Ross model," *Journal of Financial Economics*, 35, 3-42.

Buttimer, R., W. Muller and R. Reeves (1995): "An alternative trinomial formulation for one-factor term structure models," *The Journal of Financial Engineering*, 4/1, 1-10.

Cantor, R. and F. Packer (1995): "The credit rating industry," *The Journal of Fixed Income* (December), 10-34.

Carverhill, A. (1995): "A note on the models of Hull and White for pricing options on the term structure," *The Journal of Fixed Income* (September), 89-96.

Carverhill, A. and C. Strickland (1992): "Money market term structure dynamics and volatility expectations," *Working Paper.*

Carverhill, A. and K. Pang (1995): "Efficient and flexible bond option valuation in the Heath, Jarrow, and Morton framework," *The Journal of Fixed Income* (September), 70-77.

Clewlow, L. and A. Carverhill (1994): "On the simulation of contingent claims," *The Journal of Derivatives* (Winter), 66-74.

Chan, K. and G. Karolyi and F. Longstaff and A. Sanders (1992): "An empirical comparison of alternative models of the short term interest rate," *Journal of Finance*, 47/3, 1209-1227.

Chen, N. (1991): "Financial investment opportunities and the macroeconomy," *The Journal of Finance* , 46, 529-554.

Chen, R. and L. Scott (1993): "Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates," *The Journal of Fixed Income* (December), 14-30.

Chen, R. and L. Scott (1994): "Multi-factor Cox-Ingersoll-Ross models of the term structure: Estimates and tests from a Kalman filter model," *Working Paper.*

Cheyette, O. (1992): "Term structure dynamics and mortgage valuation," *The Journal of Fixed Income* (March), 28-41.

Coleman, T., L. Fisher and R. Ibbotsen (1992): "Estimating the term structure of interest rates from data that include the prices of coupon bonds," *The Journal of Fixed Income* (September), 85-116.

Curran, M. (1995): "Accelerating American option pricing in lattices," *The Journal of Derivatives* (Winter), 8-18.

Curran, M. (1996): "Recovering identity," *Risk*, 9/5.

Duan, J. (1994): "Maximum likelihood estimation using price data of the derivative contract," *Mathematical Finance*, 4/2, 155-167.

Duan, J. and J. Simonato (1995): "Estimating and testing exponential-affine term structure models by Kalman filter," *Working Paper.*

Duffee, G. (1996a): "Treasury yields and corporate bond yield spreads: An empirical analysis," *Working Paper.*

Duffee, G. (1996b): "Estimating the price of default risk," *Working Paper.*

Duffie, D. and P. Glynn (1995): "Efficient monte carlo simulation of security prices," *The Annals of Applied Probability*, 5/4, 897-905.

Duffie, D. and K. Singleton (1995): "An econometric model of the term structure of interest rate swap yields," *Working Paper.*

Fisher, M., D. Nychka and D. Zervos (1994): "Fitting the term structure of interest rates with smoothing splines," *Working Paper.*

Flesaker, B. (1993): "Testing the Heath-Jarrow-Morton/Ho-Lee model of interest rate contingent claims," *Journal of Financial and Quantitative Analysis*, 28/4, 483-495.

Fons, J. (1994): "Using default rates to model the term structure of credit risk," *Financial Analysts Journal* (September-October), 25-32.

Fons, J. and A. Kimball (1991): "Corporate bond defaults and default rates 1970-1990," *The Journal of Fixed Income* (June), 36-47.

Fridson, M. and J. Jonsson (1995): "Spread versus treasuries and the riskiness of high-yield bonds," *The Journal of Fixed Income* (December), 79-88.

Geyer, A. and S. Pichler (1995): "A state-space approach to estimate the term structure of interest rates: Some empirical evidence," *Working Paper.*

Gibbons, M. and K. Ramaswamy (1993): "A test of the Cox, Ingersoll, and Ross model of the term structure," *The Review of Financial Studies*, 6/3, 619-658.

Hamilton, J. (1994): *Time Series Analysis* Princeton University Press.

Hansen, L. and J. Scheinkman (1995): "Back to the future: Generating moment implications for continous-time Markov processes," *Econometrica*, 63, 4, 767-804.

He, H. (1990): "Convergence from discrete to continous time contingent claims prices," *The Review of Financial Studies*, 3/4, 523-546.

Heath, D., R. Jarrow, A. Morton and M. Spindel (1992): "Easier done than said," Risk, 5/9, 77-80.

Heston, S. (1995): "Discrete-time versions of continous-time interest rate models," *The Journal of Fixed Income* (September), 86-88.

Hull, J. and A. White (1989): "Coming to terms," *Risk*, 3/1.

Hull, J. and A. White (1990a): "Root and branch," *Risk*, 3/8.

Hull, J. and A. White (1990b): "Valuing derivative securities using the explicit finite difference method," *Journal of Financial and Quantitative Analysis*, 25, 87-100.

Hull, J. and A. White (1993): "One-factor interest rate models and the valuation of interest rate derivative securities," *Journal of Financial and Quantitative Analysis*, 28/2, 235-254.

Hull, J. and A. White (1994a): "Numerical procedures for implementing term structure models I: Single factor models," *The Journal of Derivatives* (Fall), 7-16.

Hull, J. and A. White (1994b): "Numerical procedures for implementing term structure models II: Two-factor models," *The Journal of Derivatives* (Winter), 37-48.

Hull, J. and A. White (1995a): "A note on the models of Hull and White for pricing options on the term structure: Response," *The Journal of Fixed Income* (September), 97-102.

Hull, J. and A. White (1995b): "The impact of default risk on the prices of options and other derivative securities," *Journal of Banking and Finance,* 19, 299-322.

Hull, J. and A. White (1996): "Using Hull-White interest rate trees," *The Journal of Derivatives* (Spring), 26-36.

Jamshidian, F. (1991): "Forward induction and construction of yield curve diffusion models," *The Journal of Fixed Income* (June), 62-74.

Jones, E., S. Mason and E. Rosenfeld (1984): "Contingent claims analysis of corporate capital structures: An empirical investigation," *The Journal of Finance*, 39/3, 611-627.

Knez, P., R. Litterman and J. Scheinkman (1994): "Explorations into factors explaining money market returns," *The Journal of Finance*, 49/5, 1861-1882.

Kopprash, R. et al. (1985): "Effective duration of callable bonds," *Working Paper.*

Lim, K. and M. Livingston (1995): "Stripping of treasury securities and segmentation in the treasury securities market," *The Journal of Fixed Income* (March), 88-94.

Litterman, R. and T. Iben (1991): "Corporate bond valuation and the term structure of credit spreads," *Financial Analysts Journal* (Spring), 52-64.

Litterman, R. and J. Scheinkman (1991): "Common factors affecting bond returns," *The Journal of Fixed Income* (June), 54-61.

Litterman, R., J. Scheinkman and L. Weiss (1991): "Volatility and the yield curve," *The Journal of Fixed Income* (June), 49-53.

Longstaff, F. and E. Schwartz (1993): "Implementation of the Longstaff-Schwartz interest rate model," *The Journal of Fixed Income* (September), 7-14.

Longstaff, F. and E. Schwartz (1995): "A simple approach to valuing risky fixed and floating rate debt," *The Journal of Finance*, 50/3, 789-819.

Koenigsberg, M., J. Showers and J. Street (1991): "The term structure of volatility and bond option valuation," *The Journal of Fixed Income* (September), 19-35.

Madan, D., F. Milne and H. Shefrin (1989): "The multinomial option pricing model and its Brownian and Poisson limits," *The Review of Financial Studies*, 2/2, 251-265.

Moro, B. (1995): "The full monte," *Risk*, 8/2.

Nawalkha, S. (1995): "The duration vector: A continous time extension to default free interest rate contingent claims," *Journal of Banking and Finance*, 19, 1359-1378.

Nelson, D. and K. Ramaswamy (1990): "Simple binomial processes as diffusion approximations in financial models," *Review of Financial Studies*, 3/3, 393-430.

Papageorgiou, A. and J. Traub (1996): "Beating Monte Carlo," *Risk*, 9/6.

Pearson, N. and T. Sun (1994): "Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll, and Ross model," *The Journal of Finance*, 49/4, 1279-1304.

Pennachi, G. (1991): "Identifying the dynamics of real interest rates and inflation: Evidence using survey data", *Review of Financial Studies*, 4/1, 53-86.

Press, W., S. Teukolsky, W. Vetterling and B. Flannery (1992): *Numerical Recipes in C* Cambridge University Press, Cambridge.

Ramaswamy, K. and S. Sundaresan (1986): "Valuation of floating rate instruments," *Journal of Financial Economics*, 17, 251-272.

Ritchken, P. and L. Sankarasubramanian (1995): "Lattice models for pricing American interest rate claims," *The Journal of Finance*, 50/2, 719-737.

Rubinstein, M. (1994): "Return to Oz," *Risk*, 7/11.

Schaefer, S. (1977): "The problem with redemption yields," *Financial Analysts Journal* (July-August), 59-67.

Shea, G. (1984): "Pitfalls in smoothing interest rate term structure data: Equilibrium models and spline approximations," *Journal of Financial and Quantitative Analysis*, 19/3, 253-268.

Singh, M. (1995): "Estimation of multifactor Cox, Ingersoll, and Ross model: Evidence on volatility structure and parameter stability," *The Journal of Fixed Income* (September), 8-28.

Titman, S. and W. Torous (1989): "Valuing commercial mortgages: An empirical investigation of the contingent-claims approach to pricing risky debt," *The Journal of Finance*, 44/2, 345-373.

Vasicek, O. and H. Fong (1982): "Term structure modeling using exponential splines," *The Journal of Finance*, 37/2, 339-325.

Weinstein, M. (1987): "A curmudgeon's view on junk bonds," *The Journal of Portfolio Management* (Spring), 76-80.

Code

Main Code

```
#include <stdio.h> #include <stdlib.h>
#include <stddef.h>
#include <math.h>
#include <malloc.h>
#include <string.h>
#include <errno.h>

#include ''h:\codelib\c-lib\nrutil.h''
#include ''h:\codelib\c-lib\locate.c''
#include ''h:\codelib\c-lib\polint.c''
#include ''h:\codelib\c-lib\covsrt.c''
#include ''h:\codelib\c-lib\gaussj.c''
#include ''h:\codelib\c-lib\mrqcof.c''
#include ''h:\codelib\c-lib\mrqmin.c''
#include ''h:\codelib\c-lib\gammln.c''
#include ''h:\codelib\c-lib\betacf.c''
#include ''h:\codelib\c-lib\betai.c''
#include ''h:\codelib\c-lib\pearsn.c''
#include ''h:\codelib\c-lib\moment.c''
#include ''h:\codelib\c-lib\indexx.c''
#include ''h:\codelib\c-lib\sort2.c''
#define MAXDAT 5
#define MAXCO 60
#define MAXCA 40
#define INTORD 2
#define MLTOL 10e-6
int t, ndat, *ng, *nas, *nac;
double **pg, **pd, **pdc, rho;

typedef struct gbonds {
 double *p,*mat,*ry;
```

```
} GBOND;
typedef struct Asbonds {
 long int*id;
 int *nco;
 double *ry,*s,*dur,*mat,*co,**com,*p_s_m,*p_s_ss,*p_s_sc;
} AS;

typedef struct Acbonds {
 long int*id;
 int *nco,*nca;
 double *ry,*s,*dur,*mat,*co,**com,**cax,**cam,*p_s_m,*p_s_sc,*p_s_ss,*p_s_cor;
 double *p_c_m,*p_c_sc,*p_c_ss,*p_c_cor,*p_ca_m,*p_ca_sc,*p_ca_ss,*p_ca_cor;
 double *ea_ca_sc,*ea_ca_ss,*ea_ca_cor,*er_ca_sc,*er_ca_ss,*er_ca_cor;
} AC;
typedef struct grate {
 int ns,js,*mid,*midi;
 double inc,*r,*pu,*pm,*pd;
} RG;

typedef struct dspread {
 int ns,js,*mid,*midi;
 double inc,*s,*pu,*pm,*pd;
} SS;
GBOND **gb;AS **as;AC **ac;RG **rg;SS **ss;
FILE *str_in, *co_in, *out_01, *out_02;
#include ''h:\codelib\code\vas01.c''
#include ''h:\codelib\code\estvas01.c''
#include ''h:\codelib\code\trevas01.c''
#include ''h:\codelib\code\privas01.c''

void main( void )
{
 int tnco,*ip4,*ip1,i,j,iter,*ncall,*naaas,*naas,*naaac,*naac;
 long int tid;
 double *x, chilast, chisq, chistart, alamda;
 double *sig, **al4, *x2, *x1, prob, z, **cov, **cov_ss, **cov_sc;
 double ave1,adev1,sdev1,var1,skew1,curt1,ave2,adev2,sdev2,var2,skew2,curt2;
```

```
double tco, tp, tmat, tdur, try, yg,ys,yc, tcam, tcax;
double mmae_ca_sc,*mae_ca_sc,mmre_ca_sc,*mre_ca_sc,mmae_ca_ss,*mae_ca_ss;

double mmre_ca_ss, *mre_ca_ss,mmae_ca_cor,*mae_ca_cor,mmre_ca_cor,*mre_ca_cor;

char inpt[10], sprt[10], drf[10], oprf[10], temp[20], **date;

ng=ivector(1,MAXDAT);naaas=ivector(1,MAXDAT);naas=ivector(1,MAXDAT);nas=ivector(1,MAXDA
naaac=ivector(1,MAXDAT);naac=ivector(1,MAXDAT);nac=ivector(1,MAXDAT);ncall=ivector(1,MA
ip4=ivector(1,4);ip1=ivector(1,1);

x1 = dvector(1, MAXDAT);x2 = dvector(1,MAXDAT);mae_ca_sc=dvector(1,MAXDAT);
mre_ca_sc=dvector(1,MAXDAT);mae_ca_ss=dvector(1,MAXDAT);mre_ca_ss=dvector(1,MAXDAT);
mae_ca_cor=dvector(1,MAXDAT);mre_ca_cor=dvector(1,MAXDAT);
al4=dmatrix(1,4,1,4);cov=dmatrix(1,4,1,4);cov_ss=dmatrix(1,4,1,4);cov_sc=dmatrix(1,1,1,


if((str_in=fopen(''h:\\data\\d5_str.txt'',''r''))==NULL)perror(''strip
error'');
if((co_in=fopen(''h:\\data\\bond_d5.txt'',''r''))==NULL)perror(''coupon
error'');
if((out_01=fopen(''h:\\data\\cout01.txt'',''a''))==NULL)perror(''out01
error'');
if((out_02=fopen(''h:\\data\\cout02.txt'',''a''))==NULL)perror(''out02
error'');

if(fseek(str_in,0L,SEEK_SET))perror(''Fseek failed on strip file'');
else if(fseek(co_in,0L,SEEK_SET))perror(''Fseek failed on bond coupon file'');
else{
date=(char**)calloc(MAXDAT,sizeof(char*));if(!date)printf(''failure in
date'');date=date-1;
t = 0;
while (EOF!=fscanf(str_in,''%s'',temp)){
if(!(strcmp(temp,''Date''))){
t=t+1;
date[t]=(char*)calloc(1,10*sizeof(char));if(!date[t])printf(''failure in
date[t]'');
```

```
fscanf(str_in,''%s'',date[t]);printf(''\n%s\n'',date[t]);
fscanf(str_in,''%*s%*s%*s%*s'');
}
else{
fscanf(str_in,''%*f%lf%*f'',&tmat);
if(tmat<=30)ng[t]=ng[t]+1;
}
}
ndat = t;
for(t=1;t<=ndat;t++)printf(''\n# coupon strips on %s:\t%i'', date[t], ng[t]);


t = 0;
while(EOF!=fscanf(co_in,''%s'',temp)){
if(!(strcmp(temp,''DATE''))){
t=t+1;
for(i=1;i<=154;i++)fscanf(co_in,''%*s'');
}
else{
fscanf(co_in,''%*li%*s%lf%s%s%s%s%*s%lf%lf%*s%i%*lf%*lf'',
&tdur,inpt,sprt,drf,oprf,&try,&tco,&tnco);

if(tdur!=9999&&!(strcmp(inpt,''FI''))&&strcmp(sprt,''9999'')&&!(strcmp(drf,''9999''))
&&(!(strcmp(oprf,''9999''))||!(strcmp(oprf,''CA'')))&&tco!=9999&&tnco<60
&&tnco!=9999&&try>5.00){

if ( !(strcmp(oprf, ''9999'')) ){
if(!(strcmp(sprt,''AAA'')))naaas[t]=naaas[t]+1;
if(!(strcmp(sprt,''AA''))||!(strcmp(sprt,''AA-''))||!(strcmp(sprt,''AA+'')))
naas[t]=naas[t]+1;
if(!(strcmp(sprt,''A''))||!(strcmp(sprt,''A-''))||!(strcmp(sprt,''A+'')))
nas[t]=nas[t]+1;
}
else if(!(strcmp(oprf, ''CA'')) ){
if(!(strcmp(sprt,''AAA'')))naaac[t]=naaac[t]+1;
if(!(strcmp(sprt,''AA''))||!(strcmp(sprt,''AA-''))||!(strcmp(sprt,''AA+'')))
naac[t]=naac[t]+1;
```

```
if(!(strcmp(sprt,''A''))||!(strcmp(sprt,''A-''))||!(strcmp(sprt,''A+'')))
nac[t]=nac[t]+1;
}
}
for(i=1;i<=60+80;i++)fscanf( co_in, ''%*s'');
}
}
/*
for(t=1;t<=ndat;t++) {
printf(''\n%s:\n'', date[t]);
printf(''# straight AAA bonds:  \t%i\n'', naaas[t]);
printf(''# callable AAA bonds:  \t%i\n'', naaac[t]);
printf(''# straight AA bonds:  \t%i\n'', naas[t]);
printf(''# callable AA bonds:  \t%i\n'', naac[t]);
printf(''# straight A bonds:  \t%i\n'', nas[t]);
printf(''# callable A bonds:  \t%i\n\n'', nac[t]);
}
*/
pg=(double**)calloc(ndat,sizeof(double*));if(!pg)printf(''failure in pg'');pg=pg-
1;
pd=(double**)calloc(ndat,sizeof(double*));if(!pd)printf(''failure in pd'');pd=pd-
1;
pdc=(double**)calloc(ndat,sizeof(double*));if(!pdc) printf(''failure in
pdc'');pdc=pdc-1;
gb=(GBOND**)calloc(ndat,sizeof(GBOND*));if(!gb)printf(''failure in gb'');gb=gb-
1;
as=(AS**)calloc(ndat,sizeof(AS*));if(!as)printf(''failure in as'');as=as-
1;
ac=(AC**)calloc(ndat,sizeof(AC*));if(!ac)printf(''failure in ac'');ac=ac-
1;

for(t=1;t<=ndat;t++){
pg[t]=dvector(1,4);pd[t]=dvector(1,4);pdc[t]=dvector(1,1);

gb[t] = (GBOND *) calloc(1, sizeof(GBOND));if (!gb[t]) printf(''failure
in gb[t] '');
```

```
gb[t]->p=dvector(1,ng[t]);gb[t]->mat=dvector(1,ng[t]);gb[t]->ry=dvector(1,ng[t]);


as[t] = (AS *) calloc(1, sizeof(AS));if (!as[t]) printf(''failure in as[t]'');
as[t]->id=lvector(1,nas[t]);as[t]->ry=dvector(1,nas[t]);
as[t]->s=dvector(1,nas[t]);as[t]->dur=dvector(1,nas[t]);
as[t]->mat=dvector(1,nas[t]);as[t]->p_s_m=dvector(1,nas[t]);
as[t]->p_s_sc=dvector(1,nas[t]);as[t]->p_s_ss=dvector(1,nas[t]);
as[t]->nco=ivector(1,nas[t]);as[t]->co=dvector(1,nas[t]);
as[t]->com=dmatrix(1,nas[t],1,MAXCO);

ac[t] = (AC *) calloc(1, sizeof(AC));if (!ac) printf(''failure in ac[t]'');

ac[t]->id=lvector(1,nas[t]);ac[t]->ry=dvector(1,nac[t]);
ac[t]->s=dvector(1,nac[t]);ac[t]->dur=dvector(1,nac[t]);
ac[t]->mat=dvector(1,nac[t]);ac[t]->nco=ivector(1,nac[t]);
ac[t]->nca=ivector(1,nac[t]);ac[t]->co=dvector(1,nac[t]);
ac[t]->com=dmatrix(1,nac[t],1,MAXCO);ac[t]->cax=dmatrix(1,nac[t],1,MAXCA);
ac[t]->cam=dmatrix(1,nac[t],1,MAXCA);ac[t]->p_s_m=dvector(1,nac[t]);
ac[t]->p_s_sc=dvector(1,nac[t]);ac[t]->p_s_ss=dvector(1,nac[t]);
ac[t]->p_s_cor=dvector(1,nac[t]);ac[t]->p_c_m=dvector(1,nac[t]);
ac[t]->p_c_sc=dvector(1,nac[t]);ac[t]->p_c_ss=dvector(1,nac[t]);
ac[t]->p_c_cor=dvector(1,nac[t]);ac[t]->p_ca_m=dvector(1,nac[t]);
ac[t]->p_ca_sc=dvector(1,nac[t]);ac[t]->p_ca_ss=dvector(1,nac[t]);
ac[t]->p_ca_cor=dvector(1,nac[t]);ac[t]->ea_ca_sc=dvector(1,nac[t]);
ac[t]->ea_ca_ss=dvector(1,nac[t]);ac[t]->ea_ca_cor=dvector(1,nac[t]);
ac[t]->er_ca_sc=dvector(1,nac[t]);ac[t]->er_ca_ss=dvector(1,nac[t]);
ac[t]->er_ca_cor=dvector(1,nac[t]);

}

if(fseek(str_in,0L,SEEK_SET))printf( ''problem reading strips\n'');
else{
t = 0;
while(EOF!=fscanf(str_in,''%s'',temp)){
if(!(strcmp(temp,''Date''))){
fscanf(str_in,''%*s%*s%*s%*s%*s%*s'');
```

```
t=t+1;
ng[t] = 0;
}
else{
fscanf(str_in,''%lf%lf%lf'',&tp,&tmat,&try);
ng[t] = ng[t] + 1;
gb[t]->p[ng[t]] = tp;
gb[t]->mat[ng[t]] = tmat;
gb[t]->ry[ng[t]] = log(DSQR(1+try/200));
}
}
}

if(fseek(co_in,0L,SEEK_SET))printf(''problem reading bond coupon file\n'');

else{
t = 0;
while(EOF!=fscanf(co_in,''%s'',temp)){
if(!(strcmp(temp,''DATE''))){
t=t+1;
printf(''\nDate %i'',t);
nas[t]=0;nac[t]=0;
for(i=1;i<=154;i++)fscanf(co_in,''%*s'');
}
else{
fscanf(co_in,''%li%*s%lf%s%s%s%s%*s%lf%lf%*s%i%lf%lf'',
&tid,&tdur,inpt,sprt,drf,oprf,&try,&tco,&tnco,&tp,&tmat);

if((tdur!=9999)&&!(strcmp(inpt,''FI''))&&(strcmp(sprt,''9999''))
&&!(strcmp(drf,''9999''))&&(!(strcmp(oprf,''9999''))||!(strcmp(oprf,''CA'')))
&&(try>0)&&(tco!=9999)&&(tnco<60)&&(tnco!=9999)&&try>5.00){

if(!(strcmp(oprf,''9999''))){
if(!(strcmp(sprt,''A''))||!(strcmp(sprt,''A-''))||!(strcmp(sprt,''A+''))){
nas[t]=nas[t] + 1;
as[t]->id[nas[t]] = tid;
as[t]->p_s_m[nas[t]] = tp;
```

```
as[t]->dur[nas[t]] = tdur;
as[t]->ry[nas[t]] = 2*log(1+try/200);
as[t]->mat[nas[t]] = tmat;
as[t]->co[nas[t]] = tco;
as[t]->nco[nas[t]] = tnco;

for(i=1;i<=DMIN((double)MAXCO,tnco);i++)fscanf(co_in,''%lf'',&as[t]->com[nas[t]][i]);
for(i=1;i<=MAXCO-DMIN((double)MAXCO,tnco);i++)fscanf(co_in,''%*s'');
for(i=1;i<=2*MAXCA;i++)fscanf(co_in,''%*s'');
}
else for(i=1;i<=MAXCO+2*MAXCA;i++)fscanf(co_in,''%*s'');
}

else{
if(!(strcmp(sprt,''A''))||!(strcmp(sprt,''A-''))||!(strcmp(sprt,''A+''))){
nac[t] = nac[t] + 1;
ac[t]->id[nac[t]] = tid;
ac[t]->p_c_m[nac[t]] = tp;
ac[t]->dur[nac[t]] = tdur;
ac[t]->ry[nac[t]] = 2*log(1+try/200);
ac[t]->mat[nac[t]] = tmat;
ac[t]->co[nac[t]] = tco;
ac[t]->nco[nac[t]] = tnco;
for(i=1;i<=DMIN((double)MAXCO,tnco);i++)fscanf(co_in,''%lf'',&ac[t]->com[nac[t]][i]);
for(i=1;i<=MAXCO-DMIN((double)MAXCO,tnco);i++)fscanf(co_in,''%*s'');
for(i=1;i<=MAXCA;i++){
fscanf( co_in, ''%lf%lf'', &tcam, &tcax);
if( (tcam != 9999) && (tcax != 9999) ){
ac[t]->nca[nac[t]] += 1;
ac[t]->cam[nac[t]][ac[t]->nca[nac[t]]] = tcam;
ac[t]->cax[nac[t]][ac[t]->nca[nac[t]]] = tcax;
}
}


}
else for(i=1;i<=MAXCO+2*MAXCA;i++)fscanf(co_in,''%*s'');
```

```
} }
else for(i=1;i<=MAXCO+2*MAXCA;i++)fscanf(co_in,''%*s'');
}
}
}
}

/*** THE ACTION ********************************************/

for(t=1;t<=ndat;t++){
pg[t][1] = 0.2;
pg[t][2] = 0.06;
pg[t][3] = 0.0004;
pg[t][4] = 0.05;
}

for(t=1;t<=ndat;t++){
sig = dvector(1, ng[t]);
x = dvector(1, ng[t]);
for(i=1;i<=4;i++) ip4[i] = 1;
for(i=1;i<=ng[t];i++){
x[i] = i;sig[i] = 1;
}
printf(''\nEstimating rg date[%i]'', t);
chilast = 10000;
for(i=1; i<=400 && !( chilast - chisq >= 0 && chilast-chisq<MLTOL ) ;i++){
if(i == 1) alamda = -1;
if(i != 1) chilast = chisq;
mrqmin(0,x,gb[t]->p,sig,ng[t],pg[t],ip4,4,cov,al4,&chisq,&ml,&alamda);

if(i == 1) chistart = chisq;
iter = i;
}
alamda = 0;
mrqmin(0,x,gb[t]->p,sig,ng[t],pg[t],ip4,4,cov,al4,&chisq,&ml,&alamda);
printf(''\n%8.8lf\t%8.8lf\t%i'', chistart, chisq, iter);
free_dvector(sig, 1, ng[t]);free_dvector(x, 1, ng[t]);
```

```
}

for(t=1;t<=ndat;t++){
printf(''\nParameters rg date[%i]\n'',t);for(i=1;i<=4;i++)printf(''\t%8.8lf'',pg[t][i])
printf(''\n'');for(j=1;j<=4;j++)printf(''\t%8.8lf'', cov[j][j]);
}
for(t=1;t<=ndat;t++){
pd[t][1] = 0.25;
pd[t][2] = 0.07;
pd[t][3] = 0.01;
pd[t][4] = 0.03;
}

for(t=1;t<=ndat;t++){
sig = dvector(1, nas[t]);
x = dvector(1, nas[t]);

for(i=1;i<=4;i++) ip4[i] = i==4?1:1;
for(i=1;i<=nas[t];i++){
x[i] = i;sig[i] = 1;
}
printf(''\nEstimating ss date[%i]'', t);
chilast = 10000;
for(i=1; i<=400 && !( chilast - chisq >= 0 && chilast - chisq < MLTOL
) ;i++){
if(i == 1) alamda = -1;
if(i != 1) chilast = chisq;
mrqmin(0,x,as[t]->p_s_m,sig,nas[t],pd[t],ip4,4,cov_ss,al4,&chisq,&ml_ss,&alamda);

if(i == 1) chistart = chisq;
iter = i;
}
alamda = 0;
mrqmin(0,x,as[t]->p_s_m,sig,nas[t],pd[t],ip4,4,cov_ss,al4,&chisq,&ml_ss,&alamda);
printf(''\n%8.8lf\t%8.8lf\t%i'', chistart, chisq, iter);
free_dvector(sig, 1, nas[t]);free_dvector(x, 1, nas[t]);
}
```

```
for(t=1;t<=ndat;t++){
printf(''\nParameters ss date[%i]\n'',t);for(i=1;i<=4;i++) printf(''\t%8.8f'',
pd[t][i]);
printf(''\n'');for(j=1;j<=4;j++) printf(''\t%8.8lf'', cov_ss[j][j]);
}

ip1[1] = 1;
for(t=1;t<=ndat;t++){
pdc[t][1] = 0.01;
sig = dvector(1, nas[t]);
x = dvector(1, nas[t]);

for(i=1;i<=nas[t];i++){
x[i] = i;sig[i] = 1;
}
printf(''\nEstimating sc date[%i]'', t);
chilast = 10000;
for(i=1; i<=100 && !( chilast - chisq >= 0 && chilast - chisq < MLTOL
) ;i++){
if(i == 1) alamda = -1;
if(i != 1) chilast = chisq;
mrqmin(0, x,as[t]->p_s_m,sig,nas[t],pdc[t],ip1,1,cov_sc,al4,&chisq,&ml_sc,&alamda);

if(i == 1) chistart = chisq;
iter = i;
}
alamda = 0;
mrqmin(0, x,as[t]->p_s_m,sig,nas[t],pdc[t],ip1,1,cov_sc,al4,&chisq,&ml_sc,&alamda);
printf(''\n%8.8lf\t%8.8lf\t%i'', chistart, chisq, iter);
free_dvector(sig, 1, nas[t]);free_dvector(x, 1, nas[t]);
}
for(t=1;t<=1;t++)printf(''\nParameters sc date[%i]:\t%8.8lf\t%8.8lf'',t,pdc[t][1],cov_s


free_ivector(ip4,1,4);free_ivector(ip1,1,1);free_dmatrix(cov,1,4,1,4);
free_dmatrix(cov_ss,1,4,1,4);free_dmatrix(cov_sc,1,1,1,1);free_dmatrix(al4,1,4,1,4);
```

```
for(t=1;t<=ndat;t++){
x1[t] = pg[t][4];x2[t] = pd[t][4];
}
pearsn(x2,x1,ndat,&rho,&prob,&z);
moment(x1,ndat,&ave1,&adev1,&sdev1,&var1,&skew1,&curt1);
moment(x2,ndat,&ave2,&adev2,&sdev2,&var2,&skew2,&curt2);
printf(''\nrho:%6.6lf\tvar1:%6.6lf\tvar2:%6.6lf'',rho,var1,var2);
/*for(t=1;t<=ndat;t++){pg[t][3]=var1;pd[t][3]=var2;}*/

/*
for(t=1;t<=ndat;t++){fprintf(out_02,''\n%s\tyg\tyc\tyg'',date[t]);
tmat = 0.5;
for(i=1;i<=40;i++){
yg = -(ag(pg[t],tmat)+bg(pg[t],tmat)*pg[t][4])/tmat;
yc = yg+pdc[t][1];
ys = yg-(ag(pd[t],tmat)+bg(pd[t],tmat)*pd[t][4])/tmat;
fprintf(out_02,''\n%6.6lf\t%6.6lf\t%6.6lf\t%6.6lf'',tmat,yg,yc,ys);
tmat += 0.5;
}
}*/

for(t=1;t<=1;t++){
printf(''\nDate%i:\n'',t);
for(i=1;i<=nac[t];i++){
if(ac[t]->nca[i]!= 0){
ncall[t] += 1;
pri_acm( i, &ac[t]->p_s_m[i], &ac[t]->p_ca_m[i]);
pri_acsc( i, &ac[t]->p_s_sc[i], &ac[t]->p_c_sc[i], &ac[t]->p_ca_sc[i]);

pri_acss( i, &ac[t]->p_s_ss[i], &ac[t]->p_c_ss[i], &ac[t]->p_ca_ss[i]);

pri_cor( i, &ac[t]->p_s_cor[i], &ac[t]->p_c_cor[i], &ac[t]->p_ca_cor[i]);

printf(''\nStraight:'');
printf(''\t%6.3lf\t%6.3lf'',ac[t]->p_s_m[i],ac[t]->p_s_sc[i]);
printf(''\t%6.3lf\t%6.3lf'',ac[t]->p_s_ss[i],ac[t]->p_s_cor[i]);
```

```
printf(''\nCallable:'');
printf(''\t%6.3lf\t%6.3lf'',ac[t]->p_c_m[i],ac[t]->p_c_sc[i]);
printf(''\t%6.3lf\t%6.3lf'', ac[t]->p_c_ss[i],ac[t]->p_c_cor[i]);
printf(''\nCall:'');
printf(''\t%6.3lf\t%6.3lf'', ac[t]->p_ca_m[i], ac[t]->p_ca_sc[i]);
printf(''\t%6.3lf\t%6.3lf'', ac[t]->p_ca_ss[i], ac[t]->p_ca_cor[i]);
}
}
}
mmae_ca_sc=0;mmre_ca_sc=0;mmae_ca_ss=0;mmre_ca_ss=0;mmae_ca_cor=0;mmre_ca_cor=0;

for(t=1;t<=1;t++){
for(i=1;i<=nac[t];i++){
if(ac[t]->nca[i] != 0){
ac[t]->ea_ca_sc[i] = ac[t]->p_ca_m[i] - ac[t]->p_ca_sc[i];
ac[t]->er_ca_sc[i] = ac[t]->ea_ca_sc[i] / ac[t]->p_ca_m[i];
ac[t]->ea_ca_ss[i] = ac[t]->p_ca_m[i] - ac[t]->p_ca_ss[i];
ac[t]->er_ca_ss[i] = ac[t]->ea_ca_ss[i] / ac[t]->p_ca_m[i];
ac[t]->ea_ca_cor[i] = ac[t]->p_ca_m[i] - ac[t]->p_ca_cor[i];
ac[t]->er_ca_cor[i] = ac[t]->ea_ca_cor[i] / ac[t]->p_ca_m[i];
mae_ca_sc[t] += fabs(ac[t]->ea_ca_sc[i]);
mre_ca_sc[t] += fabs(ac[t]->er_ca_sc[i]);
mae_ca_ss[t] += fabs(ac[t]->ea_ca_ss[i]);
mre_ca_ss[t] += fabs(ac[t]->er_ca_ss[i]);
mae_ca_cor[t] += fabs(ac[t]->ea_ca_cor[i]);
mre_ca_cor[t] += fabs(ac[t]->er_ca_cor[i]);
}
}
mae_ca_sc[t] /= ncall[t];mre_ca_sc[t] /= ncall[t];
mae_ca_ss[t] /= ncall[t];mre_ca_ss[t] /= ncall[t];
mae_ca_cor[t] /= ncall[t];mre_ca_cor[t] /= ncall[t];
mmae_ca_sc += mae_ca_sc[t];mmre_ca_sc += mre_ca_sc[t];
mmae_ca_ss += mae_ca_ss[t];mmre_ca_ss += mre_ca_ss[t];
mmae_ca_cor += mae_ca_cor[t];mmre_ca_cor += mre_ca_cor[t];
}
mmae_ca_sc /= ndat;mmre_ca_sc /= ndat;mmae_ca_ss /= ndat;
mmre_ca_ss /= ndat;mmae_ca_cor /= ndat;mmre_ca_cor /= ndat;
```

```
printf(''\nAESC:\t%6.3lf\nRESC:\t%6.3lf'', mmae_ca_sc, mmre_ca_sc);
printf(''\nAESS:\t%6.3lf\nRESS:\t%6.3lf'', mmae_ca_ss, mmre_ca_ss);
printf(''\nAECOR:\t%6.3lf\nRECOR:\t%6.3lf'', mmae_ca_cor, mmre_ca_cor);

/*for(t=1;t<=1;t++){
fprintf(out_01,''\nDate%i:\n'', t);
for(i=1;i<=nac[t];i++){
if(ac[t]->nca[i]!= 0){
fprintf(out_01,''\nAE[%i][%i]:\t%6.3lf'', t, i, ac[t]->ea_ca_sc[i]);
fprintf(out_01,''\nAE[%i][%i]:\t%6.3lf'', t, i, ac[t]->ea_ca_ss[i]);
fprintf(out_01,''\nAE[%i][%i]:\t%6.3lf'', t, i, ac[t]->ea_ca_cor[i]);
fprintf(out_01,''\nRE[%i][%i]:\t%6.3lf'', t, i, ac[t]->er_ca_sc[i]);
fprintf(out_01,''\nRE[%i][%i]:\t%6.3lf'', t, i, ac[t]->er_ca_ss[i]);
fprintf(out_01,''\nRE[%i][%i]:\t%6.3lf'', t, i, ac[t]->er_ca_cor[i]);
}
}
}*/
for(t=1;t<=ndat;t++){
free_dvector(pg[t],1,4);free_dvector(pd[t],1,4);free_dvector(pdc[t],1,1);
free_dvector(gb[t]->p,1,ng[t]);free_dvector(gb[t]->mat,1,ng[t]);
free_dvector(gb[t]->ry, 1,ng[t]);

free_lvector(as[t]->id,1,nas[t]);free_dvector(as[t]->ry,1,nas[t]);
free_dvector(as[t]->s,1,nas[t]);free_dvector(as[t]->dur,1,nas[t]);
free_dvector(as[t]->mat,1,nas[t]);free_dvector(as[t]->p_s_m,1,nas[t]);
free_dvector(as[t]->p_s_sc,1,nas[t]);free_dvector(as[t]->p_s_ss,1,nas[t]);

free_ivector(as[t]->nco,1,nas[t]);free_dvector(as[t]->co,1,nas[t]);
free_dmatrix(as[t]->com,1,nas[t],1,MAXCO);

free_lvector(ac[t]->id,1,nac[t]);free_dvector(ac[t]->ry,1,nac[t]);
free_dvector(ac[t]->s,1,nac[t]);free_dvector(ac[t]->dur,1,nac[t]);
free_dvector(ac[t]->mat,1,nac[t]);free_ivector(ac[t]->nco,1,nac[t]);
free_ivector(ac[t]->nca,1,nac[t]);free_dvector(ac[t]->co,1,nac[t]);
free_dmatrix(ac[t]->com,1,nac[t],1,MAXCO);free_dmatrix(ac[t]->cax,1,nac[t],1,MAXCA);
free_dmatrix(ac[t]->cam,1,nac[t],1,MAXCA);free_dvector(ac[t]->p_s_m,1,nac[t]);
```

```
free_dvector(ac[t]->p_s_sc,1,nac[t]);free_dvector(ac[t]->p_s_ss,1,nac[t]);

free_dvector(ac[t]->p_s_cor,1,nac[t]);free_dvector(ac[t]->p_c_m,1,nac[t]);

free_dvector(ac[t]->p_c_sc,1,nac[t]);free_dvector(ac[t]->p_c_ss,1,nac[t]);

free_dvector(ac[t]->p_c_cor,1,nac[t]);free_dvector(ac[t]->p_ca_m,1,nac[t]);

free_dvector(ac[t]->p_ca_sc,1,nac[t]);free_dvector(ac[t]->p_ca_ss,1,nac[t]);

free_dvector(ac[t]->p_ca_cor,1,nac[t]);free_dvector(ac[t]->ea_ca_sc,1,nac[t]);

free_dvector(ac[t]->ea_ca_ss,1,nac[t]);free_dvector(ac[t]->ea_ca_cor,1,nac[t]);
free_dvector(ac[t]->er_ca_sc,1,nac[t]);free_dvector(ac[t]->er_ca_ss,1,nac[t]);

free_dvector(ac[t]->er_ca_cor,1,nac[t]);

free(gb[t]);free(as[t]);free(ac[t]);free(date[t]);
}
free(date);free(pg);free(pd);free(pdc);free(gb);free(as);free(ac);
free_ivector(ng,1,MAXDAT);free_ivector(naaas,1,MAXDAT);
free_ivector(naas,1,MAXDAT);free_ivector(nas,1,MAXDAT);
free_ivector(naaac,1,MAXDAT);free_ivector(naac,1,MAXDAT);
free_ivector(nac,1,MAXDAT);free_ivector(ncall,1,MAXDAT);
free_dvector(x1,1,MAXDAT);free_dvector(x2,1,MAXDAT);
free_dvector(mae_ca_sc,1,MAXDAT);free_dvector(mre_ca_sc,1,MAXDAT);
free_dvector(mae_ca_ss,1,MAXDAT);free_dvector(mre_ca_ss,1,MAXDAT);
free_dvector(mae_ca_cor,1,MAXDAT);free_dvector(mre_ca_cor,1,MAXDAT);

if(fclose(str_in)==0)printf(''\n\tstr_in closed'');
if(fclose(co_in)==0)printf(''\n\tco_in closed'');
if(fclose(out_01)==0)printf(''\n\tout_01 file closed'');
if(fclose(out_02)==0)printf(''\n\tout_02 file closed'');
}
#undef MAXDAT
#undef MAXCO
#undef MAXCA
```

```
#undef INTORD
#undef MLTOL
```

Model Code

```
double ag(double pp[],double life) {
 double tag;
 tag = -pp[2]*life;
 tag -= pp[2]*exp(-pp[1]*life)/pp[1];
 tag += 1/(2*DSQR(pp[1]))*life*pp[3];
 tag += exp(-pp[1]*life)*pp[3]/pow(pp[1],3);
 tag -= exp(-2*life*pp[1])*pp[3]/(4*pow(pp[1],3));
 tag -= 3*pp[3]/(4*pow(pp[1],3));
 tag += pp[2]/pp[1];

 return(tag);
}
double bg(double pp[], double life)
{
 return((exp(-pp[1]*life)-1)/pp[1]);
}
double df_sc(double mat, double pp[])
{
 return(exp(ag(pp,mat)+bg(pp,mat)*pp[4]));
}
double df_ss(double mat, double ppg[], double ppd[])
{
 double df;

 df = exp(ag(ppg,mat)+bg(ppg,mat)*ppg[4]);
 df *= exp(ag(ppd,mat)+bg(ppd,mat)*ppd[4]);

 return(df);
}
int iscadat(double comat,double camat[],int ncamat)
{
```

```
 int ffi;
 for(ffi=1;ffi<=ncamat;ffi++)if(comat=camat[ffi])return(ffi);
 return(0);

}
extern GBOND **gb;
extern int t;
```

Estimation Code

```
void ml(double x,double pp[],double *y,double dyda[],int npar) {
 int xi;

 xi = (int)x;
 *y = 100*exp(ag(pp,gb[t]->mat[xi])+bg(pp,gb[t]->mat[xi])*pp[4]);

 dyda[1] = pp[2]/DSQR(pp[1])*exp(-pp[1]*gb[t]->mat[xi]);
 dyda[1] += pp[2]/pp[1]*exp(-pp[1]*gb[t]->mat[xi])*gb[t]->mat[xi];
 dyda[1] -= pp[3]/pow(pp[1],3)*gb[t]->mat[xi];
 dyda[1] -= 3*pp[3]/pow(pp[1],4)*exp(-pp[1]*gb[t]->mat[xi]);
 dyda[1] -= pp[3]/pow(pp[1],3)*exp(-pp[1]*gb[t]->mat[xi])*gb[t]->mat[xi];

 dyda[1] += pp[3]*3/(4*pow(pp[1],4))*exp(-2*pp[1]*gb[t]->mat[xi]);
 dyda[1] += pp[3]/(2*pow(pp[1],3))*exp(-2*pp[1]*gb[t]->mat[xi])*gb[t]-
>mat[xi];
 dyda[1] += 2*pp[2]/pp[1];
 dyda[1] += 9*pp[3]/(4*pow(pp[1],4));
 dyda[1] -= 3*pp[2]/DSQR(pp[1]);
 dyda[1] += pp[4]/DSQR(pp[1]);
 dyda[1] -= pp[4]*gb[t]->mat[xi]*exp(-pp[1]*gb[t]->mat[xi])/pp[1];
 dyda[1] -= pp[4]/DSQR(pp[1])*exp(-pp[1]*gb[t]->mat[xi]);
 dyda[1] *= *y;
 dyda[2] = 1/pp[1]*(1-exp(-pp[1]*gb[t]->mat[xi]))-gb[t]->mat[xi];
 dyda[2] *= *y;
 dyda[3] = gb[t]->mat[xi]/(2*DSQR(pp[1]));
 dyda[3] += exp(-pp[1]*gb[t]->mat[xi])/pow(pp[1],3);
```

```
 dyda[3] -= exp(-2*pp[1]*gb[t]->mat[xi])/(4*pow(pp[1],3));
 dyda[3] -= 3/(4*pow(pp[1],3));
 dyda[3] *= *y;
 dyda[4] = bg( pp, gb[t]->mat[xi]);
 dyda[4] *= *y;
}
extern double **pg;
extern AS **as;
extern int t;
void ml_ss(double x, double pp[], double *y, double dyda[], int npar)
{

 double ty, *tdyda;
 int xi, ti;

 xi = (int)x;
 tdyda = dvector(1, 4);

 *y = 0.0;

 for(ti=1;ti<=as[t]->nco[xi];ti++){
 ty = exp(ag(pg[t],as[t]->com[xi][ti]));
 ty *= exp(bg(pg[t],as[t]->com[xi][ti])*pg[t][4]);
 ty *= exp(ag(pp,as[t]->com[xi][ti]));
 ty *= exp(bg(pp,as[t]->com[xi][ti])*pp[4]);
 ty *= (ti==as[t]->nco[xi]) ?  as[t]->co[xi]/2 + 100.00 :  as[t]->co[xi]/2
;

 tdyda[1] = pp[2]/DSQR(pp[1])*exp(-pp[1]*as[t]->com[xi][ti]);
 tdyda[1] += pp[2]/pp[1]*exp(-pp[1]*as[t]->com[xi][ti])*as[t]->com[xi][ti];

 tdyda[1] -= pp[3]/pow(pp[1],3)*as[t]->com[xi][ti];
 tdyda[1] -= 3*pp[3]/pow(pp[1],4)*exp(-pp[1]*as[t]->com[xi][ti]);
 tdyda[1] -= pp[3]/pow(pp[1],3)*exp(-pp[1]*as[t]->com[xi][ti])*as[t]->com[xi][ti];

 tdyda[1] += pp[3]*3/(4*pow(pp[1],4))*exp(-2*pp[1]*as[t]->com[xi][ti]);
```

```
 tdyda[1] += pp[3]/(2*pow(pp[1],3))*exp(-2*pp[1]*as[t]->com[xi][ti])*as[t]-
>com[xi][ti];
 tdyda[1] += 2*pp[2]/pp[1];
 tdyda[1] += 9*pp[3]/(4*pow(pp[1],4));
 tdyda[1] -= 3*pp[2]/DSQR(pp[1]);
 tdyda[1] += pp[4]/DSQR(pp[1]);
 tdyda[1] -= pp[4]*as[t]->com[xi][ti]*exp(-pp[1]*as[t]->com[xi][ti])/pp[1];
 tdyda[1] -= pp[4]/DSQR(pp[1])*exp(-pp[1]*as[t]->com[xi][ti]);
 tdyda[2] = 1/pp[1]*(1-exp(-pp[1]*as[t]->com[xi][ti]))-as[t]->com[xi][ti];

 tdyda[3] = as[t]->com[xi][ti]/(2*DSQR(pp[1]));
 tdyda[3] += exp(-pp[1]*as[t]->com[xi][ti])/pow(pp[1],3);
 tdyda[3] -= exp(-2*pp[1]*as[t]->com[xi][ti])/(4*pow(pp[1],3));
 tdyda[3] -= 3/(4*pow(pp[1],3));
 tdyda[4] = bg( pp, as[t]->com[xi][ti]);
 dyda[1] += ty*tdyda[1];
 dyda[2] += ty*tdyda[2];
 dyda[3] += ty*tdyda[3];
 dyda[4] += ty*tdyda[4];

 *y += ty;
 }
 free_dvector(tdyda, 1, 4);
}
extern double **pg;
extern AS **as;
extern int t;
void ml_sc(double x,double pp[],double *y,double dyda[],int npar)
{
 double ty;
 int xi, ti;

 xi = (int)x;
 *y = 0.0;

 for(ti=1;ti<=as[t]->nco[xi];ti++){
 ty = exp(ag(pg[t],as[t]->com[xi][ti]));
```

```
 ty *= exp(bg(pg[t],as[t]->com[xi][ti])*pg[t][4]);
 ty *= exp(-pp[1]*as[t]->com[xi][ti]);
 ty *= (ti==as[t]->nco[xi]) ?  as[t]->co[xi]/2 + 100.00 :  as[t]->co[xi]/2
;
 dyda[1] -= as[t]->com[xi][ti]*ty;
 *y += ty;
 }
}
```

Tree Code

```
extern RG **rg; extern double **pg;
extern int t;
void set_rg(double dt[], int nstep)
{
 int fi, fj, fjj;
 double mean;
 rg = (RG **) calloc(nstep+1, sizeof(RG *));if (!rg) printf(''failure in
rg'');
 for(fi=0;fi<=nstep;fi++){
 rg[fi] = (RG *) calloc(1, sizeof(RG));if (!rg[fi]) printf(''failure in
rg[fi]'');
 rg[fi]->r = dvector(1,1+2*fi);if (!rg[fi]->r ) printf(''failure in rg[fi]-
>r'');
 rg[fi]->pu = dvector(1,1+2*fi);if (!rg[fi]->pu ) printf(''failure in rg[fi]-
>pu'');
 rg[fi]->pm = dvector(1,1+2*fi);if (!rg[fi]->pm ) printf(''failure in rg[fi]-
>pm'');
 rg[fi]->pd = dvector(1,1+2*fi);if (!rg[fi]->pd ) printf(''failure in rg[fi]-
>pd'');
 rg[fi]->mid = ivector(1,1+2*fi);if (!rg[fi]->mid ) printf(''failure in
rg[fi]->mid'');
 rg[fi]->midi = ivector(1,1+2*fi);if (!rg[fi]->midi ) printf(''failure
in rg[fi]->midi'');
 }
```

```
 /* HW step size:  2/sqrt(3)*pg[t][3]*sqrt(dt[fi]) < inc < 2*pg[t][3]*sqrt(dt[fi])
*/

 rg[0]->ns=1;rg[0]->js=0;
 for(fi=0;fi<=nstep;fi++) {
 rg[fi]->inc = sqrt(pg[t][3])*sqrt(dt[fi])*2/sqrt(3);
 if(fi<nstep){
 rg[fi+1]->ns=rg[fi]->ns+2;rg[fi+1]->js=rg[fi]->js-1;
 }
 for(fj=rg[fi]->js;fj<=rg[fi]->js+rg[fi]->ns-1;fj++) {
 fjj = 1+fj-rg[fi]->js;
 rg[fi]->r[fjj] = pg[t][4]+fj*rg[fi]->inc;
 mean = pg[t][1]*(pg[t][2] - rg[fi]->r[fjj]);
 rg[fi]->mid[fjj] = fj;
 if(mean>=0){
 while(mean*dt[fi]>(rg[fi]->mid[fjj]-fj)*rg[fi]->inc+rg[fi]->inc/2){
 rg[fi]->mid[fjj] += 1;
 if(fj==rg[fi]->js+rg[fi]->ns-1&&fi<nstep)rg[fi+1]->ns +=1;
 if(fj==rg[fi]->js && fi<nstep ){rg[fi+1]->ns-=1;rg[fi+1]->js +=1;}
 }
 }
 else{
 while(mean*dt[fi]<(rg[fi]->mid[fjj]-fj)*rg[fi]->inc-rg[fi]->inc/2){
 rg[fi]->mid[fjj] -= 1;
 if(fj==rg[fi]->js+rg[fi]->ns-1&&fi<nstep)rg[fi+1]->ns -=1;
 if(fj==rg[fi]->js && fi<nstep ){rg[fi+1]->ns += 1;rg[fi+1]->js-=1;}
 }
 }
 rg[fi]->pu[fjj] = DSQR(mean*dt[fi])+pg[t][3]*dt[fi];
 rg[fi]->pu[fjj] += (rg[fi]->mid[fjj]-fj)*(rg[fi]->mid[fjj]-1-fj)*DSQR(rg[fi]-
>inc);
 rg[fi]->pu[fjj] -= mean*dt[fi]*(2*rg[fi]->mid[fjj]-1-2*fj)*rg[fi]->inc;
 rg[fi]->pu[fjj] /= (2*DSQR(rg[fi]->inc));
 rg[fi]->pd[fjj] = DSQR(mean*dt[fi])+pg[t][3]*dt[fi];
 rg[fi]->pd[fjj] += (rg[fi]->mid[fjj]+1-fj)*(rg[fi]->mid[fjj]-fj)*DSQR(rg[fi]-
>inc);
 rg[fi]->pd[fjj] -= mean*dt[fi]*(2*rg[fi]->mid[fjj]+1-2*fj)*rg[fi]->inc;
```

```
 rg[fi]->pd[fjj] /= (2*DSQR(rg[fi]->inc));
 rg[fi]->pm[fjj] = 1-rg[fi]->pu[fjj]-rg[fi]->pd[fjj];
 rg[fi]->r[fjj] = DMAX(0.0,rg[fi]->r[fjj]);
 }
 }
 for(fi=0;fi<=nstep;fi++) {
 for(fj=1;fj<=rg[fi]->ns;fj++) {
 if(fj == 1)rg[fi]->midi[fj] = 2;
 else rg[fi]->midi[fj]=rg[fi]->midi[fj-1]+(int)fabs(rg[fi]->mid[fj]-rg[fi]-
>mid[fj-1]);
 }
 }
 /*
 for(fi=0;fi<=nstep;fi++){printf(''step[%i]:\n'', fi);
 for(fj=1;fj<=rg[fi]->ns;fj++){
 printf(''%6.6lf\t%6.6lf\t%6.6lf\t'', rg[fi]->pu[fj], rg[fi]->pd[fj], rg[fi]-
>pm[fj]);
 printf(''r:%6.4lf\n'',rg[fi]->r[fj]);
 }
 }*/
}
extern SS **ss;
extern double **pd;
extern int t;
void set_ss(double dt[], int nstep){

 int fi, fj, fjj;
 double mean;
 ss = (SS **) calloc(nstep+1, sizeof(SS *));if (!ss) printf(''failure in
ss'');
 for(fi=0;fi<=nstep;fi++){
 ss[fi] = (SS *) calloc(1, sizeof(SS));if (!ss[fi]) printf(''failure in
ss[fi]'');
 ss[fi]->s = dvector(1,1+2*fi);if (!ss[fi]->s ) printf(''failure in ss[fi]-
>s'');
 ss[fi]->pu = dvector(1,1+2*fi);if (!ss[fi]->pu ) printf(''failure in ss[fi]-
>pu'');
```

```
ss[fi]->pm = dvector(1,1+2*fi);if (!ss[fi]->pm ) printf(''failure in ss[fi]-
>pm'');
ss[fi]->pd = dvector(1,1+2*fi);if (!ss[fi]->pd ) printf(''failure in ss[fi]-
>pd'');
ss[fi]->mid = ivector(1,1+2*fi);if (!ss[fi]->mid ) printf(''failure in
ss[fi]->mid'');
ss[fi]->midi = ivector(1,1+2*fi);if (!ss[fi]->midi ) printf(''failure
in ss[fi]->midi'');
}
ss[0]->ns=1;ss[0]->js=0;

for(fi=0;fi<=nstep;fi++){
ss[fi]->inc = sqrt(pd[t][3])*sqrt(dt[fi])*2/sqrt(3);
if(fi<nstep){
ss[fi+1]->ns = ss[fi]->ns+2;ss[fi+1]->js = ss[fi]->js-1;
}
for(fj=ss[fi]->js;fj<=ss[fi]->js+ss[fi]->ns-1;fj++){
fjj = 1+fj-ss[fi]->js;
ss[fi]->s[fjj] = pd[t][4]+fj*ss[fi]->inc;
mean = pd[t][1]*(pd[t][2]-ss[fi]->s[fjj]);
ss[fi]->mid[fjj] = fj;
if(mean>=0){
while(mean*dt[fi]>(ss[fi]->mid[fjj]-fj)*ss[fi]->inc + ss[fi]->inc/2){
ss[fi]->mid[fjj] += 1;
if(fj==ss[fi]->js+ss[fi]->ns-1&&fi<nstep)ss[fi+1]->ns+=1;
if(fj==ss[fi]->js&&fi<nstep){ss[fi+1]->ns-=1;ss[fi+1]->js+=1;}
}
}
else{
while(mean*dt[fi]<(ss[fi]->mid[fjj]-fj)*ss[fi]->inc-ss[fi]->inc/2){
ss[fi]->mid[fjj]-=1;
if(fj==ss[fi]->js+ss[fi]->ns-1&&fi<nstep)ss[fi+1]->ns-=1;
if(fj==ss[fi]->js&&fi<nstep){ss[fi+1]->ns+=1;ss[fi+1]->js-=1;}
}
}

ss[fi]->pu[fjj] = DSQR(mean*dt[fi])+pd[t][3]*dt[fi];
```

```
 ss[fi]->pu[fjj] += (ss[fi]->mid[fjj]-fj)*(ss[fi]->mid[fjj]-1-fj)*DSQR(ss[fi]-
>inc);
 ss[fi]->pu[fjj] -= mean*dt[fi]*(2*ss[fi]->mid[fjj]-1-2*fj)*ss[fi]->inc;
 ss[fi]->pu[fjj] /= (2*DSQR(ss[fi]->inc));
 ss[fi]->pd[fjj] = DSQR(mean*dt[fi])+pd[t][3]*dt[fi];
 ss[fi]->pd[fjj] += (ss[fi]->mid[fjj]+1-fj)*(ss[fi]->mid[fjj]-fj)*DSQR(ss[fi]-
>inc);
 ss[fi]->pd[fjj] -= mean*dt[fi]*(2*ss[fi]->mid[fjj]+1-2*fj)*ss[fi]->inc;
 ss[fi]->pd[fjj] /= (2*DSQR(ss[fi]->inc));
 ss[fi]->pm[fjj] = 1-ss[fi]->pu[fjj]-ss[fi]->pd[fjj];
 ss[fi]->s[fjj] = DMAX(0.0,ss[fi]->s[fjj]);
 }
 }
 for(fi=0;fi<=nstep;fi++){
 for(fj=1;fj<=ss[fi]->ns;fj++){
 if(fj == 1)ss[fi]->midi[fj] = 2;
 else ss[fi]->midi[fj]=ss[fi]->midi[fj-1]+(int)fabs(ss[fi]->mid[fj]-ss[fi]-
>mid[fj-1]);
 }
 }
 /*
 for(fi=0;fi<=nstep;fi++){printf(''step[%i]:\n'', fi);
 for(fj=1;fj<=ss[fi]->ns;fj++){
 printf(''%6.6lf\t%6.6lf\t%6.6lf\t'',ss[fi]->pu[fj],ss[fi]->pd[fj],ss[fi]-
>pm[fj]);
 printf(''ss:%6.4lf\n'',ss[fi]->s[fj]);
 }
 }*/
}
extern RG **rg;
void free_rg(int nstep)
{
 int fi;

 for(fi=0;fi<=nstep;fi++){
 free_dvector(rg[fi]->r,1,1+2*fi);free_dvector(rg[fi]->pu,1,1+2*fi);
 free_dvector(rg[fi]->pm,1,1+2*fi);free_dvector(rg[fi]->pd,1,1+2*fi);
```

```
 free_ivector(rg[fi]->mid,1,1+2*fi);free_ivector(rg[fi]->midi,1,1+2*fi);
 free(rg[fi]);
 }free(rg);
}
extern SS **ss;
void free_ss(int nstep)
{
 int fi;
 for(fi=0;fi<=nstep;fi++){
 free_dvector(ss[fi]->s,1,1+2*fi);free_dvector(ss[fi]->pu,1,1+2*fi);
 free_dvector(ss[fi]->pm,1,1+2*fi);free_dvector(ss[fi]->pd,1,1+2*fi);
 free_ivector(ss[fi]->mid,1,1+2*fi);free_ivector(ss[fi]->midi,1,1+2*fi);
 free(ss[fi]);
 }free(ss);
}
```

Pricing Code

```
extern GBOND **gb; extern AS **as;
extern AC **ac;
extern int t, *nas, *ng;
void pri_acm(int ffi, double *straight, double *call)
{
 int fj, fk, imat, *duri;
 double iry_g, iry_d, idry_g, idry_d,*sort_dur,*sort_ry,m_c;

 duri=ivector(1,nas[t]);sort_dur=dvector(1,nas[t]);sort_ry=dvector(1,nas[t]);

 for(fj=1;fj<=nas[t];fj++){sort_dur[fj] = as[t]->dur[fj];sort_ry[fj] =
as[t]->ry[fj];}
 sort2( nas[t], sort_dur, sort_ry);

 if(ac[t]->dur[ffi]<=gb[t]->mat[1]) iry_g = gb[t]->ry[1];
 else if(ac[t]->dur[ffi]>=gb[t]->mat[ng[t]]) iry_g = gb[t]->ry[ng[t]];

 else{locate( gb[t]->mat, ng[t], ac[t]->dur[ffi], &imat);
```

```
fk = IMIN(IMAX(imat-(INTORD-1)/2,1),ng[t]+1-INTORD);
polint( &gb[t]->mat[fk-1], &gb[t]->ry[fk-1],INTORD,ac[t]->dur[ffi],&iry_g,
&idry_g);
}
if(ac[t]->dur[ffi]<=sort_dur[1]) iry_d = sort_ry[1];
else if(ac[t]->dur[ffi]>=sort_dur[nas[t]]) iry_d = sort_dur[nas[t]];
else{locate( sort_dur, nas[t], ac[t]->dur[ffi], &imat);
fk = IMIN(IMAX(imat-(INTORD-1)/2,1),nas[t]+1-INTORD);
polint(&sort_dur[fk-1],&sort_ry[fk-1],INTORD,ac[t]->dur[ffi],&iry_d,&idry_d);
}
ac[t]->s[ffi] = iry_d - iry_g;

*call = -ac[t]->p_c_m[ffi];
for(fj=1;fj<=ac[t]->nco[ffi];fj++){
if(ac[t]->com[ffi][fj]<=gb[t]->mat[1]) iry_g = gb[t]->ry[1];
else if(ac[t]->com[ffi][fj]>=gb[t]->mat[ng[t]]) iry_g = gb[t]->ry[ng[t]];

else{locate( gb[t]->mat, ng[t], ac[t]->com[ffi][fj], &imat);
fk = IMIN(IMAX(imat-(INTORD-1)/2,1),ng[t]+1-INTORD);
polint(&gb[t]->mat[fk-1],&gb[t]->ry[fk-1],INTORD,ac[t]->com[ffi][fj],&iry_g,&idry_g);
}
m_c = exp(-ac[t]->com[ffi][fj]*(iry_g+ac[t]->s[ffi]));
m_c *= (fj==ac[t]->nco[ffi]) ?  ac[t]->co[ffi]/2 + 100.00 :  ac[t]->co[ffi]/2;
*straight += m_c;
}
*call += *straight;
free_dvector(sort_dur,1,nas[t]);free_dvector(sort_ry,1,nas[t]);free_ivector(duri,1,nas[
}
extern RG **rg;
extern AC **ac;
extern double **pg,**pdc;
extern int t;
void pri_acsc(int ffi, double *straight, double *callable, double *call)
{
int fi, fj, nstep, ex_ind;
double **p_s, **p_c, **p_ca, *ppp, *dt;
```

```
ppp = dvector(1,4);
for(fi=1;fi<=4;fi++) ppp[fi] = pg[t][fi];
nstep = ac[t]->nco[ffi]-1;

dt = dvector(0, nstep);
for(fi=0;fi<=nstep;fi++) dt[fi] = 0.5;
set_rg(dt, nstep);

p_s = (double **) calloc(nstep+1, sizeof(double *));if (!p_s) printf(''failure
in p_s'');
p_c = (double **) calloc(nstep+1, sizeof(double *));if (!p_c) printf(''failure
in p_c'');
p_ca = (double **) calloc(nstep+1, sizeof(double *));if (!p_ca) printf(''failure
in p_ca'');

for(fi=nstep;fi>=0;fi--){
p_s[fi] = dvector(1, rg[fi]->ns);
p_c[fi] = dvector(1, rg[fi]->ns);
p_ca[fi] = dvector(1, rg[fi]->ns);

if(fi==nstep){
for(fj=1;fj<=rg[fi]->ns;fj++){
p_s[fi][fj] = 100 + ac[t]->co[ffi]/2;
p_c[fi][fj] = 100 + ac[t]->co[ffi]/2;
p_ca[fi][fj] = 0;
}
}
else{
for(fj=1;fj<=rg[fi]->ns;fj++){
ppp[4] = rg[fi]->r[fj];

p_s[fi][fj] = rg[fi]->pu[fj]*p_s[fi+1][rg[fi]->midi[fj]+1];
p_s[fi][fj] += rg[fi]->pm[fj]*p_s[fi+1][rg[fi]->midi[fj]];
p_s[fi][fj] += rg[fi]->pd[fj]*p_s[fi+1][rg[fi]->midi[fj]-1];
p_s[fi][fj] *= df_sc(dt[fi+1], ppp)*exp(-dt[fi+1]*pdc[t][1]);
p_s[fi][fj] += ac[t]->co[ffi]/2;
```

```
p_c[fi][fj] = rg[fi]->pu[fj]*p_c[fi+1][rg[fi]->midi[fj]+1];
p_c[fi][fj] += rg[fi]->pm[fj]*p_c[fi+1][rg[fi]->midi[fj]];
p_c[fi][fj] += rg[fi]->pd[fj]*p_c[fi+1][rg[fi]->midi[fj]-1];
p_c[fi][fj] *= df_sc(dt[fi+1], ppp)*exp(-dt[fi+1]*pdc[t][1]);
p_c[fi][fj] += ac[t]->co[ffi]/2;

p_ca[fi][fj] = rg[fi]->pu[fj]*p_ca[fi+1][rg[fi]->midi[fj]+1];
p_ca[fi][fj] += rg[fi]->pm[fj]*p_ca[fi+1][rg[fi]->midi[fj]];
p_ca[fi][fj] += rg[fi]->pd[fj]*p_ca[fi+1][rg[fi]->midi[fj]-1];
p_ca[fi][fj] *= df_sc(dt[fi+1], ppp)*exp(-dt[fi+1]*pdc[t][1]);

ex_ind = iscadat(ac[t]->com[ffi][fi], ac[t]->cam[ffi], ac[t]->nca[ffi]);


if( ex_ind != 0 ){
p_c[fi][fj] = DMIN(p_c[fi][fj], ac[t]->cax[ffi][ex_ind]);
p_ca[fi][fj] = p_s[fi][fj] - p_c[fi][fj];
}
}
}
if(fi!=nstep){
free_dvector(p_s[fi+1],1,rg[fi+1]->ns);
free_dvector(p_c[fi+1],1,rg[fi+1]->ns);
free_dvector(p_ca[fi+1],1,rg[fi+1]->ns);
}


}
ppp[4] = pg[t][4];

*straight = p_s[0][1]*df_sc(ac[t]->com[ffi][1],ppp)*exp(-ac[t]->com[ffi][1]*pdc[t][1])
*callable = p_c[0][1]*df_sc(ac[t]->com[ffi][1],ppp)*exp(-ac[t]->com[ffi][1]*pdc[t][1])

*call = *straight-*callable;

free_dvector(p_s[0],1,1);free_dvector(p_c[0],1,1);free_dvector(p_ca[0],1,1);
free(p_s);free(p_c);free(p_ca);
free_rg(nstep);free_dvector(dt,0,nstep);free_dvector(ppp,1,4);
```

```
}
extern double **pg, **pd;
extern RG **rg;
extern SS **ss;
extern AC **ac;
extern int t;
void pri_acss(int ffi, double *straight, double *callable, double *call)
{
 double **p_s, **p_c, **p_ca, *ppg, *ppd, *dt;
 double pdd, pdm, pdu, pmd, pmm, pmu, pud, pum, puu;
 int fi, fj, fi1, fi2, nstep, index, ex_ind;

 ppg = dvector(1,4);
 ppd = dvector(1,4);

 for(fi=1;fi<=4;fi++){ppg[fi] = pg[t][fi];ppd[fi] = pd[t][fi];}

 nstep = ac[t]->nco[ffi]-1;
 dt = dvector(0, nstep);
 for(fi=0;fi<=nstep;fi++) dt[fi] = 0.5;

 set_rg(dt, nstep);set_ss(dt, nstep);

 p_s = (double **) calloc(nstep+1, sizeof(double *));if (!p_s) printf(''failure
in p_s'');
 p_c = (double **) calloc(nstep+1, sizeof(double *));if (!p_c) printf(''failure
in p_c'');
 p_ca = (double **) calloc(nstep+1, sizeof(double *));if (!p_ca) printf(''failure
in p_ca'');

 for(fi=nstep;fi>=0;fi--){

 p_s[fi] = dvector(1, rg[fi]->ns*ss[fi]->ns);
 p_c[fi] = dvector(1, rg[fi]->ns*ss[fi]->ns);
 p_ca[fi] = dvector(1, rg[fi]->ns*ss[fi]->ns);

 if(fi==nstep){
```

```
for(fj=1;fj<=rg[fi]->ns*ss[fi]->ns;fj++){
p_s[fi][fj] = 100 + ac[t]->co[ffi]/2;
p_c[fi][fj] = 100 + ac[t]->co[ffi]/2;
p_ca[fi][fj] = 0;
}
}
else{
index = 0;
for(fi1=1;fi1<=rg[fi]->ns;fi1++){
for(fi2=1;fi2<=ss[fi]->ns;fi2++){
index += 1;
ppg[4] = rg[fi]->r[fi1];
ppd[4] = ss[fi]->s[fi2];

pdd = rg[fi]->pd[fi1]*ss[fi]->pd[fi2];
pdm = rg[fi]->pd[fi1]*ss[fi]->pm[fi2];
pdu = rg[fi]->pd[fi1]*ss[fi]->pu[fi2];
pmd = rg[fi]->pm[fi1]*ss[fi]->pd[fi2];
pmm = rg[fi]->pm[fi1]*ss[fi]->pm[fi2];
pmu = rg[fi]->pm[fi1]*ss[fi]->pu[fi2];
pud = rg[fi]->pu[fi1]*ss[fi]->pd[fi2];
pum = rg[fi]->pu[fi1]*ss[fi]->pm[fi2];
puu = rg[fi]->pu[fi1]*ss[fi]->pu[fi2];

p_s[fi][index] = pdd*p_s[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]-
1)];
p_s[fi][index] += pdm*p_s[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2])];
p_s[fi][index] += pdu*p_s[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]+1)];
p_s[fi][index] += pmd*p_s[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]-
1)];
p_s[fi][index] += pmm*p_s[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_s[fi][index] += pmu*p_s[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]+1)];
p_s[fi][index] += pud*p_s[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]-
1)];
p_s[fi][index] += pum*p_s[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2])];
p_s[fi][index] += puu*p_s[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]+1)];
p_s[fi][index] *= df_ss(dt[fi+1], ppg, ppd);
```

```
p_s[fi][index] += ac[t]->co[ffi]/2;

p_c[fi][index] = pdd*p_c[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]-
1)];
p_c[fi][index] += pdm*p_c[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2])];
p_c[fi][index] += pdu*p_c[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]+1)];
p_c[fi][index] += pmd*p_c[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]-
1)];
p_c[fi][index] += pmm*p_c[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_c[fi][index] += pmu*p_c[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]+1)];
p_c[fi][index] += pud*p_c[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]-
1)];
p_c[fi][index] += pum*p_c[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2])];
p_c[fi][index] += puu*p_c[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]+1)];
p_c[fi][index] *= df_ss(dt[fi+1], ppg, ppd);
p_c[fi][index] += ac[t]->co[ffi]/2;

p_ca[fi][index] = pdd*p_ca[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]-
1)];
p_ca[fi][index] += pdm*p_ca[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2])];
p_ca[fi][index] += pdu*p_ca[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]+1)];
p_ca[fi][index] += pmd*p_ca[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]-
1)];
p_ca[fi][index] += pmm*p_ca[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_ca[fi][index] += pmu*p_ca[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_ca[fi][index] += pud*p_ca[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]-
1)];
p_ca[fi][index] += pum*p_ca[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2])];
p_ca[fi][index] += puu*p_ca[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]+1)];
p_ca[fi][index] *= df_ss(dt[fi+1], ppg, ppd);

ex_ind = iscadat(ac[t]->com[ffi][fi], ac[t]->cam[ffi], ac[t]->nca[ffi]);
if( ex_ind != 0 ){
p_c[fi][index] = DMIN(p_c[fi][index], ac[t]->cax[ffi][ex_ind]);
p_ca[fi][index] = p_s[fi][index] - p_c[fi][index];
}
}
```

```
}
}
if(fi!=nstep){
free_dvector(p_s[fi+1], 1, rg[fi+1]->ns * ss[fi+1]->ns);
free_dvector(p_c[fi+1], 1, rg[fi+1]->ns * ss[fi+1]->ns);
free_dvector(p_ca[fi+1], 1, rg[fi+1]->ns * ss[fi+1]->ns);
}
}

 *straight = p_s[0][1]*df_ss(ac[t]->com[ffi][1], pg[t], pd[t]);
 *callable = p_c[0][1]*df_ss(ac[t]->com[ffi][1], pg[t], pd[t]);
 *call = *straight - *callable;

 free_dvector(p_s[0],1,1);free_dvector(p_c[0],1,1);free_dvector(p_ca[0],1,1);

 free(p_s);free(p_c);free(p_ca);
 free_rg(nstep);free_ss(nstep);
 free_dvector(dt, 0, nstep);
 free_dvector(ppg, 1, 4);
 free_dvector(ppd, 1, 4);
}

extern double **pg, **pd, rho;
extern RG **rg;
extern SS **ss;
extern AC **ac;
extern int t;
void pri_cor(int ffi, double *straight, double *callable, double *call)
{
 double **p_s, **p_c, **p_ca, *ppg, *ppd, *dt, eta;
 double pdd, pdm, pdu, pmd, pmm, pmu, pud, pum, puu;
 int fi, fj, fi1, fi2, nstep, index, ex_ind;

 ppg = dvector(1,4);ppd = dvector(1,4);

 for(fi=1;fi<=4;fi++){ppg[fi]=pg[t][fi];ppd[fi]=pd[t][fi];}
```

```
nstep = ac[t]->nco[ffi]-1;
dt = dvector(0, nstep);
for(fi=0;fi<=nstep;fi++) dt[fi] = 0.5;

set_rg(dt, nstep);set_ss(dt, nstep);

eta = rho > 0 ?  rho/36 :  -rho/36;

p_s = (double **) calloc(nstep+1, sizeof(double *));if (!p_s) printf(''failure
in p_s'');
p_c = (double **) calloc(nstep+1, sizeof(double *));if (!p_c) printf(''failure
in p_c'');
p_ca = (double **) calloc(nstep+1, sizeof(double *));if (!p_ca) printf(''failure
in p_ca'');

for(fi=nstep;fi>=0;fi--){
p_s[fi] = dvector(1, rg[fi]->ns*ss[fi]->ns);
p_c[fi] = dvector(1, rg[fi]->ns*ss[fi]->ns);
p_ca[fi] = dvector(1, rg[fi]->ns*ss[fi]->ns);

if(fi==nstep){
for(fj=1;fj<=rg[fi]->ns*ss[fi]->ns;fj++){
p_s[fi][fj] = 100 + ac[t]->co[ffi]/2;
p_c[fi][fj] = 100 + ac[t]->co[ffi]/2;
p_ca[fi][fj] = 0;
}
}
else{
index = 0;
for(fi1=1;fi1<=rg[fi]->ns;fi1++){
for(fi2=1;fi2<=ss[fi]->ns;fi2++){
index += 1;
ppg[4] = rg[fi]->r[fi1];
ppd[4] = ss[fi]->s[fi2];

pdd = rg[fi]->pd[fi1]*ss[fi]->pd[fi2]; pdd += rho > 0 ?  5*eta :  -eta;
```

```
pdm = rg[fi]->pd[fi1]*ss[fi]->pm[fi2]; pdm += rho > 0 ?  -4*eta :  -4*eta;

pdu = rg[fi]->pd[fi1]*ss[fi]->pu[fi2]; pdu += rho > 0 ?  -eta :  +5*eta;

pmd = rg[fi]->pm[fi1]*ss[fi]->pd[fi2]; pmd += rho > 0 ?  -4*eta :  -4*eta;

pmm = rg[fi]->pm[fi1]*ss[fi]->pm[fi2]; pmm += rho > 0 ?  8*eta :  8*eta;

pmu = rg[fi]->pm[fi1]*ss[fi]->pu[fi2]; pmu += rho > 0 ?  -4*eta :  -4*eta;

pud = rg[fi]->pu[fi1]*ss[fi]->pd[fi2]; pud += rho > 0 ?  -eta :  5*eta;

pum = rg[fi]->pu[fi1]*ss[fi]->pm[fi2]; pum += rho > 0 ?  -4*eta :  -4*eta;

puu = rg[fi]->pu[fi1]*ss[fi]->pu[fi2]; puu += rho > 0 ?  5*eta :  -eta;


p_s[fi][index] = pdd*p_s[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]-
1)];
p_s[fi][index] += pdm*p_s[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2])];
p_s[fi][index] += pdu*p_s[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]+1)];
p_s[fi][index] += pmd*p_s[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]-
1)];
p_s[fi][index] += pmm*p_s[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_s[fi][index] += pmu*p_s[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]+1)];
p_s[fi][index] += pud*p_s[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]-
1)];
p_s[fi][index] += pum*p_s[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2])];
p_s[fi][index] += puu*p_s[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]+1)];
p_s[fi][index] *= df_ss(dt[fi+1], ppg, ppd);
p_s[fi][index] += ac[t]->co[ffi]/2;

p_c[fi][index] = pdd*p_c[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]-
1)];
p_c[fi][index] += pdm*p_c[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2])];
p_c[fi][index] += pdu*p_c[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]+1)];
```

```
p_c[fi][index] += pmd*p_c[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]-
1)];
p_c[fi][index] += pmm*p_c[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_c[fi][index] += pmu*p_c[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]+1)];
p_c[fi][index] += pud*p_c[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]-
1)];
p_c[fi][index] += pum*p_c[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2])];
p_c[fi][index] += puu*p_c[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]+1)];
p_c[fi][index] *= df_ss(dt[fi+1], ppg, ppd);
p_c[fi][index] += ac[t]->co[ffi]/2;

p_ca[fi][index] = pdd*p_ca[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]-
1)];
p_ca[fi][index] += pdm*p_ca[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2])];
p_ca[fi][index] += pdu*p_ca[fi+1][(rg[fi]->midi[fi1]-1)*(ss[fi]->midi[fi2]+1)];
p_ca[fi][index] += pmd*p_ca[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2]-
1)];
p_ca[fi][index] += pmm*p_ca[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_ca[fi][index] += pmu*p_ca[fi+1][(rg[fi]->midi[fi1])*(ss[fi]->midi[fi2])];
p_ca[fi][index] += pud*p_ca[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]-
1)];
p_ca[fi][index] += pum*p_ca[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2])];
p_ca[fi][index] += puu*p_ca[fi+1][(rg[fi]->midi[fi1]+1)*(ss[fi]->midi[fi2]+1)];
p_ca[fi][index] *= df_ss(dt[fi+1], ppg, ppd);

ex_ind = iscadat(ac[t]->com[ffi][fi], ac[t]->cam[ffi], ac[t]->nca[ffi]);
if( ex_ind != 0 ){
p_c[fi][index] = DMIN(p_c[fi][index], ac[t]->cax[ffi][ex_ind]);
p_ca[fi][index] = p_s[fi][index] - p_c[fi][index];
}
}
}
}
if(fi!=nstep){
free_dvector(p_s[fi+1], 1, rg[fi+1]->ns * ss[fi+1]->ns);
free_dvector(p_c[fi+1], 1, rg[fi+1]->ns * ss[fi+1]->ns);
free_dvector(p_ca[fi+1], 1, rg[fi+1]->ns * ss[fi+1]->ns);
```

```
 }
 }

 *straight = p_s[0][1]*df_ss(ac[t]->com[ffi][1], pg[t], pd[t]);
 *callable = p_c[0][1]*df_ss(ac[t]->com[ffi][1], pg[t], pd[t]);
 *call = *straight - *callable;
 free_dvector(p_s[0],1,1);free_dvector(p_c[0],1,1);free_dvector(p_ca[0],1,1);

 free(p_s);free(p_c);free(p_ca);free_rg(nstep);free_ss(nstep);
 free_dvector(dt,0,nstep);free_dvector(ppg,1,4);free_dvector(ppd,1,4);
}
```

Notation

$R_t(x)$	Recovery rate at time t
$w_t(x) = 1 - R_t(x)$	Loss rate at time t
$B_{t,T}$	Zero coupon bond at time t maturing at time T
$P_{t,T}$	Coupon bond at time tmaturing at time T
$f_{t,T}$	Instantaneous forward rate at time t for maturity T
r_t	Instantaneous spot rate at time t
$\lambda_{t,T}$	Instantaneous forward probability of default at time t for maturity T
λ_t	Instantaneous spot probability of default at time t
$L_{t,T}$	Money market rate at time t for maturity T
$Y_{t,T}$	Zero coupon yield at time t maturing at time T
Z_t^*	Density process defining the spot risk measure
Z_t^T	Density process defining the T-maturity forward risk measure
C_t	Correlation matrix
M_t^c	Continous martingale at time t
M_t^d	Discontinous martingale at time t
D_t	Proportional dividend process
$\delta_{t,T}$	Interest accrual factor for a money market deposit at time t maturin
F_t	Forward price at time t
H_t	Futures price at time t
c_t	Coupon at time t
P	Pricipal

V_t	Value of claim at time t, including contractual and non-contractual payoffs
Φ_t	Distribution of random variables determining the recovery rate outcome
θ^c	Market price of risk for continuous risk sources
θ^d	Market price of risk for discontinuous risk sources
y_t	Convenience yield accruing to investors with long positions in the treasury asset
N_t	Doubly stochastic Poisson process indicating the time of default
τ	Time of default
F_t^X	Filtration at time t generated by the random variable X
$\rho_t^{i,j}$	Instantaneous correlation between variables i and j
Q_t	Cholesky decomposition of correlation matrix C_t
h_t	Function capturing time dependency in recovery rates
P	Empirical probability measure
P^*	Probability measure associated with the money market deposit numeraire
P^T	Probability measure associated with the discount bond with maurity T as numeraire

X_t	Conditional current exposure at time t
$X^e_{t,T}$	Conditional expected exposure at time T
$X^m_{t,T}$	Conditional maximal exposure at time T
X^n_t	Conditional current net exposure at time t
$X^{ne}_{t,T}$	Conditional net expected exposure at time T
$X^{nm}_{t,T}$	Conditional conditional net maximal exposure at time T
X^c_t	Conditional current collateralized exposure at time t
$X^{ce}_{t,T}$	Conditional collateralized expected exposure at time T
$X^{cm}_{t,T}$	Conditional collateralized maximal exposure at time T
$X^{nc}_{t,T}$	Conditional collateralized net exposure at time T
$X^{nce}_{t,T}$	Conditional collateralized net expected exposure at time T
$X^{ncm}_{t,T}$	Conditional collateralized net maximal exposure at time T
$L^e_{t,T}$	Conditional expected loss at time T
$L^m_{t,T}$	Conditional maximal loss at time T
$L^{ne}_{t,T}$	Conditional net expected loss at time T
$L^{nm}_{t,T}$	Conditional net maximal loss at time T
$L^{ce}_{t,T}$	Conditional expected collateralized loss at time T
$L^{cm}_{t,T}$	Conditional maximal collateralized loss at time T
$L^{nce}_{t,T}$	Conditional net expected collateralized loss at time T
$L^{ncm}_{t,T}$	Conditional net maximal collateralized loss at time T