Advance Selling to Ease Financial Distress

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Left unable to provide service during the COVID-19 pandemic, many small businesses have experimented with alternative ways of generating income. One approach that has gained traction is the use of advance selling, whereby the firm asks consumers in its local community to support the business by paying in advance for consumption at a future date. In this paper, we develop a game theoretic model to investigate whether and how advance selling schemes can be successfully implemented by firms facing financial distress. In cases of high distress (i.e., where obtaining bank financing is infeasible given the firm's financial need), we show that advance selling in its classic implementation can help the firm secure its survival in some scenarios, but may suffer from significant inefficiencies associated with strategic consumer behavior and firm moral hazard. We demonstrate that two modifications of the classic scheme currently observed in practice—namely, (i) the introduction of an "all-or-nothing" clause, and (ii) selling future discount coupons as opposed to the full service—can expand the set of scenarios in which survival is ensured, while also allowing the firm to extract higher profit. In cases of moderate financial distress (i.e., where bank financing is a feasible but inefficient option), we find that simple advance selling schemes typically fail to make an impact. However, we show that a more complex scheme, combining both of the aforementioned modifications simultaneously, can be used in conjunction with bank financing to generate a substantial improvement in firm profit.

Key words: OM-Finance interface, social operations management, advance selling, strategic consumers, moral hazard

1. Introduction

The global COVID-19 pandemic has led to the permanent closure of many small business, and has left many more on the brink of bankruptcy.[1] The challenge faced by these businesses is unprecedented: while lockdowns and other restrictions have severely limited their ability to generate income by serving their customers, expenses such as rent, employee salaries and other fixed operating costs continue to pile up. Faced with the prospect of bankruptcy, business owners have been forced to

[1] According to the National Restaurant Association, approximately 17% of restaurants in the United States have shut their doors permanently or long-term (National Restaurant Association 2020).

experiment with various approaches to raising capital, in an attempt to keep their businesses afloat until restrictions are eased and consumers return for service.

One such approach is the practice of *advance selling*, whereby a business sells its service in advance for consumption at a future date. The idea is simple: By selling service in advance, the business gains access to much-needed capital today, while the consumers, by helping the business avoid bankruptcy, maintain the ability to enjoy service in the future. The restaurant industry offers a prime example of such efforts. In Germany, the non-profit initiative "Pay Now Eat Later" allows consumers to purchase meal tickets from their favorite local restaurants, which can be redeemed once normal business resumes (PayNowEatLater 2023). In India, the "Rise For Restaurants" program facilitates the purchase of future discount coupons for a modest upfront fee (Vaswani 2020). And in the United States, many restaurant owners have turned to popular crowdfunding platforms such as GoFundMe and Kickstarters in an effort to raise funds to cover employee salaries and operating expenses during lockdowns (Roe and Smith 2021, Yang and Koh 2022).

While all of the aforementioned schemes share the same goal of allowing small businesses to lean on their consumers for financial support, it is interesting to observe the differences in their implementation. Some schemes conduct advance selling in its classic form (full upfront payment for future service), while others divide consumer payments into two components (one paid upfront and one at the time of consumption); some schemes reward participating consumers with a future price discount, while others simply ask for donations without any reward; and some schemes utilize crowdfunding platforms, adopting a threshold funding approach, while others raise funds through their own websites and keep whatever they can raise. As the experimentation with various versions of advance selling continues to unfold in practice, our goal in this paper is to generate some basic insights regarding the use and optimal implementation of advance selling programs aiming to ease financial distress.

To do so, we study a modified version of the classic model of credit rationing presented in Tirole (2009). In the classic model, a firm faces financial distress, requiring a fixed amount of capital to avoid bankruptcy, and can only secure the necessary funds through a bank loan. If successful in securing the loan, the firm exerts costly effort in order to attract consumers and repay the bank. This model highlights the effects of moral hazard on the interaction between the firm and the bank, identifying three qualitatively different cases: (i) when the firm's financial distress is low, the firm is able to secure a bank loan at a zero (i.e., normalized risk-free) interest rate and subsequently exerts an efficient level effort; (ii) when the firm's financial distress is moderate, the firm is able to secure a loan, albeit at a positive interest rate, and subsequently exerts a less-than-efficient level of effort; and (iii) when the firm's financial distress is high, the firm is credit-rationed and goes bankrupt. We modify the classic model by allowing the firm to conduct an advance selling campaign to raise funds which can be used either instead of, or in combination with, a bank loan. Our analysis focuses on establishing whether and how such a campaign can improve the firm's financial outlook. Our results are summarized as follows.

First, we analyze cases of high financial distress where, in the absence of a successful advance selling scheme, the firm is unable to secure bank financing and goes out of business. We begin by considering a classic advance selling scheme, whereby the firm attempts to sell service to its consumers (potentially at a discount) before the time of their consumption. We show that such a scheme can help the firm survive when the consumers' valuation for the firm's service is sufficiently high. However, we observe that this approach suffers from consumer free-riding and induces firm moral hazard, two effects which taken together severely restrict the firm's ability to generate profit.

We then consider two modifications of the classic scheme, motivated by practical observations of advance selling programs implemented during the pandemic. The first modification is the addition of an "all-or-nothing" clause, similar to the threshold mechanism encountered in many popular crowdfunding platforms, while the second entails breaking up the consumers' payments into two parts, one paid in advance and one in spot (similar to the approach of selling future discount coupons or simply requesting donations in advance). We show that under both modifications, the firm can combine funds from advance selling with a bank loan to increase its equilibrium profit and expand the set of scenarios in which survival is ensured. In comparing the modified versions of advance selling, we find that the all-or-nothing mechanism tends to be the most profitable approach for the firm, apart from cases in which moral hazard is a significant concern for the consumers and at the same time the consumers' valuations for service are moderate; in such cases, we show that the preferred approach is a coupon mechanism consisting of simple donations.

Next, we consider the more complex case of moderate financial distress. In this case, even in the absence of funds raised through advance selling, the firm maintains a feasible option for survival in the form of a bank loan. Our model highlights that the availability of this option makes it harder for the firm to design a beneficial advance selling scheme; in particular, the consumers, realizing that the firm will be able to secure survival even without their help, are significantly less willing to participate in the firm's advance selling scheme. As a result, we find that in most cases neither the classic advance selling scheme, nor any of the two modifications described above can help improve the firm's position.

However, our analysis of the shortcomings of the modified schemes suggests that the approach of incorporating both modifications simultaneously (namely, a scheme consisting of two payments with an all-or-nothing clause applied to the advance payment) might provide a solution. Indeed, we demonstrate that a scheme of this kind allows the firm to achieve a substantial profit increase by combining crowdsourced funds with a significantly reduced bank loan. We find, in particular, that when the consumers' valuation for the firm's service is moderate, the optimal financing approach combines bank financing with an advance selling scheme taking the form of a "threshold discount" (whereby consumers receive a discount provided enough of their peers participate in the program). Surprisingly, we further establish that in cases where advance selling is beneficial, the firm's profit can be non-monotone in its financing need, which implies that the firm can in fact be better off in cases of higher financial distress.

The rest of the paper is organized as follows. In §2 we review the related literature. In §3 we analyze the benchmark model of bank financing without advance selling and in §4 we investigate the use of advance selling to improve the firm's financial position. §5 concludes.

2. Related literature

This paper relates primarily to two active streams of literature: (i) the literature that analyzes the use of advance selling in various contexts; and (ii) the literature that studies issues on the interface between OM and Finance.

The literature on advance selling focuses predominantly on highlighting the various economics benefits of such programs. Earlier work in this area, including Gale and Holmes (1993), Dana (1998), and Desiraju and Shugan (1999), focuses on the role of advance selling in achieving price discrimination. DeGraba (1995) shows that advance selling can be profitable for the seller when facing consumers with ex ante unknown valuations. Xie and Shugan (2001), Swinney (2011), and Yu et al. (2015) further enrich this theory by examining the implications of capacity constraints, operational flexibility, and consumer valuation interdependence. Png (1989), Liu and van Ryzin (2008) and Nasiry and Popescu (2012) examine the benefit of advance selling when consumers are averse to availability risk, while Tang et al. (2004), Prasad et al. (2011), and Li and Zhang (2013) show that advance selling may induce more efficient operational decisions by generating early demand information. Other papers in this literature investigate the interaction between advance selling and other market characteristics, such as competition (McCardle et al. 2004), sellers' private information on product quality (Yu et al. 2014), and social learning (Papanastasiou and Savva 2017). Our paper contributes to this literature by identifying a novel use of advance selling, namely, that of helping the firm avoid bankruptcy in cases of financial distress. As our analysis demonstrates, conducting advance selling in such cases presents additional challenges, and thus calls for innovative forms of implementation.

This work also relates to the literature on reward-based crowdfunding, which can be viewed as a form of advance selling for new-to-the-world products. Chakraborty and Swinney (2021) characterize how pricing and the choice of funding target can be used to signal product quality in crowdfunding campaigns. Alaei et al. (2022), Du et al. (2022), and Zhang et al. (2017) study the dynamic aspects of backer behavior in crowdfunding campaigns. Babich et al. (2021) examine the interaction between crowdfunding and venture capital financing in the presence of double-sided moral hazard, whereas Xu et al. (2020) focus on the implications of social learning for crowdfunding. Strausz (2017) studies the design of crowdfunding mechanisms to counteract possible funds misappropriation by the entrepreneur, while Belavina et al. (2020) further advance this theory by adding insights relating to performance opacity and the interaction between the two. In this paper, we aim to develop insights on how to optimally combine crowdsourced funds with bank financing, as opposed to choosing between the two or optimizing the crowdfunding process in isolation.

The literature on the interface of operations management and finance focuses on the interplay between firms' operational and financial decisions (e.g., Babich and Sobel 2004, Gaur and Seshadri 2005, Kazaz et al. 2005, Swinney and Netessine 2009, Boyabatlı and Toktay 2011, Dong and Tomlin 2012, Lai et al. 2012, Yang et al. 2015, Turcic et al. 2015, Alan and Gaur 2018, de Véricourt and Gromb 2018, Ning and Babich 2018, and Luo and Shang 2019). From this literature, our work is most related to studies focusing on financing means that leverage supply-chain interactions, such as trade credit (Kouvelis and Zhao 2012, Yang and Birge 2018), receivable financing (Tunca and Zhu 2018, Kouvelis and Xu 2021), logistic financing (Chen et al. 2018), pre-shipment financing (Tang et al. 2018, Reindorp et al. 2018) and peer-to-peer lending (Gao et al. 2018). Our paper adds to this line of work by focusing on financing through a group of consumers, each with their own individual incentives, as opposed to through larger and more concentrated supply-chain partners. Our analysis suggests that in this case, customers' interactions and strategic behavior play an important role in the firm's ability to secure financing. In this respect, our paper is also related to Birge et al. (2017), who consider inventory and pricing decisions to mitigate the negative impact of consumers' strategic waiting behavior, albeit assuming that the firm already has financing in place.

3. Financing Without Advance Selling

We begin by analyzing a simple model of bank financing without advance selling. The results of this section establish a baseline performance against which the use of advance selling can be subsequently compared.

3.1. Model Description

We consider a fixed investment model of credit rationing in the presence of moral hazard, as presented in Tirole (2009), Chapter 3. There is a small capital-constrained firm (e.g., coffee shop, restaurant, dry cleaner, grocery store) that sells a good or service to a local community at a regular price p_r and a marginal cost normalized to zero. The firm is in financial distress and requires external funding at level $I \geq 0$ in order to continue operating, where I may represent rental costs, employee salaries, necessary repairs, maintenance, or other operating expenses.[2] We segment time into two representative periods, indexed by $t \in \{f, r\}$. Period $t = f$ is the "financing period," during which the firm seeks to raise the necessary funds in the form of a bank loan. If the firm fails to secure a loan, it goes bankrupt. If the firm obtains financing, it proceeds to the "repayment period," during which it experiences demand for its service, and either repays the bank in full (if total revenues exceed the loan plus interest) or goes bankrupt (if not). The financial market is assumed to be perfectly competitive (i.e., the bank loan is fairly priced) and the risk-free interest rate is normalized to zero. Moreover, the firm bears limited liability for the loan and is assumed to be creditworthy (i.e., the firm always repays the loan to the extent possible).

In the repayment period, we assume that with probability $1 - \beta \in [0, 1]$ the firm is unable to generate revenue due to factors outside of the firm's control (e.g., further pandemic-related closures) and is forced into bankruptcy. If the firm is able to resume operations (i.e., with probability β), the market demand the firm experiences in the repayment period depends on an unverifiable (for the bank) level of effort exerted by the firm; for instance, this may represent direct effort exerted to provide a high-quality service, or the time the business owner spends on improving the firm's operations (e.g., by searching for better staff, revising the menu, supervising employees, etc.) versus the time she spends on alternative activities. In particular, we assume that a level of effort $e \in [0, 1]$ results in a binary service quality outcome $z \in \{s, n\}$, representing "success" or "no success," with a higher level of effort resulting in a higher probability of success, but incurring a higher cost; for simplicity, we further assume that exerting effort e results in a probability of success $P(z = s \mid e) = e$ and comes at a cost $c(e) = ae^2$. The outcome of the firm's effort is probabilistically related to the demand it experiences in the repayment period. In particular, we assume that there are two customer segments indexed by $j = \{i, o\}$, each consisting of infinitesimally small consumers with total mass m_j, where we normalize $m_i = 1$.[3] Segment-i represents the firm's "inner circle" of loyal/regular customers, while segment o represents the more general "outer" market. During the repayment period, each customer has a stochastic valuation for service $V_j \in \{0, v_j\}$ where $v_i = v \geq 1 = v_o$ and $P(V_j = v_j \mid z) = \lambda^z$. We assume that $\lambda^s > \lambda^n$ (i.e., a successful effort outcome results in higher expected demand); moreover, to simplify the exposition, throughout the analysis we set $\lambda^s = 1$ and $\lambda^n = \lambda \in (0, 1)$.

[2] During the COVID-19 pandemic, governments implemented various subsidies (e.g., the paycheck protection program in the US) to alleviate the financial pressure faced by small businesses. In our model, such a subsidy can be captured by reducing the level of financial distress I.

[3] That is, we assume that the unit mass of i-type consumers consists of k individuals each having a mass $1/k$, and we conduct our analysis in the limit $k \to \infty$.

The sequence of events is as follows. In the financing period, the firm applies for a bank loan at level I, and the bank determines whether to grant the loan as well as the interest rate r (if the loan is to be granted). If the loan is secured, the firm moves to the repayment period. With probability β, the firm is able to resume operations and chooses a level of effort e and the selling price p_r. The effort outcome $z \in \{s, n\}$ is then realized along with the corresponding demand (parameterized by λ_j^z, $z \in \{s, n\}$ and $j \in \{i, o\}$).[4] Finally, the firm uses its sales revenues to pay back the loan (to the extent possible) and any leftover revenues are retained as profit.

Before proceeding with the analysis, we place the following assumptions on our model parameters. Define $v_m = \max\{(m_i + m_o)v_o, m_i v_i\} = \max\{1 + m_o, v\}$ (i.e., v_m is the maximum achievable revenue in the repayment period).

Assumption 1. $a \geq \frac{(1-\lambda)v_m}{2}$.

Assumption 2. $I < I_{\max} = \beta\left[\lambda v_m + \frac{(1-\lambda)^2 v_m^2}{4a}\right]$.

The first assumption ensures that the cost of effort is sufficiently high to rule out boundary solutions with $e = 1$. The second assumption ensures that, in the absence of a financial constraint, the firm's business represents a risky investment, but has a positive net present value.

Throughout the analysis that follows, we focus on Pareto equilibria in pure symmetric strategies. Moreover, we say that a particular approach to firm financing *dominates* another whenever the former results in higher expected profit for the firm.

3.2. Analysis

We solve the benchmark game between the bank and the firm via backward induction. First, we solve for the firm's optimal selling price in the repayment period. We next derive the firm's optimal level of effort under a loan with a given interest rate, and then solve for the bank's loan decision and interest rate. Let

$$I_l = \beta\lambda v_m \text{ and } I_h = I_l + \frac{\beta(1-\lambda)^2 v_m^2}{8a}.$$

Bypassing the technical details, the unique equilibrium of the game is described in Proposition 1.

Proposition 1. *Without advance selling, the equilibrium selling price is*

$$p_r^* = \begin{cases} v & \text{if } v \geq 1 + m_o, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover:

[4] Alternatively, p_r can be chosen after the realization of the effort outcome; this has no impact on our results.

(i) When $I \leq I_l$, the equilibrium interest rate, effort, and firm profit are given by

$$r^*_{FB} = \frac{1}{\beta} - 1, \; e^*_{FB} = \frac{(1-\lambda)v_m}{2a}, \; \text{and } \pi^*_{FB} = \beta\left[\frac{(1-\lambda)^2 v_m^2}{4a} + \lambda v_m\right] - I.$$

(ii) When $I_l < I \leq I_h$, the equilibrium interest rate, effort, and firm profit are given by

$$r^*_B = \frac{(1+\lambda)v_m - \sqrt{(1-\lambda)^2 v_m^2 - 8a\left(\frac{I}{\beta} - \lambda v_m\right)}}{2I} - 1 \geq r^*_{FB},$$

$$e^*_B = \frac{(1-\lambda)v_m + \sqrt{(1-\lambda)^2 v_m^2 - 8a\left(\frac{I}{\beta} - \lambda v_m\right)}}{4a} \leq e^*_{FB}, \; \text{and}$$

$$\pi^*_B = \beta a \left(e^*_B\right)^2 \leq \pi^*_{FB}$$

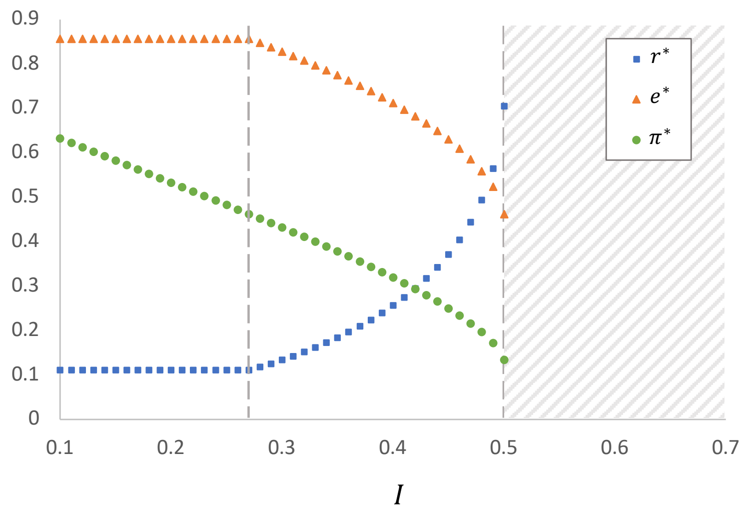
(iii) When $I > I_h$, the firm fails to secure a bank loan and goes bankrupt.

The result is illustrated in Figure 1. Proposition 1 highlights three qualitatively different cases, depending on the level of the firm's financial distress. When the firm is under low distress (i.e., $I \leq I_l$), the loan repayment can be fully guaranteed by the firm's sales revenue, and the loan is risky only with respect to the firm's ability to resume operations—this is reflected in the bank's interest rate, which depends only on β and is otherwise risk-free. In our model, the case of low distress serves as the first-best case, since the firm effectively behaves as if there is no financial constraint, exerting an efficient level of effort. When the firm is under moderate distress (i.e., $I_l < I \leq I_h$), the bank loan cannot always be fully guaranteed by the firm's sales revenue and is therefore risky also with respect to the demand experienced by the firm in the repayment period. In this case, the bank grants the loan at a higher interest rate to balance against scenarios where the firm is unable to repay the funds in full. Note that under bank financing, the firm's incentive to exert effort is weaker, because the firm only extracts part of the benefit of exerting this effort (when the experienced demand is low, in particular, the firm goes bankrupt and all of the collected revenues go to the bank). Notice also that, as the firm's financing need I increases within this region, the interest rate increases, the firm's effort decreases, and the firm's profit also decreases. Finally, when the firm is under high financial distress (i.e., $I > I_h$), the loan becomes prohibitively risky for the bank, and the firm is credit-rationed.

4. Financing With Advance Selling

In cases where bank financing is either infeasible or inefficient, the firm may be able to secure some or all of the funds it requires by conducting advance selling. Rather than applying for a bank loan for the full amount needed, the firm first attempts to raise funds by selling service to consumers in advance of their consumption date. In this section, we modify the base model of §3 by allowing the firm to conduct advance selling in the financing period to customers belonging to the "inner

Figure 1 **Equilibrium interest rate r^*, effort e^* and firm profit π^*, as a function of the firm's funding need I in the benchmark model without advance selling. Parameter values:** $a = 0.7$, $\lambda = 0.2$, $v = 1.5$, $\beta = 0.9$, $m_o = 0.5$ **(note also that** $I_l = 0.27$, $I_h = 0.50$, **and** $I_{max} = 0.73$**).**



circle". These segment-i consumers are assumed to be forward-looking and strategic: in deciding whether to purchase in advance, they compare their expected utility from doing so against the expected utility of waiting until the repayment period (at which time they may purchase in the spot market, or not purchase at all). Under any specific advance selling scheme employed by the firm, each customer's expected utility from purchasing in advance versus waiting depends not only on their own action, but also on the actions of the other consumers, as well as the firm's actions in the repayment period (i.e., effort level and spot price). Our analysis focuses on symmetric, pure strategy, rational expectations equilibria. Thus, in equilibrium, segment-i customers will either all purchase in advance or will all wait. For a purchasing equilibrium to be established, we require the necessary and sufficient conditions that (i) all customers purchasing in advance is an equilibrium (i.e., no individual consumer can benefit from waiting when all others purchase in advance), and (ii) all customers waiting is either not an equilibrium, or results in lower expected utility for the consumers as compared to the purchase equilibrium.[5]

The funds secured by the firm through advance selling are then used either in conjunction with, or instead of, a bank loan. Our analysis considers the classic version of advance selling scheme as well as the two main modifications observed in practice during the COVID-19 pandemic.

We note that it is straightforward to show that the first-best case of low financial distress ($I \leq I_l$) cannot be improved upon. Thus, in the analysis that follows, we focus on the remaining

[5] We present the technical versions of these necessary and sufficient conditions in Lemma A.1.

qualitatively different cases of moderate ($I_l < I \leq I_h$) and high ($I > I_h$) financial distress, starting with the simpler analytically, albeit more precarious for the firm, case of high distress.

4.1. High Financial Distress ($I > I_h$)

We analyze first the case of high financial distress, which is defined by $I > I_h$ (see Proposition 1). The defining characteristic of the high financial distress case is that, without any funds raised through advance selling, the firm is unable to secure a bank loan and is forced to declare bankruptcy.

4.1.1. Advance Selling at Full Price ("F"). We begin the analysis of high financial distress by considering advance selling in its classic implementation, referred to here as advance selling at full price (abbreviated "F"). In the financing period, the firm chooses an advance selling price p_{aF} and the segment-i customers choose whether or not to purchase in advance. If the firm is able to secure enough funds in advance to survive, and assuming the firm is able to resume operations, it then exerts effort and chooses the regular selling price p_{rF}.

Define the valuation threshold

$$\overline{v}_F := \frac{I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]}.$$

Our first result characterizes the conditions under which advance selling at full price can help the firm avoid bankruptcy, and describes the scheme's optimal implementation.

PROPOSITION 2. *Suppose $I > I_h$. Under advance selling at full price ("F"):*

(i) If $v \geq \overline{v}_F$, the firm conducts advance selling at price $p^*_{aF} = I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]$ and sets the regular price to $p^*_{rF} = 1$. The firm's equilibrium effort and profit are given by

$$e^*_F = \frac{(1-\lambda)m_o}{4a} \text{ and } \pi^*_F = \frac{\beta(1-\lambda)^2 m_o^2}{16a}.$$

(ii) If $v < \overline{v}_F$, the firm fails to survive.

Recall that without a successful advance selling scheme, in cases of high financial distress the firm fails to secure bank financing and goes bankrupt (see Proposition 1). Proposition 2 suggests that, provided the segment-i consumers' valuation for the firm's service is sufficiently high, the firm can secure part of the funds it needs through advance selling and source the remainder of the funds from the bank, ensuring that the firm avoids bankruptcy.[6]

Although advance selling in its classic implementation can help the firm survive in certain scenarios, Proposition 2 also highlights two significant sources of inefficiency. The first is the firm's

[6] To see that only part of the required funds are raised through advance selling, observe that the advance selling price is strictly less than I, while the mass of segment-i consumers is one.

inability to conduct advance selling at a price which better reflects the consumers' valuation for service. Indeed, observe that the expression for p_{aF}^* is unrelated to the consumers' valuation v and is instead tied to the minimum amount of funding necessary for the firm to survive. This is a direct result of strategic consumer behavior. To see how this behavior restricts the firm's ability to raise funds, suppose that the firm in equilibrium was able to charge a higher advance selling price (thus securing more than the minimum amount of funding necessary for survival). Since the contribution of each individual consumer is small on its own, any given consumer could deviate from participating in the advance selling campaign without putting the firm's survival in jeopardy; at the same time, for the consumer, deviating would come with the benefit of observing her realized valuation for service (recall that $V_i \in \{0, v\}$) before making a purchase decision. It follows that such an equilibrium cannot be sustained; instead, the only possible equilibrium is one where the funds raised through advance selling are the minimum required to secure the firm's survival, and are otherwise unrelated to the consumers' willingness to pay for the firm's service.[7]

The second source of inefficiency is the firm's decreased level of effort (we note that $e_F^* < e_{FB}^*$, where e_{FB}^* is the efficient level of effort described in Proposition 1). The decrease in effort occurs because, having sold service to consumers in advance, the firm is left with less of an incentive to exert effort resulting in high-quality service in the repayment period. Indeed, observe that in equilibrium the firm's effort is driven exclusively by the potential to generate additional revenue in the repayment period by selling service to the broader market (i.e., the equilibrium effort is directly proportional to the mass of segment-o consumers). This implies that in extreme cases where the firm's demand consists only of segment-i consumers (e.g., a neighborhood dry-cleaning service), the effort in the repayment period drops to zero and so does the firm's profit (although it is important to note that even in these cases advance selling remains beneficial for the firm, in that it helps ensure the firm's survival).

In summary, the preceding analysis suggests that while advance selling in its classic implementation can help ensure the firm's survival (provided the consumers' valuation for service is sufficiently high), the scheme suffers from inefficiencies associated with strategic behavior on the consumer side and moral hazard on the firm side. In §4.1.2 and §4.1.3, we demonstrate how two modifications of the classic approach which have been observed in practice may help address these limitations, improving the firm's position. We then perform a comparison between the three schemes in §4.1.4.

[7] Note that our assumption of infinitesimally small consumers does not play a critical role in this argument: If individual consumers had nonzero mass, the maximum amount the firm could raise under this mechanism would be instead constrained to be no more than the minimum amount required plus the contribution of a single consumer; otherwise, deviation would be profitable for any consumer following the same logic as described in the main text.

4.1.2. Modification I: Advance Selling with an All-or-Nothing Clause ("A"). We consider next a modified version of advance selling, where the firm adds an "all-or-nothing" clause to the scheme (abbreviated "A"), similar to those encountered in popular crowdfunding platforms. Under this scheme, the firm chooses an advance selling price p_{aA}, but the firm effectively commits to refund the consumers' advance purchases in the event that the firm falls short of its funding goal.[8]

Define the two valuation thresholds $\underline{v}_A$ and $\overline{v}_A$, where

$$\underline{v}_A := \frac{2\sqrt{4a^2\lambda^2 + \frac{2a(1-\lambda)^2 I}{\beta}} - 4a\lambda}{(1-\lambda)^2} - m_o \text{ and } \overline{v}_A := \frac{I - \beta\lambda m_o}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]},$$

and the threshold on the size of the outer market

$$M_A := \frac{\sqrt{4a^2\lambda^2 + \frac{2a(1-\lambda)^2 I}{\beta}} - 2a\lambda}{(1-\lambda)^2}$$

Our next result describes the optimal implementation of advance selling and the resulting equilibrium when the firm utilizes an "all-or-nothing" clause.

PROPOSITION 3. *Suppose $I > I_h$. Under advance selling with an all-or-nothing clause ("A"):*

(i) If $v \geq \overline{v}_A$, the firm conducts advance selling at price $p^*_{aA} = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]$ and sets the regular selling price at $p^*_{rA} = 1$. The firm's equilibrium effort and profit are given by

$$e^*_A = \frac{(1-\lambda)m_o}{2a} \text{ and } \pi^*_A = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right] + \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right] - I.$$

(ii) If $\underline{v}_A \leq v < \overline{v}_A$ and $m_o > M_A$, the firm conducts advance selling at price

$$p^*_{aA} = \frac{\beta v\left[(1-\lambda)^2(v+m_o) + 4a\lambda\right] + \beta v(1-\lambda)\sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}$$

and sets the regular selling price at $p^*_{rA} = 1$. The firm's equilibrium effort and profit are

$$e^*_A = \frac{(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a} \text{ and } \pi^*_A = \beta a(e^*_A)^2.$$

(iii) In all other cases, the firm goes bankrupt.

[8] We note that it is possible to further optimize such a scheme by optimizing the advance selling quantity. Here, our focus is to illustrate the qualitative properties of all-or-nothing schemes, which apply also when the quantity is optimized. For a more detailed analysis of such schemes see Astashkina and Marinesi (2022).

Proposition 3 describes two scenarios in which advance selling with an all-or-nothing clause can help the firm. We first point out that the two scenarios share a common feature: the addition of the all-or-nothing clause allows the firm to charge an advance selling price which is related to the consumers' valuation for service v, instead of the firm's financing need I (as was the case in Proposition 2). In particular, the introduction of the all-or-nothing clause implies that if any individual consumer were to deviate from purchasing in advance, the entire advance selling scheme would collapse, forcing the firm to declare bankruptcy. As a result, each consumer is now willing to participate in the scheme up to a price which reflects her expected surplus, conditional on the firm's survival and subsequent equilibrium effort (since the alternative of deviating results in firm bankruptcy and thus zero surplus). In this way, the all-or-nothing clause significantly mitigates the inefficiency identified under scheme "A" with respect to strategic consumer behavior.

Next, we note that the two cases of Proposition 3 differ in terms of the firm's equilibrium effort and profit. In the first case, the consumers' valuation v is sufficiently high ($v \geq \overline{v}_A$) so that a successful advance selling campaign either raises enough funds for the firm to survive without using a bank loan (this occurs when v is very high), or allows the firm to secure a loan from the bank at the first-best interest rate (this occurs when v is moderately high). It is worthwhile to note here that although the loan is free from default risk (apart from the exogenous market risk captured by β), the resulting equilibrium effort is lower than the first best effort e^*_{FB}, because the consumers' advance purchases reduces the firm's incentive to exert effort.

In the second case, the consumers' valuation is not as high ($\underline{v}_A \leq v \leq \overline{v}_A$), but the firm enjoys good potential to generate revenue in the repayment period due to the large size of the outer market ($m_o > M_A$). The combination of these two conditions implies that the firm can raise enough funds through advance selling for the bank to be willing to grant a loan, but the loan in this case is risky: the firm will be able to repay the bank in full only if the effort outcome is a success. In turn, this causes the bank to charge a higher interest rate (see Lemma A.2 for the corresponding analytical expression) and the firm to exert a further decreased level of effort.

4.1.3. Modification II: Advance Selling with Discount Coupons ("C"). The next modification we consider is one where the firm sells discount coupons in advance (abbreviated "C"), as opposed to selling its service in full. This approach attempts to break up the segment-i consumers' payments into two components, a payment p_{aC} which is paid in advance, and a spot payment p_{sC} which is paid at the time of consumption (in the event that the consumer chooses to seek service). Under this scheme, the firm first announces the coupon price plan $\{p_{aC}, p_{sC}\}$ and consumers choose whether to purchase in advance. Then, assuming the firm is able to raise the funds it needs, the firm chooses an effort level and the regular price $p_{rC} \geq p_{sC}$, which applies to consumers who do not possess a coupon.

Define the valuation threshold

$$\overline{v}_C := \frac{I - \beta\lambda m_o + \frac{\beta(1-\lambda)^2(1-m_o^2)}{8a}}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{4a}\right]}.$$

The following result describes the optimal advance selling scheme and the resulting equilibrium when the firm employs advance selling with coupons.

PROPOSITION 4. *Suppose $I > I_h$. Under advance selling with discount coupons ("C"):*

(i) If $v \geq \overline{v}_C$, the firm conducts advance selling at a coupon price $p^*_{aC} = I - \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$ and a spot price $p^*_{sC} = 1$, and sets the regular price to $p^*_{rC} = 1$. The firm's equilibrium effort and profit are given by

$$e^*_C = \frac{(1-\lambda)(1+m_o)}{4a} \text{ and } \pi^*_C = \frac{\beta(1-\lambda)^2(1+m_o)^2}{16a}.$$

(ii) If $v < \overline{v}_C$, the firm goes bankrupt.

Proposition 4 exhibits the same structure as the previous results: advance selling can help the firm only provided the consumers' valuation for service is sufficiently high. The most important feature of the coupon scheme is that it maintains the firm's incentive to exert effort in the repayment period. Even though consumers purchase coupons in advance that help the firm survive, the firm's revenues in the repayment period still depend on providing a high level of service, through the scheme's spot component (to see the impact of this feature, observe that the firm's equilibrium effort depends not only on the mass of the outer market m_o, but also on the mass of consumers who purchase in advance $m_i = 1$). This significantly alleviates the consumers' concerns regarding the firm's moral hazard, and increases their willingness to participate in the scheme.

Next, we point out that Proposition 4 suggests that whenever advance selling with coupons is beneficial for the firm, the optimal implementation of this scheme reduces to a simple request for donations from segment-i consumers. That is, the firm does not offer any discount to consumers who participate in the scheme (i.e., the optimal prices are $p^*_{sC} = p^*_{rC} = 1$); instead, the firm asks consumers for donations that add up to the minimum amount required for the firm to be able to secure a bank loan and secure its survival (we note that the bank loan here is risky and is only repaid by the firm in the event of high demand in the repayment period). We note that securing a higher portion of the necessary funds from the consumers cannot be achieved, because doing so would leave individual consumers with a strong incentive to deviate: knowing that the firm can secure partial financing from the bank even without her participation, a consumer could benefit by electing not to purchase a coupon. To avoid such deviations, the total funds raised by the firm in advance must not exceed the amount needed to ensure survival. In the end, under the optimal scheme, consumers effectively pay for the option of enjoying the firm's service in the future.

4.1.4. Scheme Comparison. To conclude our analysis of the high distress case, we perform a comparison between the three schemes considered in the preceding analysis, with the goal of understanding which scheme is preferable for the firm at different values of our model parameters.

The first comparison we perform is between classic advance selling ("F") and advance selling with an all-or-nothing clause ("A").

LEMMA 1. *Advance selling with an all-or-nothing clause ("A") always dominates classic advance selling ("F").*

The result is relatively straightforward to obtain by comparing Propositions 2 and 3. Doing so leads to two observations: First, the set of parameters under which "A" ensures the firm's survival subsumes the corresponding set of parameters under "F" (in particular, we note that $\overline{v}_A \leq \overline{v}_F$, which implies that the first case of Proposition 3 alone includes all parameter combinations covered in Proposition 2). Second, even in those cases where the firm survives under both schemes, the firm's equilibrium profit under the all-or-nothing scheme "A" is strictly higher. Intuitively, the addition of the all-or-nothing clause can only benefit the firm: by restricting the consumers' ability to free-ride (i.e., setting their expected utility from deviation to zero), the clause allows the firm to charge a higher price in advance.

Lemma 1 thus suggests that the optimal advance selling scheme, if it exists, takes the form of either "A" or "C." To see which of the two schemes is more beneficial for the firm in different market conditions, it is instructive to recall their respective properties. In particular, we note from the preceding analysis that scheme "A" mainly targets the negative effects of strategic consumer behavior (i.e., free-riding), while scheme "C" mainly targets the consumers' concerns with respect to firm moral hazard (i.e., decreased future effort). We further note that, according to the preceding analysis, a low level of effort is more of a concern for consumers when the firm's ability to generate revenue from the outer market is limited (because the firm in this case is left with little incentive to exert effort); we consider first these cases, where m_o is small.

PROPOSITION 5. *If $m_o \leq 1$, advance selling with an all-or-nothing clause ("A") dominates advance selling with coupons ("C") if and only if $v \geq \overline{v}$, where*

$$\overline{v} := \frac{I - \beta\lambda m_o + \frac{\beta(1-\lambda)^2(1+3m_o)(1-m_o)}{16a}}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]}.$$

In particular, the optimal advance selling scheme is described as follows:

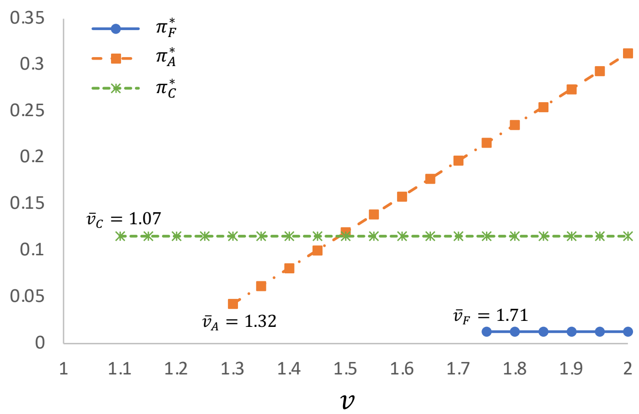
(i) If $v > \overline{v}$, the firm conducts advance selling with an all-or-nothing clause ("A").

(ii) If $\overline{v}_C \leq v < \overline{v}$, the firm conducts advance selling with coupons ("C").

(iii) If $v < \overline{v}_C$, the firm goes bankrupt.

We note first that when the outer market potential is small $m_o \leq 1$, the minimum valuation required for scheme "A" to be feasible is strictly higher than that required by scheme "C". Moreover, Proposition 5 suggests that scheme "A" dominates when the consumers' valuation is sufficiently high (in which case the firm can use the scheme to raise significant funds in advance, lowering the need for a bank loan significantly), while scheme "C" dominates when the consumers' valuation is moderate. The result is illustrated in Figures 2 and 3. In Figure 2, we observe that (i) the minimum valuations above which each of the three advance selling schemes is feasible satisfy $\bar{v}_C < \bar{v}_A < \bar{v}_F$; (ii) schemes "A" and "C" always dominate scheme "F"; and (iii) scheme "A" dominates scheme "C" when $v > \bar{v} = 1.5$, while scheme "C" is the preferred scheme when $\bar{v}_C \leq v \leq \bar{v}$. Figure 3 further illustrates how the thresholds $\bar{v}_C$ and $\bar{v}$ depend on λ. Note that since we have fixed $\lambda_s = 1$, the plot demonstrates that campaign "C" (respectively, campaign "A") tends to be the dominant form of advance selling when the firm's service effort leads to a significant (respectively, modest) improvement in the consumers' expected utility. This is consistent with the preceding analysis, which highlights the relative advantage of the coupon campaign "C" in terms of retaining the firm's incentive to exert effort.

Figure 2 **Equilibrium firm profit under the three advance selling schemes, as a function of the consumer valuation** v**. Parameter values:** $I = 0.6$, $a = 0.7$, $\lambda = 0.2$, $m_o = 0.5$, $\beta = 0.9$.

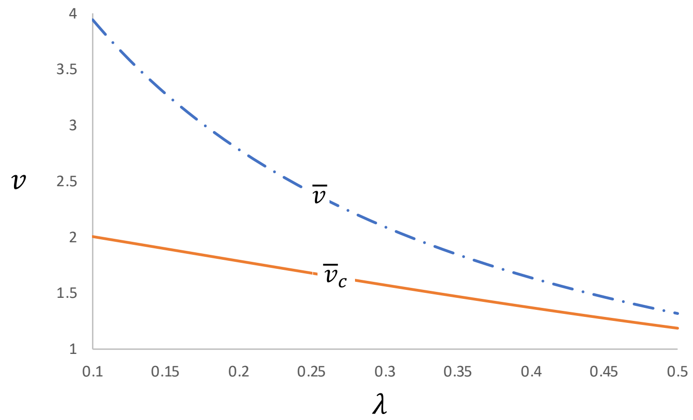


To complete the analysis, we next consider the remaining cases where the size of the outer market is relatively large.

Proposition 6. *If* $m_o > 1$*, advance selling with an all-or-nothing clause ("A") always dominates advance selling with coupons ("C"). In particular, the optimal advance selling scheme is described as follows:*

(i) If $v \geq \bar{v}_A$*, or if* $\underline{v}_A \leq v < \bar{v}_A$ *and* $m_o > M_A$*, the firm conducts advance selling with an all-or-nothing clause ("A").*

Figure 3 **Thresholds $\overline{v}_c$ and $\overline{v}$ as a function of λ. Parameter values:** $a = 0.5$, $I = 0.7$, $\beta = 0.9$, $m_o = 0.2$.



(ii) In all other cases, the firm goes bankrupt.

When $m_0 > 1$, the minimum valuation required for scheme A to be feasible is strictly lower than that required by scheme C. Furthermore, Proposition 6 establishes that when moral hazard is not a significant issue (owing to the large size of the outer market), the advantage of the coupon approach ("C") is limited, and the firm in these cases is always better off by implementing an advance selling campaign with an all-or-nothing clause ("A").

4.2. Moderate Financial Distress

We continue our analysis with the case of moderate financial distress, which is defined by $I_l \leq I < I_h$ (see Proposition 1). The defining characteristic of these cases is that, in the absence of any fund raised through advance selling, the firm is still able to secure a bank loan and can therefore avoid bankruptcy. The goal of the analysis that follows is to understand whether advance selling can increase the firm's expected profit relative to the case where the firm uses only bank financing.

4.2.1. Advance Selling at Full Price ("F"). We consider first the classic implementation of advance selling, where the firm sells its service in advance, potentially at a discount. Recall that in the case of high financial distress, this approach was able to improve the firm's outlook by ensuring survival in cases where bank financing was infeasible (under the sufficient condition that the consumers' valuation for service is sufficiently high). Our first results suggests that this mode of advance selling fails to make an impact in cases of moderate financial distress.

Proposition 7. *Suppose $I_l < I \leq I_h$. Advance selling at full price ("F") is dominated by pure bank financing.*

In the case of moderate financial distress, it is possible for the firm to secure financing through a bank loan and subsequently extract positive profit (see Proposition 1). The availability of this financing channel makes it hard for the firm to implement an advance selling scheme that is beneficial. The main difficulty is that consumers realize that even without their support, the firm will be able to survive by securing a bank loan, so that their utility from not participating in the advance selling scheme is no longer zero (as was the case under high financial distress). Furthermore, consumers also reason that by participating in an advance selling scheme, the firm will have less of an incentive to exert effort in the consumption period, which will result in a lower quality of service (this effect becomes even more pronounced when the firm's outer market potential is small). Therefore, as a consequence of the consumers' strategic behavior, for an advance purchase to be favored by the consumers, the service would need to be offered at a significant discount; from the firm's perspective, such an approach is dominated by the option of securing all the funds needed through bank financing.

4.2.2. Modifications. Although advance selling in its classic implementation is not an advantageous approach for the firm when facing moderate distress, modified versions of advance selling may prove to be more successful. We consider here the two modifications discussed in the preceding sections, namely, the addition of an all-or-nothing clause ("A") and the approach of selling coupons ("C"). The following result suggests that the effectiveness of such modifications relative to pure bank financing is also limited.

PROPOSITION 8. *Suppose $I_l < I \leq I_h$.*

(i) Advance selling with discount coupons ("C") is always dominated by pure bank financing.

(ii) There exists a threshold $\overline{M}_A > 0$ such that:

(a) If $m_o \leq \overline{M}_A$, advance selling with an all-or-nothing clause ("A") is dominated by pure bank financing.

(b) If $m_o > \overline{M}_A$, advance selling with an all-or-nothing clause ("A") dominates pure bank financing if and only if $\underline{w}_A \leq v \leq \overline{w}_A$, for some $1 \leq \underline{w}_A \leq \overline{w}_A$.

We discuss each part of the proposition in turn. The first part suggests that there are no circumstances under which advance selling with coupons can improve the firm's profit. Note that this result is intuitive given the preceding analysis, to the extent that the availability of the bank financing channel significantly exacerbates the consumers' strategic behavior: knowing that the firm's survival is guaranteed, the consumers' tendency to delay their purchase increases. As a result, for consumers to participate in an advance selling campaign with coupons, the firm must offer a substantial discount in the repayment period relative to the regular selling price. As Proposition 8 suggests, however, rather than offer such a steep discount, the firm prefers to rely fully on bank financing.

The second part of the proposition suggests that in a limited range of scenarios, advance selling with an all-or-nothing clause may improve the firm's profit. Recall that this approach has the benefit of curbing strategic consumer behavior: If an individual consumer elects to deviate, the entire advance selling scheme collapses, sending the game into the alternative equilibrium of pure bank financing. The challenge then is to design an advance selling scheme that is preferred by both the consumers and the firm, relative to the pure bank financing equilibrium. Proposition 8 identifies two conditions under which this is the case: first, when the firm's outer market potential is sufficiently high so that firm moral hazard is not a significant concern for consumers participating in advance selling; second, when the consumers' valuation is moderate. With regards to the second condition, we note that when the consumers valuation is moderate or high, it is possible to design a scheme that is preferred by the consumers; however, such a scheme is preferred by the firm over pure bank financing only in the case of moderate consumer valuations (if, instead, the consumers' valuation is high, the firm is able to set a higher spot price under pure bank financing, making this the preferred financing approach).

4.2.3. A Hybrid Approach ("H"). Although the scenarios in which the modified versions of advance selling can improve firm profit are limited, one potential solution which follows naturally from the discussion above is the somewhat more complex approach of combining the two modifications considered in our analysis. The main idea behind this approach is to combine the beneficial aspects of the all-or-nothing modification (with respect to curbing strategic consumer behavior) with those of the coupon scheme (with respect to alleviating moral hazard concerns). Accordingly, in this section we consider a hybrid approach ("H") consisting of a coupon campaign with an all-or-nothing clause (which applies to coupon sales).[9]

The complete analysis of the hybrid approach is cumbersome and is presented in full in Appendix A.3.[10] Here, we focus on two representative scenarios which seek to illustrate the following two significant points that emerge from the general analysis:

1. The hybrid approach can be beneficial for the firm even when the firm's outer market potential is small (i.e., this is in contrast to the scenarios described in Proposition 8, which require a large outer market).
2. When the firm's outer market potential is large, the hybrid approach can turn the firm's financial distress into an advantage, in the sense that the firm's profit becomes *increasing* in its financial distress.

[9] This mechanism is similar to the "threshold discounting" approach used in the past by platforms such as GroupOn.

[10] We also provide in Appendix A.2 the analysis of the hybrid approach for the case of high financial distress.

To illustrate the first point, we consider the limiting case of $m_o \to 0$, where there is effectively no outer market. Recall that in this case, classic advance selling as well as the two modified versions considered above fail to make an impact.

Define

$$\overline{w}_H^s := 1 + \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]}.$$

PROPOSITION 9. *Suppose $I_l < I \le I_h$ and $m_o > 0$ is small. When $1 + m_o < v < \overline{w}_H^s$ and $I \le 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2(1+m_o)^2}{2a}$, advance selling with discount coupons and an all-or-nothing clause ("H") dominates pure bank financing. The optimal prices are: $p_{rH}^{s*} = 1$,*

$$p_{sH}^{s*} = \frac{(v-m_o)}{2} + \frac{\sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{2(1-\lambda)}, \quad p_{aH}^{s*} = I - \beta\lambda\left(p_{sH}^{s*} + m_o\right).$$

and the firm's equilibrium effort and expected profit are:

$$e_H^{s*} = \frac{(1-\lambda)\left(p_{sH}^{s*} + m_o\right)}{2a}, \quad \pi_H^{s*} = \beta a\left(e_H^{s*}\right)^2.$$

The use of the hybrid approach expands the set of scenarios in which advance selling can help the firm significantly. As a way to implement a favorable coupon campaign (which on its own fails due to free-riding effects), Proposition 9 suggests that the use of an all-or-nothing clause, this time implemented with respect to the coupon sales, can be effective. The key point is that, with the addition of the all-or-nothing clause, if any individual consumer deviates from purchasing the coupon, the deviating consumer sends the game into a less-preferred equilibrium (for all the consumers as well as the firm) of pure bank financing. In particular, Proposition 9 presents a special case where the consumers' valuation is moderate (we note that the full analysis of all cases can be found in Appendix A.3). In this case, we observe that the firm sells discount coupons, or equivalently, solicits donations in exchange for a discounted service. Even though consumers pay p_{aH}^* upfront, the future discount is necessary in order to ensure that consumers are better off in the equilibrium with advance selling as compared to the equilibrium with pure bank financing.

With regards to the optimal prices under the advance selling scheme, we note that these must be carefully chosen taking the interaction between the firm's moral hazard and the consumers' strategic behavior into account. Although the analytical expressions are complicated, it can be shown that the spot price p_{sH}^{s*} is increasing in parameters $\{v, \beta, \lambda\}$ and is decreasing in parameters $\{I, a\}$, while the advance selling price p_{aH}^{s*} is increasing in parameters $\{I, a\}$ and is decreasing in parameters $\{v, \beta, \lambda\}$.

To illustrate our second main point, we next consider a special case where the firm's outer market potential is relatively high; in particular, we set $m_o = 1$ and $v \in (1,2)$. We first illustrate the use of the hybrid approach in this case, and then show that the firm's profit under the hybrid approach becomes increasing in the level of financial distress I.

Define

$$\underline{w}_H^l := 1 + \frac{2\sqrt{(1-\lambda)^2 - 2a\left(\frac{I}{\beta} - 2\lambda\right)}}{1-\lambda}, \text{ and}$$
$$I_m := \beta\left[2\lambda + \frac{3(1-\lambda)^2}{8a}\right] \in (I_l, I_h).$$

PROPOSITION 10. *Suppose $I_l < I \leq I_h$, $m_o = 1$ and $v \in (1,2)$. Under advance selling with discount coupons and an all-or-nothing clause ("H"),*

(i) If $I > I_m$ and $v \geq \underline{w}_H^l$, the firm conducts hybrid advance selling. The equilibrium prices are given by $p_{rH}^{l*} = 1$, and

$$p_{sH}^{l*} = v - 1 - \frac{\sqrt{(1-\lambda)^2 - 2a\left(\frac{I}{\beta} - 2\lambda\right)}}{1-\lambda}, \quad p_{aH}^{l*} = I - \beta\lambda\left(p_{sH}^{l*} + 1\right).$$

The corresponding equilibrium effort and expected profit are given by

$$e_H^{l*} = \frac{(1-\lambda)\left(p_{sH}^{l*} + 1\right)}{2a}, \quad \pi_H^{l*} = \beta a\left(e_H^{l*}\right)^2.$$

(ii) Otherwise, the firm uses pure bank financing.

We note that the case of a large outer market is qualitatively similar to that of the small outer market discussed above, with the difference that in this case the hybrid approach tends to be beneficial relative to pure bank financing under the additional condition that the firm's financial distress is relatively high (within the moderate distress region).[11] We further note that in this case, it can be shown that p_{sH}^{l*} is increasing in parameters $\{v, a, I\}$ and is decreasing in parameters $\{\beta, \lambda\}$, while p_{aH}^{l*} is decreasing in parameters $\{v, a\}$ and can be either increasing or decreasing in parameters $\{I, \beta, \lambda\}$.

Perhaps more interestingly, in those cases where the hybrid approach is beneficial, the following phenomenon occurs.

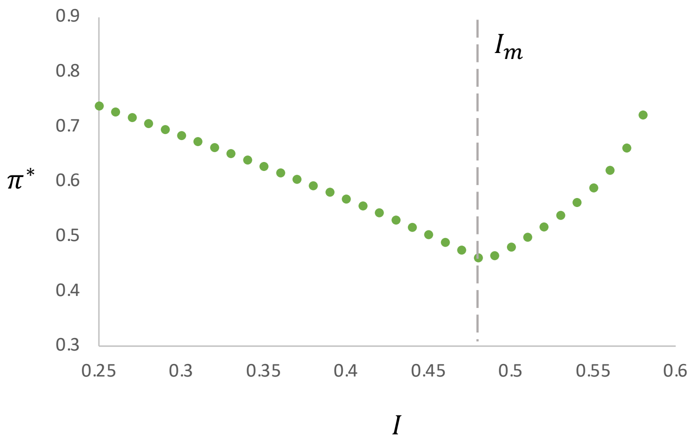
PROPOSITION 11. *Suppose $I_l < I \leq I_h$, $m_o = 1$ and $\underline{w}_H^l \leq v < 2$. The firm's expected profit is strictly decreasing in $I_l < I \leq I_m$ and strictly increasing in $I_m < I \leq I_h$.*

The result is illustrated in Figure 4. Observe that while the firm opts for pure bank financing, the firm's equilibrium profit decreases in the level of financial distress I (as was the case in the

[11] We note that while the above proposition does not exhibit cases where the hybrid scheme's spot price does not include a discount, such cases do arise in the general version of the result; see Proposition A.7 in Appendix A.3.

benchmark model without advance selling). However, once I crosses the threshold above which the hybrid approach is preferred by the firm, the firm's profit becomes increasing in I. In particular, observe from Proposition 10 that, as I increases, the firm's effort in equilibrium increases. As a result, the consumers' expected utility from service increases. By employing the hybrid mechanism, the firm is able to jointly optimize the advance price and the spot price so as to extract this additional surplus. Moreover, we note that in equilibrium the consumers's surplus is the same under the hybrid mechanism and under pure bank financing, which implies that the firm's adoption of the hybrid advance selling mechanism represents a Pareto improvement.

Figure 4 Equilibrium firm profit (π^*) as a function of the firm's funding need I in the case of moderate financial distress. Parameter values: $a = 0.9$, $\lambda = 0.1$, $v = 1.9$, $m_o = 1$, $\beta = 0.9$ **(note also that** $I_l = 0.18$, $I_h = 0.585$, $I_m = 0.484$, **and** $I_{max} = 0.99$**).**



5. Conclusion

This paper studies whether and how different forms of advance selling can be used to help alleviate a firm's financial distress. We find that in cases of high financial distress (where bank financing is not an option for the firm), simple advance selling schemes can help the firm survive when otherwise it would not be possible. In its simplest form, advance selling suffers from inefficiencies associated with strategic consumer behavior and firm moral hazard. Our analysis demonstrates that modified versions of advance selling which are consistent with implementations observed in practice (such as advance selling with an all-or-nothing clause and advance selling with discount coupons) can be designed to target these inefficiencies, leading to an increase in firm profit. In cases of moderate financial distress (where bank financing is a viable option), we find that simple advance selling mechanisms typically fail to make an impact; however, we find that more complex schemes may be able to align the consumers' and the firm's interests, allowing both parties to extract higher surplus. Surprisingly, we find that using such mechanisms may even turn the firm's financial distress into a positive, in the sense that a firm which is under a higher level of distress may be able to extract higher profit.

Our results relate to the different implementations of advance selling used by small businesses in practice to alleviate financial distress during the COVID-19 pandemic. Our analysis provides insight as to which type of advance selling scheme may be more appropriate, depending on situation characteristics such as the level of financial distress experienced by the firm, the consumers' valuation for the firm's service, and the impact of the firm's effort on the consumers' service experience. For instance, in cases of high financial distress, our analysis suggests that the standard approach of advance selling at full price can be improved upon with the addition of an all-or-nothing clause, similar as is observed in threshold discounting and crowdfunding platforms; furthermore, when the consumers' valuation for service is relatively low and/or the firm's service is more of a commodity (i.e., less dependent on service effort by the firm), the approach of selling future discount coupons may provide a better alternative than selling the full service in advance. In cases of moderate financial distress, our analysis suggests that simple advance selling schemes will likely fail to benefit the firm, and that more complex schemes may be necessary in order to align the firm's and the consumers' incentives; moreover, in these cases we find that when the consumers' valuation for service is sufficiently high, the firm need not offer a discount to consumers participating in the scheme, and may instead benefit from the solicitation of simple donations.

Apart from the results pertaining to firm survival and expected profit, it is interesting to note that in most cases the financing scheme which maximizes the firm's expected profit is also the one that maximizes the consumers' expected surplus. In particular, it can be shown that this is always the case for moderate financial distress scenarios, while it is also the case for high financial distress scenarios unless the consumers' valuation for service is high (in these cases, the consumers' surplus is maximized under the coupon scheme, while the firm prefers the all-or-nothing scheme as described in our analysis).

Our work makes several simplifications which may represent avenues for future work. For instance, in order to focus on the interaction between firm's moral hazard and strategic consumer behavior, we have assumed that there is no information asymmetry between the two. This may be a valid assumption in the presence of intermediaries such as crowdfunding platforms and when firms can credibly disclose their financial situation to consumers. However, in other cases, the assumption that customers possess the same information as the firm and/or the bank may be less realistic. Moreover, our analysis has not explicitly captured the role of government interventions (such as paycheck protection programs) to alleviate financial distress during the pandemic. Simpler interventions can be captured in our model by reducing the financial distress parameter, but more complex approaches may affect the strategic interactions between the firm, the bank, and the consumers. We expect that future work can build on the current model to investigate the impact of such programs in more detail.

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Appendix A: Supplemental Results

A.1. Sufficient and Necessary Conditions for Segment-i Customers to Purchase in Advance

Let $\mathbb{E}[u_{bb}]$ (resp. $\mathbb{E}[u_{bw}]$) represent the expected utility of a segment-i consumer who purchases in advance (resp. a segment-i consumer who waits) when all the other consumers purchase in advance; Similarly, let $\mathbb{E}[u_{wb}]$ (resp. $\mathbb{E}[u_{ww}]$) represent the expected utility of a segment-i consumer who purchases in advance (resp. a segment-i consumer who waits) when all the other segment-i consumers wait.

Lemma A.1 *Under all advance selling mechanisms considered in the paper ("F", "A", "C", and "H"), the segment-i consumers will purchase in advance if and only if both of the following two conditions hold:*

1. $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$, *and*
2. $\mathbb{E}[u_{wb}] > \mathbb{E}[u_{ww}]$ *or* $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{ww}]$.

A.2. Advance Selling under the Hybrid Approach: the High Financial Distress Case

In this section, we establish the optimal contract and performance under advance selling with all-or-nothing and discount coupon (the Hybrid approach, "H") when $I > I_h$. We first define the following threshold levels:

$$\overline{v}_H := 1 + \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]}$$

and

$$\underline{v}_H := \begin{cases} \overline{v}_A, & \text{for } I \leq 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2 m_o^2}{2a} \\ \underline{v}_A, & \text{for } I \in \left(2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(m_o)^2}{2a},\ 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2(1+m_o)^2}{2a}\right] \\ \overline{v}_H, & \text{for } I > 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2(1+m_o)^2}{2a} \end{cases}$$

We note that the condition for the first scenario for $\underline{v}_H$ is equivalent to $m_o \geq M_A$, where M_A is the threshold defined before Proposition 3, that is, $M_A = \frac{1}{2}(\underline{v}_A + m_o) = \frac{\sqrt{4a^2\lambda^2 + 2a(1-\lambda)^2\frac{I}{\beta}} - 2a\lambda}{(1-\lambda)^2}$.

Proposition A.1 *Suppose $I > I_h$, under advance selling with all-or-nothing and discount coupon ("H"),*

(i) if $v > \overline{v}_H$, the firm would advance sell with regular price $p^*_{rH} = 1$, spot price $p^*_{sH} = 1$, and advance price $p^*_{aH} = \beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right](v-1)$. The firm's equilibrium effort is $e^*_H = \frac{(1-\lambda)(1+m_o)}{2a}$ and the expected profit is

$$\pi^*_H = \beta\left[\frac{(1-\lambda)^2(m_o^2-1)}{4a} + \lambda m_o\right] + \beta v\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right] - I.$$

(ii) if $\underline{v}_H \leq v \leq \overline{v}_H$, the firm would advance sell with the following prices: $p^*_{rH} = 1$, and

$$p^*_{aH} = I - \frac{\beta\lambda\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]}{2(1-\lambda)};$$

$$p^*_{sH} = \frac{(1-\lambda)(v-m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{2(1-\lambda)}.$$

The equilibrium effort and expected profit are:

$$e^*_H = \frac{(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a};\ \ \pi^*_H = \beta\left(e^*_H\right)^2.$$

(iii) *if* $\underline{v}_A \le v < \underline{v}_H$ *and* $m_o \ge M_A$*, the firm would advance sell with the prices following* $p^*_{rH} = 1$, $p^*_{sH} \in [0,1]$*, and*

$$p^*_{aH} = \frac{\beta(v - p^*_{sH})\left[4a\lambda + (1-\lambda)^2(v+m_o) + (1-\lambda)\sqrt{(1-\lambda)^2(v+m_o)^2 + 8a\lambda(v+m_o) - \frac{8aI}{\beta}}\right]}{4a}.$$

The firm's equilibrium effort and expected profit are:

$$e^*_H = \frac{(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}; \;\; \pi^*_H = \beta\left(e^*_H\right)^2$$

(iv) *if otherwise, the firm fails to secure financing and goes bankrupt.*

Proposition A.2 *Suppose* $I > I_h$*. Comparing the hybrid scheme ("H") with "A" leads to:*

1. *when* $m_o \le M_A$*, the firm is able to advance sell under "H" over a larger region than under "A" (*$\underline{v}_H < \overline{v}_A$*). For* $v > \overline{v}_A$ *(advance selling can be achieved under "A"), the firm's effort and profit are both higher under "H" than under "A".*
2. *when* $m_o > M_A$*, the region over which the firm could advance sell under "H" is identical to that under "A". Further, the firm's effort and profit under "H" and "A" are identical for* $v \in [\underline{v}_A, \overline{v}_A]$*. For* $v > \overline{v}_A$*, the firm's effort and profit is higher under "H" than under "A".*

Remark. As shown, when m_o is small, the firm has a greater incentive to shirk. Thus, adding the coupon component to the all-or-nothing clause could both increase the region for advance selling success and increase the firm's effort level and profit. On the other hand, with a larger m_o, the firm is better incentivized to exert effort in order to attract the outer market customers. Thus, the all-or-nothing clause alone is sufficient to ensure the success of advance selling. Adding the coupon component will not expand the firm's survival region. That said, when customer valuation v is sufficiently high, the coupon component allows the firm with an additional lever to extract surplus from customers, thus boosting the firm's profit.

A.3. General Results for the Case of Moderate Distress

In this appendix, we present the results related to moderate financial distress with full technical details and general parameters.

Proposition A.3 *Suppose* $m_o < v - 1$ *and* $I_l < I \le I_h$*. Define* $\overline{v}^s_A := \frac{I - \beta\lambda m_o}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]}$*. Under advance selling with all-or-nothing clause ("A"),*

1. *if* $v > \overline{v}^s_A$*, the firm is able to advance sell, and the optimal results are as follows:* $p^{s*}_{rA} = 1$, $p^{s*}_{aA} = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]$*, and the firm's expected profit is* $\pi^{s*}_A = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o + v\left(\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right)\right] - I$*. Moreover,* $\pi^{s*}_A - \pi^*_B$ *is decreasing in* v*.*
2. *if* $v < \overline{v}^s_A$*, the firm fails to advance sell.*

Proposition A.4 *Suppose* $m_o \ge v - 1$*. Consider advance selling with an all-or-nothing clause ("A"). Let*

$$I^A_m := \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o}{2a}\right].$$

1. *Advance selling is dominated by pure bank financing if (i)* $m_o \le 1$ *or (ii)* $m_o > 1$ *and* $I_l < I < I_m^A$.

2. *For* $m_o > 1$ *and* $I_m^A < I \le I_h$*, define*

$$\underline{v}_A^l := 1 + \frac{\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{1-\lambda},$$

$$\overline{v}_A^l := \frac{4a\left[\frac{I}{\beta} - \lambda(1+m_o)\right] - (1-\lambda)\left[(1-\lambda)(1+m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}{(1-\lambda)\left[(1-\lambda)(m_o-1) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}.$$

   (i) If $v < \underline{v}_A^l$*, the dominant form of financing is a pure bank loan.*

   (ii) If $v \ge \underline{v}_A^l$*, the firm conducts advance selling, and*

   (a) if $v > \overline{v}_A^l$*, the equilibrium prices, effort and profit are given by* $p_{rA}^{l*} = 1$, $p_{aA}^{l*} = \beta\left[\left(\frac{(1-\lambda)m_o}{2a} - e_B^*\right)(1-\lambda)v + [e_B^* + (1-e_B^*)\lambda]\right]$, $e_A^{l*} = \frac{(1-\lambda)m_o}{2a}$ *and*

$$\pi_A^{l*} = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o + \left[\frac{(1-\lambda)m_o}{2a} - e_B^*\right](1-\lambda)v + [e_B^* + (1-e_B^*)\lambda]\right] - I.$$

   (b) if $\underline{v}_A^l \le v \le \overline{v}_A^l$*, the equilibrium prices, effort and profit are given by* $p_{rA}^{l*} = 1$, $p_{aA}^{l*} = [\underline{\phi}_A^l]^{-1}(v)$, $e_A^{l*} = e_{bb}^l(p_{aA}^{l*})$ *and* $\pi_A^{l*} = \beta a[e_{bb}^l(p_{aA}^{l*})]^2$*, where* $e_{bb}^l(\cdot)$ *is defined in Lemma A.2 and* $[\underline{\phi}_A^l]^{-1}(v)$ *denotes the large root to the equation of* $\underline{\phi}_A^l(p_a) := \frac{\frac{p_a}{\beta} - [e_B^* + (1-e_B^*)\lambda]}{[e_{bb}^l(p_a) - e_B^*](1-\lambda)} = v$.

Proposition A.5 *For* $I_l < I \le I_h$*, advance selling with coupon ("C") is dominated by pure bank financing.*

Proposition A.6 *Suppose* $m_o < v - 1$ *and* $I_l < I < I_h$*. Define* $\overline{w}_H^s := \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1$*. Under advance selling with discount coupons and an all-or-nothing clause ("H"),*

1. *if* $v > \overline{w}_H^s$*, the firm is able to advance sell and the equilibrium prices are:* $p_{rH}^{s*} = 1$, $p_{sH}^{s*} = 1$, $p_{aH}^{s*} = \beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right](v-1)$*, and the firm's equilibrium effort and profit are:* $e_H^{s*} = \frac{(1-\lambda)(1+m_o)}{2a}$ *and*

$$\pi_H^{s*} = \beta\left[\frac{(1-\lambda)^2(m_o^2-1)}{4a} + \lambda m_o + \left(\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right)v\right] - I.$$

   Moreover, $\pi_H^{s*} - \pi_B^*$ *decreases in* v.

2. *if* $v < \overline{w}_H^s$ *and* $I \le 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2(1+mo)^2}{2a}$*, the firm is able to advance sell and the equilibrium prices are:* $p_{rH}^{s*} = 1$ *and*

$$p_{sH}^{s*} = \frac{(v-m_o)}{2} + \frac{\sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{2(1-\lambda)}, \quad p_{aH}^{s*} = I - \beta\lambda\left(p_{sH}^{s*} + m_o\right).$$

   The firm's equilibrium effort and profit are:

$$e_H^{s*} = \frac{(1-\lambda)\left(p_{sH}^{s*} + m_o\right)}{2a}, \quad \pi_H^{s*} = \beta a\left(e_H^{s*}\right)^2 \ge \pi_B^*.$$

3. *if otherwise, the firm fails to advance sell.*

Proposition A.7 *Suppose $m_o \geq v-1$ and $I_l < I \leq I_h$. Define $\underline{w}_H^l := 1 + \frac{\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{1-\lambda}$ and*

$$\overline{w}_H^l := \min\left\{1+m_o, 1+\frac{(1-\lambda)(1+m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{2(1-\lambda)}\right\}.$$

Further define $I_m := \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2(1+2m_o)}{8a}\right] \in (I_l, I_h)$. Under advance selling with discount coupons and an all-or-nothing clause ("H"), the dominant form of financing is a pure bank loan for $I_l < I < I_m$.

For $I_m < I \leq I_h$,

(i) if $v < \underline{w}_H^l$, the dominant form of financing is a pure bank loan.

(ii) if $v \geq \underline{w}_H^l$, the firm conducts advance selling, and

(a) if $\underline{w}_H^l \leq v \leq \overline{w}_H^l$, the equilibrium prices are $p_{rH}^{l*} = 1$,

$$p_{sH}^{l*} = \frac{(1-\lambda)(2v-1-m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{2(1-\lambda)}, \quad p_{aH}^{l*} = I - \beta\lambda\left(p_{sH}^{l*} + m_o\right).$$

The firm's equilibrium effort and profit are:

$$e_H^{l*} = \frac{(1-\lambda)\left(p_{sH}^{l*}+m_o\right)}{2a}, \quad \pi_H^{l*} = \beta a\left(e_H^{l*}\right)^2$$

(b) if $v > \overline{w}_H^l$, the equilibrium prices are $p_{rH}^{l*} = 1$, $p_{sH}^{l*} = 1$, and

$$p_{aH}^{l*} = \frac{\beta(1-\lambda)(v-1)\left[(1-\lambda)(1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}\right]}{4a},$$

and the firm's equilibrium effort and profit are:

$$e_H^{l*} = \frac{(1-\lambda)(1+m_o)}{2a}, \quad \pi_H^{l*} = \beta a\left(e_H^{l*}\right)^2 + \beta\lambda(1+m_o) + p_{aH}^{l*} - I.$$

Proposition A.8 *Suppose $I_l < I \leq I_h$. Under advance selling with discount coupons and an all-or-nothing clause ("H"),*

1. *the firm's expected profit (π_H^{s*}) decreases in I if $m_o < v-1$;*
2. *the firm's expected profit (π_H^{l*}) increases in I if $m_o \geq v-1$.*

A.4. Technical Lemmas

Lemmas A.2–A.4 in the following apply to the types of contracts where customers purchase in advance face a zero spot price ($p_s = 0$), such as full-price advance selling ("F") and advance selling with an all-or-nothing clause ("A").

Lemma A.2 *Let $p_{bb}^h := I - \beta\lambda m_o$ and $p_{bb}^l := I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]$. Given p_a, provided that all k segment-i customers purchase in advance and the firm continues in the second period,*

1. *if $p_a \geq p_{bb}^h$ (**High-price strategy**), the firm's optimal effort and expected profit are $e_{bb}^h = \frac{(1-\lambda)m_o}{2a}$ and*

$$\pi_{bb}^h(p_a) = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right] + p_a - I.$$

In this case, if $p_a \geq I$, the firm finances solely through advance selling. Otherwise, the firm obtains financing through both advance selling and a risky bank loan, with interest rate $r_{bb}^h = \frac{1}{\beta} - 1$.

2. *if* $p_{bb}^l \le p_a < p_{bb}^h$ ***(Low-price strategy)****, the firm's optimal effort and expected profit are:*

$$e_{bb}^l(p_a) = \frac{(1-\lambda)m_o + \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{4a},$$

and $\pi_{bb}^l(p_a) = \beta a\left[e_{bb}^l(p_a)\right]^2$. *In this case, the firm obtains financing through both advance selling and a risky bank loan, and the interest rate is*

$$r_{bb}^l(p_a) = \frac{(1+\lambda)m_o - \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{2(I-p_a)} - 1.$$

3. *if* $0 < p_a < p_{bb}^l$*, the firm fails to secure financing through advance selling.*

Lemma A.3 *Let* $p_{bw}^h := \frac{I-\beta\lambda\left(m_o+\frac{1}{k}\right)}{1-\frac{1}{k}}$ *and* $p_{bw}^l := \frac{I-\beta\left[\lambda\left(m_o+\frac{1}{k}\right)+\frac{(1-\lambda)^2\left(m_o+\frac{1}{k}\right)^2}{8a}\right]}{1-\frac{1}{k}}$. *Given* p_a*, if all but one segment-i consumer purchase in advance and the firm continues in the second period,*

1. *if* $p_a \ge p_{bw}^h$ *, then the optimal effort is* $e_{bw}^h = \frac{(1-\lambda)\left(m_o+\frac{1}{k}\right)}{2a}$.
2. *if* $p_{bw}^l \le p_a < p_{bw}^h$ *, then the optimal effort is*

$$e_{bw}^l(p_a) = \frac{(1-\lambda)\left(m_o+\frac{1}{k}\right) + \sqrt{(1-\lambda)^2\left(m_o+\frac{1}{k}\right)^2 - 8a\left[\frac{I-p_a\left(1-\frac{1}{k}\right)}{\beta} - \lambda\left(m_o+\frac{1}{k}\right)\right]}}{4a}.$$

3. *if* $p_a < p_{bw}^l$*, then the firm fails to secure finnancing through advance selling.*

Lemma A.4 *Let* $\hat{I}_l := \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o(2+m_o)}{8a}\right]$. *For* $I > I_l$,

1. *if* $I \ge \hat{I}_l$*, then we have* $p_{bb}^l < p_{bw}^l < p_{bb}^h < p_{bw}^h$ *and* $p_{bw}^l \to p_{bb}^l$ *as* $k \to \infty$*;*
2. *if* $I < \hat{I}_l$*, then we have* $p_{bw}^l < p_{bb}^l < p_{bb}^h < p_{bw}^h$ *and* $p_{bw}^l \to p_{bb}^l$ *as* $k \to \infty$*;*

Lemmas A.5–A.7 in the following apply to the types of the contracts where customers purchase in advance may face a non-zero spot price p_s, such as advance selling with coupons ("C") and advance selling with the hybrid approach (an all-or-nothing clause and coupons, "H").

Lemma A.5 *Let* $p_{bb}^h(p_s) := I - \beta\lambda(p_s + m_o)$ *and* $p_{bb}^l(p_s) := I - \beta\left[\lambda(p_s+m_o) + \frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]$. *Given* (p_a, p_s)*, provided that all k segment-i consumers purchase in advance and the firm continues in the second period,*

1. *if* $p_a \ge p_{bb}^h(p_s)$ ***(High-price strategy)****, the firm's optimal effort and expected profit are* $e_{bb}^h(p_s) = \frac{(1-\lambda)(p_s+m_o)}{2a}$ *and*

$$\pi_{bb}^h(p_a, p_s) = \beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a} + \lambda(p_s+m_o)\right] + p_a - I.$$

If $p_a \ge I$*, the firm obtains financing solely through advance selling. Otherwise, the firm obtains financing through both advance selling and a risky bank loan, with the interest rate* $r_{bb}^h = \frac{1}{\beta} - 1$.

2. *if* $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$ *(**Low-price strategy**), the firm's optimal effort and expected profit are*

$$e^l_{bb}(p_a,p_s) = \frac{(1-\lambda)(p_s+m_o) + \sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]}}{4a},$$

and $\pi^l_{bb}(p_a,p_s) = \beta a\left[e^l_{bb}(p_a,p_s)\right]^2$. *In this case, the firm uses a combination of customer financing and bank financing, and the interest rate of the bank loan is*

$$r^l_{bb}(p_a,p_s) = \frac{(1+\lambda)(p_s+m_o) - \sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]}}{2(I-p_a)} - 1.$$

3. *If* $0 < p_a < p^l_{bb}(p_s)$, *the firm fail to secure financing through advance selling.*

Lemma A.6 *Let* $p^h_{bw}(p_s) := \frac{I-\beta\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]}{1-\frac{1}{k}}$ *and* $p^l_{bw}(p_s) := \frac{I-\beta\left[\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]+\frac{(1-\lambda)^2\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]^2}{8a}\right]}{1-\frac{1}{k}}$. *Given* (p_a,p_s), *if all but one segment-i consumer purchase in advance and the firm continues in the second period,*

1. *if* $p_a \ge p^h_{bw}(p_s)$, *then the optimal effort is*

$$e^h_{bw}(p_s) = \frac{(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]}{2a}.$$

2. *if* $p^l_{bw}(p_s) \le p_a < p^h_{bw}(p_s)$, *then the optimal effort is*

$$e^l_{bw}(p_a,p_s) = \frac{(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right] + \sqrt{(1-\lambda)^2\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]^2 - 8a\left[\frac{I-p_a\left(1-\frac{1}{k}\right)}{\beta} - \lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]\right]}}{4a}.$$

3. *If* $p_a < p^l_{bw}(p_s)$, *then the firm fails to secure financing through advance selling.*

Lemma A.7 *Let* $\hat{I}_h := \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$ *and* $\hat{I}_l$ *as defined in Lemma A.4. For* $p_s \in (0,1]$ *and* $I > I_l$, *there are three relevant cases below:*

1. *When* $I > \hat{I}_h$, *we have* $p^l_{bb}(p_s) < p^l_{bw}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ *and* $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ *as* $k \to \infty$.
2. *When* $\hat{I}_l \le I \le \hat{I}_h$, *let* $I_t(p_s) =: \hat{I}_h - \frac{\beta(1-\lambda)^2(1-p_s)^2}{8a}$ *and evidently there exists a unique* $\ddot{p}_s$ *such that* $I_t(\ddot{p}_s) = I$. *Depending on* p_s, *we have the following two subcases:*
   (a) *If* $p_s \in (0,\ddot{p}_s)$, *then* $p^l_{bb}(p_s) < p^l_{bw}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ *and* $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ *as* $k \to \infty$.
   (b) *If* $p_s \in [\ddot{p}_s,1]$, *then* $p^l_{bw}(p_s) < p^l_{bb}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ *and* $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ *as* $k \to \infty$.
3. *When* $I < \hat{I}_l$, *we have* $p^l_{bw}(p_s) < p^l_{bb}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ *and* $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ *as* $k \to \infty$.

Appendix B: Proofs

Proof of Proposition 1. With backward induction, we first consider the firm's optimal regular price. Given effort e, the firm either succeeds or fails. In the case of succeeding, each segment-i (Resp. segment-o) customer has a valuation v (Resp. 1) for the product with probability 1. Evidently, the firm will either sets $p_r = 1$ or $p_r = v$ for optimality. Accordingly, the market demand in the repayment period, D_r, is 1 if the firm sets $p_r = v$, and is $1 + m_o$ if the firm sets $p_r = 1$. Correspondingly, the firm obtain a sales revenue of v by setting $p_r = v$, and a sales revenue of $1 + m_o$ by setting $p_r = 1$. In the case of failing, each segment-i (Resp. segment o) customer has a valuation v (Resp. 1)for the product with probability λ and a valuation of 0 with probability $1-\lambda$. Thus, the market demand in the repayment period, D_r, is λ if the firm sets $p_r = v$, and is $\lambda(1+m_o)$ if setting $p_r = 1$. Accordingly, the firm obtains a sales revenue of λv by setting $p_r = v$, and a sales revenue of $\lambda(1+m_o)$ by setting $p_r = 1$. Thus, to maximize his sales revenue, in either case, the firm will set $p_r = 1$ if $v < 1 + m_o$ and $p_r = v$ if otherwise.

Next, we continue to consider the firm's optimal effort level. At the end of selling season, the firm collects sales revenue $S_r := D_r p_r$, using which to repay the principle plus the interest of bank loan $I(1+r)$ to the extent possible due to the assumption of limited liability. According to the game sequence, after obtaining bank finance, the firm will be able to operate with a probability of β and fail with a probability of $1-\beta$. Given that the firm fails, the firm earns a zero profit, i.e., $\pi_B = 0$. By contrast, given the firm succeeds, she will put an effort e in the following daily operations. Accordingly, the firm's expected profit is

$$\pi_B(e) = \mathbb{E}[S_r - I(1+r)]^+ - ae^2, \tag{1}$$

where S_r approximately follows a binary distribution:

$$S_r = \begin{cases} v_m := \max\{v, 1+m_o\}, & \text{with probability } e; \\ \lambda v_m, & \text{with probability } 1-e. \end{cases} \tag{2}$$

We note that $I(1+r) < v_m$ should be hold since the bank loan will be declined if otherwise. Thus, from (1) and (2), given that the firm does not suffer from random shock, her profit can be rewritten as

$$\pi_B(e) = -ae^2 + v_m - I(1+r) - [\lambda v_m - I(1+r)]^+\}e + [\lambda v_m - I(1+r)]^+.$$

By maximizing $\pi_B(e)$, the optimal effort level is derived as:

$$e_B = \frac{v_m - I(1+r) - [\lambda v_m - I(1+r)]^+}{2a}. \tag{3}$$

Finally, we consider the bank's pricing decision. By lending I to the firm, if the firm is able to operate, then the repayment collected from the firm, defined as Γ, would be $\min\{S_r, I(1+r)\}$; if the firm suffers from random shock and thus goes bankrupt, then the repayment Γ is zero. Thus, in the repayment period, Γ approximately follows the following distribution:

$$\Gamma = \begin{cases} I(1+r), & \text{with probability } \beta e_B; \\ \min\{\lambda v_m, I(1+r)\}, & \text{with probability } \beta(1-e_B); \\ 0, & \text{with probability } 1-\beta. \end{cases}$$

According to the fair pricing principle, the interest rate r is uniquely determined by the following equation:

$$I = \mathbb{E}[\Gamma] = \beta e_B I(1+r) + \beta(1-e_B)\min\{\lambda v_m, I(1+r)\}. \tag{4}$$

Depending on the relationship between λv_m and $I(1+r)$, we solve the problem in the following two cases:

1. If $I(1+r) \le \lambda v_m$, then substituting (3) into (4) leads to $r_B^* = \frac{1}{\beta} - 1$. To ensure $I(1+r) \le \lambda v_m$ holds, it should be satisfied that $I \le \beta\lambda v_m =: I_l$. Accordingly, the optimal effort is $e_B^* = \frac{(1-\lambda)v_m}{2a}$. Correspondingly, the firm earns an expected profit
$$\pi_B^* = \beta\left[-a(e_B^*)^2 + (1-\lambda)v_m e_B^* + \lambda v_m - I(1+r_B^*)\right] = \beta\left[\frac{(1-\lambda)^2 v_m^2}{4a} + \lambda v_m\right] - I.$$
2. If $I(1+r) > \lambda v_m$, then substituting (3) into (4) leads to $I = \frac{\beta I(1+r)[v_m - I(1+r)]}{2a} + \beta\lambda v_m\left[1 - \frac{v_m - I(1+r)}{2a}\right]$, or equivalently
$$\frac{[I(1+r)]^2}{2a} - \frac{(1+\lambda)v_m I(1+r)}{2a} + \frac{I}{\beta} - \left(1 - \frac{v_m}{2a}\right)\lambda v_m = 0, \tag{5}$$
which is quadratic in r. Thus, the bank will lend to the firm if and only if there exists a solution r satisfying $I(1+r) > \lambda v_m$ to Eq. (5), which is equivalent to
$$I \le \beta\left[\lambda v_m + \frac{(1-\lambda)^2 v_m^2}{8a}\right] =: I_h. \tag{6}$$
When Eq. (6) is met, solving Eq. (5), we derive the equilibrium interest rate, which is equal to the smaller root due to the competitiveness of the bank credit market, as follows
$$r_B^* = \frac{(1+\lambda)v_m - \sqrt{(1-\lambda)^2 v_m^2 - 8a\left(\frac{I}{\beta} - \lambda v_m\right)}}{2I} - 1. \tag{7}$$
Substituting (7) into (3), we derive the equilibrium effort level: $e_B^* = \frac{(1-\lambda)v_m + \sqrt{(1-\lambda)^2 v_m^2 - 8a\left(\frac{I}{\beta} - \lambda v_m\right)}}{4a}$. Correspondingly, the firm earns an expected profit $\pi_B^* = \beta\left[-a(e_B^*)^2 + (v_m - I(1+r_B^*))\, e_B^*\right] = \beta a(e_B^*)^2$. Otherwise, that is, if $I > I_h$, the firm fails to obtain bank loan and thus goes bankrupt. □

Proof of Proposition 2. As in Appendix A.1, in this proof and the following ones, we let $\mathbb{E}[u_{bb}]$ (resp. $\mathbb{E}[u_{bw}]$) represent the expected utility of a segment-i consumer who purchases in advance (resp. a segment-i consumer who waits) when all the other consumers purchase in advance; Similarly, let $\mathbb{E}[u_{wb}]$ (resp. $\mathbb{E}[u_{ww}]$) represent the expected utility of a segment-i consumer who purchases in advance (resp. a segment-i consumer who waits) when all the other segment-i consumers wait.

Next, we turn to the proof. We first consider the firm's optimal pricing on the regular selling price p_r. Provided that all segment-i customers advance buy and the random shock does not happen, the firm will set $p_{rF}^* = 1$ since only outer customer buy at the regular selling price in the repayment period. In the remaining cases (i.e., all segment-i customers wait or random shock happens), the firm fails to continue and thus p_r becomes irrelevant.

We continue to analyze the firm's optimal pricing on p_a and the segment-i customers' decision behavior. For $I > I_h$, according to Proposition 1, the firm fails to secure bank financing and thus goes bankrupt without advance selling. Thus, if every segment-i consumer waits, the expected surplus of this consumer is equal to zero, i.e., $\mathbb{E}[u_{ww}] = 0$. As such, according to Lemma A.1, the equivalent condition for the firm to induce purchase in advance degenerates to
$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{bw}]. \tag{8}$$

Further, from Lemma A.2, the firm can advance sell with either high-price strategy (i.e., $p_a \geq p_{bb}^h$) or low-price strategy ($p_{bb}^l \leq p_a < p_{bb}^h$). In what follows, we consider these two pricing strategies, respectively.

High-price strategy. Given $p_a \geq p_{bb}^h$, assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e_{bb}^h = \frac{(1-\lambda)m_o}{2a}$ in accordance with Lemma A.2. Accordingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e_{bb}^h + (1-e_{bb}^h)\lambda] - p_a = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right] - p_a. \tag{9}$$

However, given $p_a \geq p_{bb}^h$, when $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, according to Lemma A.3, the firm's optimal effort level $e_{bw}(p_a)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.4, $p_{bw}^h > p_{bb}^h > p_{bw}^l > p_{bb}^l$ holds when $k \to \infty$ since $I > I_h > \hat{I}_l$ holds. Thus, with high-price strategy, the firm might set $p_a \geq p_{bw}^h$, or $p_{bb}^h \leq p_a < p_{bw}^h$. In what follows, we consider these two pricing scenarios respectively.

Scenario I. If the firm sets

$$p_a \geq p_{bw}^h, \tag{10}$$

then given p_a, assuming that $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, the firm would put an effort of $e_{bw}^h = \frac{(1-\lambda)(m_o+\frac{1}{k})}{2a}$ according to Lemma A.3. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e_{bw}^h + (1-e_{bw}^h)\lambda\right](v-p_r) = \beta\left[\lambda + \frac{(1-\lambda)^2(m_o+\frac{1}{k})}{2a}\right](v-1), \tag{11}$$

where $p_r = 1$, because one segment-i customer's deviation will not change the firm's optimal pricing on the regular selling price, since the size of one segment-i customer is negligible compared to that of segment-o customers. Anticipating this, from the equivalent condition (8) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (9) and $\mathbb{E}[u_{bw}]$ in (11) into this equivalent condition gives

$$p_a \leq \beta\left[v\left(e_{bb}^h - e_{bw}^h\right)(1-\lambda) + e_{bw}^h(1-\lambda) + \lambda\right]. \tag{12}$$

The constraints of (10) and (12) jointly indicates

$$p_{bw}^h \leq \beta\left[v\left(e_{bb}^h - e_{bw}^h\right)(1-\lambda) + e_{bw}^h(1-\lambda) + \lambda\right],$$

or equivalently,

$$v \leq 1 + km_o - \frac{2ak[I - \beta\lambda(1+m_o)]}{\beta\left(1-\frac{1}{k}\right)(1-\lambda)^2} \leq 1 - \frac{k(1-m_o)^2 + 4m_o}{4\left(1-\frac{1}{k}\right)} < 1,$$

where the second "$\leq$" holds due to $I > I_h$. This contradicts with $v \geq 1$. Therefore, the firm fails to advance sell in this case.

Scenario II. If the firm sets

$$p_{bb}^h \leq p_a < p_{bw}^h, \tag{13}$$

then given p_a, assuming that $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, the firm would put an effort of $e^l_{bw}(p_a)$, according to Lemma A.3. Accordingly, the expected surplus of the segment-i consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e^l_{bw}(p_a) + (1-e^l_{bw}(p_a))\lambda\right](v-p_r) = \beta\left[e^l_{bw}(p_a) + (1-e^l_{bw}(p_a))\lambda\right](v-1), \tag{14}$$

where $p_r = 1$ for similar reason as analyzed in the Scenario I. Anticipating this, from the equivalent condition (8) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (9) and $\mathbb{E}[u_{bw}]$ in (14) into this equivalent condition gives

$$p_a \leq \beta\left[v\left(e^h_{bb} - e^l_{bw}(p_a)\right)(1-\lambda) + e^l_{bw}(p_a)(1-\lambda) + \lambda\right]. \tag{15}$$

As $k \to \infty$, we have $p^h_{bw} \to p^h_{bb}$ and $e^l_{bw}(p_a) \to e^l_{bb}(p_a)$. Thus, the constraints of (13) and (15) are transformed into

$$\begin{cases} p_a = p^h_{bb} & (16) \\ p_a \leq \beta\left[v\left(e^h_{bb} - e^l_{bb}(p_a)\right)(1-\lambda) + e^l_{bb}(p_a)(1-\lambda) + \lambda\right]. & (17) \end{cases}$$

We note that the feasible region of p_a satisfying the above constraints (16)-(17) is empty. Accordingly, the firm fails to advance sell in this case.

Low-price strategy. Given $p^l_{bb} \leq p_a < p^h_{bb}$, assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e^l_{bb}(p_a) = \frac{(1-\lambda)m_o + \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{4a}$ in accordance with Lemma A.2. Accordingly, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e^l_{bb}(p_a) + (1-e^l_{bb}(p_a))\lambda] - p_a. \tag{18}$$

However, given $p^l_{bb} \leq p_a < p^h_{bb}$, when $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, according to Lemma A.3, the firm's optimal effort level $e_{bw}(p_a)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.4, $p^h_{bw} > p^h_{bb} > p^l_{bw} > p^l_{bb}$ holds when $k \to \infty$ since $I > I_h > \hat{I}_l$ holds. Thus, with low-price strategy, the firm might set $p^l_{bw} \leq p_a < p^h_{bb}$, or $p^l_{bb} \leq p_a < p^l_{bw}$. In what follows, we consider these two pricing scenarios respectively.

Scenario I. If the firm sets

$$p^l_{bw} \leq p_a < p^h_{bb}, \tag{19}$$

then given p_a, assuming that $k-1$ consumers purchase in advance but one consumer deviates to wait, the firm would put an effort of $e^l_{bw}(p_a)$, according to Lemma A.3. Accordingly, the expected surplus of the segment-i consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e^l_{bw}(p_a) + (1-e^l_{bw}(p_a))\lambda\right](v-p_r) = \beta\left[e^l_{bw}(p_a) + (1-e^l_{bw}(p_a))\lambda\right](v-1). \tag{20}$$

where $p_r = 1$ for similar reason as analyzed in the Scenario I. Anticipating this, from the equivalent condition (8) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (18) and $\mathbb{E}[u_{bw}]$ in (20) into this equivalent condition gives

$$p_a \leq \beta\left[v\left(e^l_{bb}(p_a) - e^l_{bw}(p_a)\right)(1-\lambda) + e^l_{bw}(p_a)(1-\lambda) + \lambda\right]. \tag{21}$$

As $k \to \infty$, we have $p^l_{bw} \to p^l_{bb}$ and $e^l_{bw}(p_a) \to e^l_{bb}(p_a)$. Thus, the constraints of (19) and (21) are transformed into

$$\begin{cases} p^l_{bb} \le p_a < p^h_{bb} \\ p_a \le \beta\left[e^l_{bb}(p_a)(1-\lambda)+\lambda\right]. \end{cases} \tag{22}$$

The condition (22) can be rewritten as

$$2a\left(\frac{p_a}{\beta}-\lambda\right)^2 - (1-\lambda)^2(1+m_o)\left(\frac{p_a}{\beta}-\lambda\right) + (1-\lambda)^2\left[\frac{I}{\beta}-\lambda(1+m_o)\right] \le 0,$$

which is quadratic in p_a. There is no feasible p_a satisfying the this constraint since $(1-\lambda)^4(1+m_o)^2 - 8a(1-\lambda)^2\left[\frac{I}{\beta}-\lambda(1+m_o)\right] < 0$ for $I > I_h$. Thus, the firm fails to advance sell in this case.

Scenario II. If the firm sets

$$p^l_{bb} \le p_a < p^l_{bw}, \tag{23}$$

then given p_a, assuming that $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, the firm fails to secure bank financing and has to declare bankruptcy, according to Lemma A.3. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = 0. \tag{24}$$

Anticipating this, by substituting $\mathbb{E}[u_{bb}]$ in (18) and $\mathbb{E}[u_{bw}]$ in (24) into the equivalent condition (8) of inducing all k consumers to purchase in advance, the segment-i consumers will purchase in advance if and only if $p_a \le \beta v[e^l_{bb}(p_a) + (1-e^l_{bb}(p_a))\lambda]$, or equivalently,

$$v \ge \frac{p_a}{\beta[e^l_{bb}(p_a) + (1-e^l_{bb}(p_a))\lambda]}. \tag{25}$$

On the other aspect, as $k \to \infty$, we have $p^l_{bw} \to p^l_{bb}$, and thus the constraint (23) is transformed into

$$p_a = p^l_{bb} =: I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]. \tag{26}$$

The constraints of (25) and (26) jointly imply that

$$v \ge \frac{p^l_{bb}}{\beta[e^l_{bb}(p^l_{bb}) + (1-e^l_{bb}(p^l_{bb}))\lambda]} = \frac{I-\beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]} =: \overline{v}_F.$$

That is, the firm is able to advance sell only when $v \ge \overline{v}_F$. As such, the firm will set advance selling price as $p^*_{aF} = p^l_{bb} =: I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]$. The corresponding effort level is $e^*_F = \frac{(1-\lambda)m_o}{4a}$ and the expected profit is $\pi^*_F = \beta a(e^*_F)^2 = \frac{\beta[(1-\lambda)m_o]^2}{16a}$. Summarizing the results in the above pricing scenarios lead to the conclusion in Proposition 2. □

Proof of Proposition 3. With similar analysis done in the proof of Proposition 2, we have $p^*_{rA} = 1$. Under advance selling with all-or-nothing clause, the firm would cancel advance selling, and thus it degenerates to the benchmark case of pure bank financing, unless all consumers purchase in advance. Moreover, according to Proposition 1, when $I > I_h$, without advance selling, the firm fails to obtain bank financing and goes bankrupt. Accordingly, the consumers achieve zero surplus under pure bank financing. That is, $\mathbb{E}[u_{bw}] = \mathbb{E}[u_{ww}] = 0$, which indicates that the firm can induce all consumers to purchase in advance if and only if

$$\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{ww}] = 0 \tag{27}$$

under advance selling with all-or-nothing clause for $I > I_h$, in accordance with Lemma A.1.

Next, note that according to Lemma A.2, the firm can advance sell with either high-price strategy (i.e., $p_a \geq p^h_{bb}$) or low-price strategy ($p^l_{bb} \leq p_a < p^h_{bb}$). In what follows, we consider these two pricing strategies, respectively.

High-price strategy. Given

$$p_a \geq p^h_{bb}, \tag{28}$$

assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e^h_{bb} = \frac{(1-\lambda)m_o}{2a}$ in accordance with Lemma A.2. Thus, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e^h_{bb} + (1 - e^h_{bb})\lambda] - p_a = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right] - p_a. \tag{29}$$

Combining (27) and (29), the segment-i consumers would advance buy if and only if

$$p_a \leq \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]. \tag{30}$$

To ensure the feasible region of p_a which satisfies (28) and (30) to be non-empty, we must have

$$v \geq \frac{I - \beta\lambda m_o}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]} =: \overline{v}_A. \tag{31}$$

When the condition (31) holds, the firm is able to advance sell with high-price strategy by setting $p_a \in \left[p^h_{bb}, \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]\right]$. According to Lemma A.2, under advance selling with high-price strategy, the firm's expected profit is $\pi^h_{bb}(p_a) = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right] + p_a - I$, which increases in p_a. Therefore, the optimal advance selling price, denoted as p^*_{aA}, should be the maximum value in the feasible region of p_a. That is, $p^*_{aA} = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]$. Correspondingly, the effort level is $e^*_A = \frac{(1-\lambda)m_o}{2a}$ and the firm's expected profit is $\pi^*_A = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right] + \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right] - I$.

Low-price strategy. Given

$$p^l_{bb} \leq p_a < p^h_{bb}, \tag{32}$$

assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e^l_{bb}(p_a) = \frac{(1-\lambda)m_o + \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{4a}$ in accordance with Lemma A.2. Correspondingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e^l_{bb}(p_a) + (1 - e^l_{bb}(p_a))\lambda] - p_a. \tag{33}$$

Combining (27) and (33), the segment-i consumers would advance buy if and only if $p_a \le \beta v[e^l_{bb}(p_a) + (1 - e^l_{bb}(p_a))\lambda]$ or equivalently

$$v \ge \frac{p_a}{\beta[e^l_{bb}(p_a) + (1 - e^l_{bb}(p_a))\lambda]} =: \underline{\phi}_A(p_a). \tag{34}$$

Constraints (32) and (34) together imply that the firm is able to advance sell with low-price strategy as long as v is sufficiently large. Specifically, the firm is able to advance sell with low-price strategy if and only if $v \ge \min_{p_a \in [p^l_{bb}, p^h_{bb})} \underline{\phi}_A(p_a)$. In what follows, we solve for this threshold.

The first-order derivative of $\underline{\phi}_A(p_a)$ with respect to p_a is

$$\frac{d\underline{\phi}_A(p_a)}{dp_a} = \frac{\eta(p_a)}{\beta^2\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}[\lambda + (1-\lambda)e^l_{bb}(p_a)]^2},$$

where

$$\eta(p_a) =: \beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)} + (1-\lambda)p_a + \frac{\beta(1-\lambda)^3 m_o^2}{4a} - 2(1-\lambda)(I - \beta\lambda m_o).$$

We note that $\eta(p_a)$ increases in p_a. Further, at $p_a = p^l_{bb}$, we have

$$\eta(p^l_{bb}) = -(1-\lambda)\left[I - \beta\lambda m_o - \beta\frac{(1-\lambda)^2 m_o^2}{8a}\right] < 0$$

for $I > I_h$. On the other hand, at $p_a = p^h_{bb}$,

$$\eta(p^h_{bb}) = -(1-\lambda)\left[I - \beta\lambda \cdot 2m_o - \beta\frac{(1-\lambda)^2(2m_o)^2}{8a}\right].$$

Depending on the value of m_o, $\eta(p^h_{bb})$ can be positive or negative. Next, we discuss these two cases separately.

1. If $\eta(p^h_{bb}) \le 0$, that is, $I \ge \beta\lambda \cdot 2m_o + \beta\frac{(1-\lambda)^2(2m_o)^2}{8a}$, or equivalently,

$$m_o \le \frac{\sqrt{4a^2\lambda^2 + 2a(1-\lambda)^2\frac{I}{\beta}} - 2a\lambda}{(1-\lambda)^2} =: M_A,$$

   then $\eta(p_a) \le 0$ for $p_a \in [p^l_{bb}, p^h_{bb})$ since $\eta(p_a)$ increases in p_a. This further implies that $\frac{d\underline{\phi}_A(p_a)}{dp_a} \le 0$ for $p_a \in [p^l_{bb}, p^h_{bb})$ since the denominator of $\frac{d\underline{\phi}_A(p_a)}{dp_a}$ is positive. Therefore, $\underline{\phi}_A(p_a)$ achieves the minimum at $p_a = p^h_{bb}$ and thus $\underline{\phi}_A(p^h_{bb}) = \frac{I - \beta\lambda m_o}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]} = \overline{v}_A$. That is, the lower bound of v beyond which the firm can advance sell with low-price strategy is identical to that with high-price strategy. Thus, in this case, the firm will always prefer to advance sell with high-price strategy.

2. If $\eta(p^h_{bb}) > 0$, that is, $m_o > M_A$, then there exists a unique root, denoted as $p^0_a \in (p^l_{bb}, p^h_{bb})$, such that $\eta(p^0_a) = 0$, since $\eta(p_a)$ increases in p_a and $\eta(p^l_{bb}) < 0$. That is,

$$\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p^0_a}{\beta} - \lambda m_o\right)} + (1-\lambda)p^0_a + \frac{\beta(1-\lambda)^3 m_o^2}{4a} - 2(1-\lambda)(I - \beta\lambda m_o) = 0,$$

   or equivalently,

$$p^0_a = \frac{8a\beta\lambda^2 + 4(1-\lambda)^2 I - [4a\lambda + (1-\lambda)^2 m_o]\sqrt{4\beta^2\lambda^2 + \frac{2\beta(1-\lambda)^2 I}{a}}}{2(1-\lambda)^2}.$$

Under p_a^0, we have $\eta(p_a) \le 0$ for $p_a \in [p_{bb}^l, p_a^0]$ and $\eta(p_a) > 0$ for $p_a \in (p_a^0, p_{bb}^h)$. Accordingly, $\frac{d\underline{\phi}_A(p_a)}{dp_a} \le 0$ for $p_a \in [p_{bb}^l, p_a^0]$ and $\frac{d\underline{\phi}_A(p_a)}{dp_a} > 0$ for $p_a \in (p_a^0, p_{bb}^h)$ since the denominator of $\frac{d\underline{\phi}_A(p_a)}{dp_a}$ is positive. This implies that $\underline{\phi}_A(p_a)$ values minimum at $p_a = p_a^0$. Thus,

$$\underline{v}_A := \underline{\phi}_A(p_a^0) = \frac{p_a^0}{\beta[e_{bb}^l(p_a^0) + (1 - e_{bb}^l(p_a^0))\lambda]} = \frac{2\sqrt{4a^2\lambda^2 + 2a\beta(1-\lambda)^2\frac{I}{\beta}} - 4a\lambda}{(1-\lambda)^2} - m_o < \overline{v}_A.$$

That is, the lower bound of v beyond which the firm is able to advance sell with low-price strategy is smaller than that with high-price strategy. Rearranging the above scenarios leads to the following three cases:

(a) If $v \ge \overline{v}_A$, the firm will advance sell with high-price strategy;

(b) If $\underline{v}_A \le v < \overline{v}_A$, the firm would advance with low-price strategy. Specifically, the optimal advance selling price p_{aA}^* would be equal to the larger root to the equation $\underline{\phi}_A(p_a) = v$, i.e.,

$$p_{aA}^* = \frac{\beta v\left[(1-\lambda)^2(v+m_o) + 4a\lambda\right] + \beta v(1-\lambda)\sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}.$$

Correspondingly, the effort level is

$$e_A^* = e_{bb}^l(p_{aA}^*) = \frac{(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}. \tag{35}$$

and the firm's expected profit is $\pi_A^* = \beta a(e_A^*)^2$.

(c) If $v < \underline{v}_A$, then the firm fails to advance selling even with the all-or-nothing clause. $\square$

Proof of Proposition 4. With similar analysis done in the proof of Proposition 2, we have $p_{rC}^* = 1$. At $I > I_h$, if all customers wait without purchasing coupons, according to Proposition 1, the firm fails to obtain bank financing and goes bankrupt. Accordingly the consumers achieve zero surplus, that is, $\mathbb{E}[u_{ww}] = 0$. Therefore, according to Lemma A.1, the firm can induce all consumers to purchase in advance if and only if

$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{bw}]. \tag{36}$$

Next, according to Lemma A.5, the firm can advance sell either with high-price strategy (i.e., $p_{bb}^l(p_s) \le p_a < p_{bb}^h(p_s)$) or with low-price strategy (i.e., $p_a \ge p_{bb}^h(p_s)$). In the following, we consider these two pricing strategies in turn. Moreover, note that although the coupon price p_a and spot price p_s are determined simultaneously by the firm, in what follows, we first characterize the optimal coupon price p_a for given p_s, and then derive the optimal spot price p_s.

High-price strategy. Given $p_a \ge p_{bb}^h(p_s)$, assuming that all k consumers purchase in advance, the firm would exert an effort of $e_{bb}^h(p_s) = \frac{(1-\lambda)(p_s+m_o)}{2a}$ in accordance with Lemma A.5. Thus, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta(v - p_s)[e_{bb}^h(p_s) + (1 - e_{bb}^h(p_s))\lambda] - p_a. \tag{37}$$

However, given $p_a \ge p_{bb}^h(p_s)$, when $k-1$ consumers purchase coupons in advance but one consumer waits, according to Lemma A.6, the firm's optimal effort level $e_{bw}(p_a, p_s)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.7, we have $p^l_{bb}(p_s) < p^l_{bw}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ when $k \to \infty$ for $I > I_h$. Thus, with risk-free strategy (i.e., $p_a \geq p^h_{bb}(p_s)$), the firm can set either $p_a \geq p^h_{bw}(p_s)$ or $p^h_{bb}(p_s) \leq p_a < p^h_{bw}(p_s)$. Subsequently, we consider these two pricing intervals, separately.

(1) Given $p_s \in [0,1]$ and $p_a \geq p^h_{bw}(p_s)$, assuming that $k-1$ consumers purchase coupons in advance but one consumer deviates to wait, the firm would put an effort of $e^h_{bw}(p_s) = \frac{(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]}{2a}$ according to Lemma A.6. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e^h_{bw}(p_s) + (1-e^h_{bw}(p_s))\lambda\right](v-p_r) = \beta\left[e^h_{bw}(p_s) + (1-e^h_{bw}(p_s))\lambda\right](v-1). \tag{38}$$

where $p_r = 1$ for similar reason as analyzed in the proof of Proposition 2. Anticipating this, from the equivalent condition (36) of inducing all k consumers to purchase coupons in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (37) and $\mathbb{E}[u_{bw}]$ in (38) into this equivalent condition gives

$$p_a \leq \beta(v-p_s)[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda] - \beta(v-1)\left[e^h_{bw}(p_s) + (1-e^h_{bw}(p_s))\lambda\right].$$

As $k \to \infty$, we have $p^h_{bw}(p_s) \to p^h_{bb}(p_s)$ and $e^h_{bw}(p_s) \to e^h_{bb}(p_s)$. Thus, the above associated conditions can be transformed and summarized as follows:

$$\begin{cases} p_s \in [0,1] & (39) \\ p_a \geq p^h_{bb}(p_s) & (40) \\ p_a \leq \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda](1-p_s) & (41) \end{cases}$$

We define the feasible region of (p_a, p_s) bounded by (39)-(41) as $\overline{\Delta}_C$, in which the firm is able to advance sell. It can be observed from (39)-(41) that $\overline{\Delta}_C$ is nonempty if and only if

$$\beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda](1-p_s) - p^h_{bb}(p_s) = \frac{\beta(1-\lambda)^2(p_s+m_o)(1-p_s)}{2a} + \beta\lambda(1+m_o) - I =: \delta(p_s) \geq 0. \tag{42}$$

Since $\delta(p_s)$ is quadratic on $p_s \in [0,1]$ and

$$\delta(p_s) \leq \frac{\beta(1-\lambda)^2(1+m_o)^2}{8a} + \beta\lambda(1+m_o) - I < I_h - I < 0$$

for $I > I_h$, which contradicts with (42). Therefore, $\overline{\Delta}_C$ is empty for $I > I_h$ and thus the firm fails to advance sell in this case.

(2) Given $p_s \in [0,1]$ and $p^h_{bb}(p_s) \leq p_a < p^h_{bw}(p_s)$, assuming that $k-1$ consumers purchase in advance but one consumer deviates to wait, the firm would put an effort of $e^l_{bw}(p_a, p_s)$, as given in Lemma A.6. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta[e^l_{bw}(p_a,p_s) + (1-e^l_{bw}(p_a,p_s))\lambda](v-p_r) = \beta[e^l_{bw}(p_a,p_s) + (1-e^l_{bw}(p_a,p_s))\lambda](v-1), \tag{43}$$

where $p_r = 1$ for similar reason as analyzed in the proof of Proposition 2. Anticipating this, from the equivalent condition (36) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (37) and $\mathbb{E}[u_{bw}]$ in (43) into this equivalent condition gives

$$p_a \leq \beta(v-p_s)[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda] - \beta(v-1)\left[e_{bw}^l(p_a,p_s) + (1-e_{bw}^h(p_a,p_s))\lambda\right].$$

When $k \to \infty$, we have $p_{bw}^h(p_s) \to p_{bb}^h(p_s)$ and thus $p_a \to p_{bb}^h(p_s)$, which further indicates $e_{bw}^l(p_a,p_s) \to e_{bb}^l(p_a,p_s) \to e_{bb}^h(p_s)$. Thus, the above conditions are transformed and summarized into the following:

$$\begin{cases} p_s \in [0,1] & (44) \\ p_a = p_{bb}^h(p_s) & (45) \\ p_a \leq \beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda](1-p_s) & (46) \end{cases}$$

Obviously, the feasible region of (p_a,p_s) constrained by (44)-(46) is a subset of $\overline{\Delta}_C$. Therefore, similar to the previous case with $p_a \geq p_{bw}^h(p_s)$, the firm also fails to advance sell in this case.

Low-price strategy. Given $p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s)$, assuming that all k consumers purchase in advance, the firm would exert an effort of $e_{bb}^l(p_a,p_s) = \frac{(1-\lambda)(p_s+m_o)+\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}{4a}$ in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta(v-p_s)[e_{bb}^l(p_a,p_s) + (1-e_{bb}^l(p_a,p_s))\lambda] - p_a. \tag{47}$$

However, given $p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s)$, when $k-1$ consumers purchase in advance but one consumer deviates to wait, according to Lemma A.6, the firm's optimal effort level $e_{bw}(p_a,p_s)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval that p_a locates in. In accordance with Lemma A.7, we have $p_{bb}^l(p_s) < p_{bw}^l(p_s) < p_{bb}^h(p_s) < p_{bw}^h(p_s)$ when $k \to \infty$ for $I > I_h$. Thus, with low-price strategy (i.e., $p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s)$), the firm can set either $p_{bw}^l(p_s) \leq p_a < p_{bb}^h(p_s)$ or $p_{bb}^l(p_s) \leq p_a < p_{bw}^l(p_s)$. Next, we consider these two pricing intervals in turn.

1. Given $p_s \in [0,1]$ and $p_{bw}^l(p_s) \leq p_a < p_{bb}^h(p_s)$, assuming that $k-1$ consumers purchase in advance but one consumer deviates to wait, the firm would put an effort of $e_{bw}^l(p_a,p_s)$, as given in Lemma A.6. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta[e_{bw}^l(p_a,p_s) + (1-e_{bw}^l(p_a,p_s))\lambda](v-p_r) = \beta[e_{bw}^l(p_a,p_s) + (1-e_{bw}^l(p_a,p_s))\lambda](v-1), \tag{48}$$

where $p_r = 1$ for similar reason as analyzed in the proof of Proposition 2. Anticipating this, from the equivalent condition (36) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (47) and $\mathbb{E}[u_{bw}]$ in (48) into this equivalent condition gives

$$p_a \leq \beta(v-p_s)[e_{bb}^l(p_a,p_s) + (1-e_{bb}^l(p_a,p_s))\lambda] - \beta(v-1)[e_{bw}^l(p_a,p_s) + (1-e_{bw}^l(p_a,p_s))\lambda].$$

As $k \to \infty$, we have $p_{bw}^l(p_s) \to p_{bb}^l(p_s)$ and $e_{bw}^l(p_a,p_s) \to e_{bb}^l(p_a,p_s)$. Therefore, the above conditions can be transformed and summarized into the following:

$$\begin{cases} p_s \in [0,1] \\ p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s) \\ p_a \leq \beta[e_{bb}^l(p_a,p_s) + (1-e_{bb}^l(p_a,p_s))\lambda](1-p_s) \end{cases} \tag{49}$$

in which Eq. (49) can be reformulated as:

$$2ap_a^2-\beta(1-p_s)[4a\lambda+(1-\lambda)^2(1+m_o)]p_a+[2a\beta^2\lambda^2+\beta(1-\lambda)^2I](1-p_s)^2\le 0. \tag{50}$$

Note that the expression in the left side of (50) is quadratic in p_a, so it has no feasible solution if $I>\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$, which holds for $I>I_h$. Therefore, the firm fails to advance sell in the pricing interval of $p_{bw}^l(p_s)\le p_a<p_{bb}^h(p_s)$ as $k\to\infty$ for $I>I_h$.

2. Given $p_s\in[0,1]$ and $p_{bb}^l(p_s)\le p_a<p_{bw}^l(p_s)$, assuming that $k-1$ consumers purchase in advance but one consumer deviates to wait, the firm fails to advance sell according to Lemma A.6. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}]=\mathbb{E}[u_{ww}]=0. \tag{51}$$

Anticipating this, from the equivalent condition (36) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}]\ge\mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (37) and $\mathbb{E}[u_{bw}]$ in (51) into this equivalent condition gives $p_a\le\beta\,[e_{bb}^l(p_a,p_s)+(1-e_{bb}^l(p_a,p_s))\lambda]\,(v-p_s)$ or equivalently, $v\ge\frac{p_a}{\beta\left[(1-\lambda)e_{bb}^l(p_a,p_s)+\lambda\right]}+p_s$. As $k\to\infty$, we have $p_{bw}^l(p_s)\to p_{bb}^l(p_s)$. Thus, the above constraints can be transformed and summarized into the following:

$$\begin{cases} p_s\in[0,1] & (52)\\ p_a=p_{bb}^l(p_s) & (53)\\ v\ge\dfrac{p_{bb}^l(p_s)}{\beta\left[(1-\lambda)e_{bb}^l(p_{bb}^l(p_s),p_s)+\lambda\right]}+p_s=\dfrac{I-\beta\left[\lambda(p_s+m_o)+\frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]}{\beta\left[\frac{(1-\lambda)^2(p_s+m_o)}{4a}+\lambda\right]}+p_s=:\underline{\phi}_C(p_s). & (54)\end{cases}$$

The firm is able to sell in advance if and only if (p_a,p_s) locates in the feasible region bounded by the above constraints (52)–(54), according to which, the firm is able to advance sell when v is sufficiently large. In what follows, we solve the threshold v beyond which the firm can successfully advance sell. We denote this threshold v as $\overline{v}_C$, which satisfies $\overline{v}_C=\min_{p_s\in[0,1]}\underline{\phi}_C(p_s)$. The first-order derivative of $\underline{\phi}_C(p_s)$ with respect to p_s is

$$\frac{d\underline{\phi}_C(p_s)}{dp_s}=\frac{-\left[I-\lambda(p_s+m_o)-\frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]\frac{(1-\lambda)^2}{4a}}{\beta\left[\frac{(1-\lambda)^2(p_s+m_o)}{4a}+\lambda\right]^2}\le 0.$$

Thus, $\underline{\phi}_C(p_s)$ values minimum at $p_s=1$, and

$$\overline{v}_C=\frac{I-\beta\lambda m_o+\beta\frac{(1-\lambda)^2(1-m_o^2)}{8a}}{\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{4a}\right]},$$

which increases in I.

Provided $v\ge\overline{v}_C$, the firm is able to advance sell. According to Lemma A.5, the optimal effort level is $e_{bb}^l(p_a,p_s)=e_{bb}^l(p_{bb}^l(p_s),p_s)=\frac{(1-\lambda)(p_s+m_o)}{4a}$, and accordingly, the firm's expected profit is $\pi_{bb}^l(p_a,p_s)=\pi_{bb}^l(p_{bb}^l(p_s),p_s)=\frac{\beta(1-\lambda)^2(p_s+m_o)^2}{16a}$. To maximize $\pi_{bb}^l(p_a,p_s)$, the firm will set optimal coupon price and spot price as $p_{sC}^*=1$ and $p_{aC}^*=I-\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$, respectively. Accordingly, the equilibrium effort is $e_C^*=\frac{(1-\lambda)(1+m_o)}{4a}$, and the firm's expected profit is $\pi_C^*=\frac{\beta(1-\lambda)^2(1+m_o)^2}{16a}$.

Summarizing the results derived above for high-price and low-price strategies, we conclude that the firm will advance sell with low-price strategy provided that $v\ge\overline{v}_C$. □

Proof of Lemma 1. First, by comparing $\overline{v}_F$ and $\overline{v}_A$, we have that $\overline{v}_A \leq \overline{v}_F$ if and only if $I \geq \beta\lambda m_o + \frac{\beta}{2}\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]$, which always holds for $I \geq I_h$ as $v_m \geq 1 + m_o$. Thus, in the region when the firm can implement advance selling under full price ("F"), she can also implement advance selling with the all-or-nothing clause ("A").

Second, by comparing the profit function under these two mechanisms, we could clearly see that π_A^* under all-or-nothing ("A") with $v \geq \overline{v}_A$ (Scenario 1 in Proposition 3) is greater than π_F^* under full price ("F") with $v \geq \overline{v}_F$ (Scenario 1 in Proposition 2) if and only if

$$v \geq \frac{I - \beta\left[\lambda m_o + \frac{3(1-\lambda)^2 m_o^2}{16a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]},$$

which holds when $v \geq \overline{v}_F$ and $I \geq I_h$. That is, the firm's profit under "A" is always greater than that under "F" when "F" is feasible.

Combining the above two points, we can conclude that for $I \geq I_h$, advance selling with the all-or-nothing clause ("A") dominates the classic one ("F"). □

Proof of Proposition 5. When $m_o \leq 1$, we could not have $m_o > M_A$ for $I > I_h$. Thus, Scenario (a) in Case 2 in Proposition 3 is irrelevant. This means that the firm will succeed in advance selling if $v > \overline{v}_A$ and fails to advance sell if otherwise under all-or-nothing clause. Further, it can be derived that $\overline{v}_A > \overline{v}_C$ since $I > I_h$ and $m_o \leq 1$.

Next, a comparison of $\pi_A^* = \beta\left[\frac{[(1-\lambda)m_o]^2}{4a} + \lambda m_o\right] + \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right] - I$ and $\pi_C^* = \frac{\beta(1-\lambda)^2(1+m_o)^2}{16a}$ leads to that scheme "A" dominates scheme "C" when

$$v > \frac{I - \beta\lambda m_o + \frac{\beta(1-\lambda)^2(1+3m_o)(1-m_o)}{16a}}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]} =: \overline{v} > \overline{v}_A.$$

Combining these results lead to the statements in the proposition. □

Proof of Proposition 6. When $m_o > 1$, depending on the relationship between m_o and M_A (or equivalently, between I and $\beta\left[2\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{2a}\right]$, we compare scheme "A" and scheme "C" in the following two cases:

1. If $m_o \leq M_A$, under scheme "A", according to Proposition 3, the firm would advance sell with high-price strategy if $v \geq \overline{v}_A$ and fail to advance sell otherwise. Further, we have $\overline{v}_A < \overline{v}_C$ since $m_o \leq M_A$ and $m_o > 1$, which means scheme "A" applies in a wider region of parameter combinations than "C". Finally, we note that $\overline{v}_C > \overline{v}$, and thus $\pi_A^* > \pi_C^*$ for $v > \overline{v}_C$. Therefore, the firm would always use scheme "A";

2. If $m_o > M_A$, under scheme "A", according to Proposition 3, the firm would advance sell with low-price strategy if $\underline{v}_A \leq v < \overline{v}_A$ and with high-price strategy if $v \geq \overline{v}_A$. It can be derived that $\overline{v}_C > \underline{v}_A$, which means scheme "A" applies in a wider region of parameter combinations than "C". Moreover, the firm

earns a lower expected profit under scheme "C" than under scheme "A" even with low-price strategy, i.e.,

$$\pi_A^* = \beta a \left[\frac{(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a} \right]^2$$
$$> \beta a \left[\frac{(1-\lambda)(1+m_o)}{4a}\right]^2 = \pi_C^*$$

Thus, the firm would always use scheme "A" in this case. □

Proof of Proposition 7. According to Proposition 1, for $I_l < I \le I_h$, the firm can obtain bank financing even without advance selling. In this proof, when analyzing the firm's pricing strategies, we consider only the first condition to induce advance buy in Lemma A.1, that is,

$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{bw}], \tag{55}$$

and will show that even under this case with relaxed conditions, the firm is either unable or unwilling to advance sell.

From Lemma A.2, the firm can advance sell with either high-price strategy (i.e., $p_a \ge p_{bb}^h$) or low-price strategy ($p_{bb}^l \le p_a < p_{bb}^h$). In what follows, we consider these two pricing strategies, respectively.

High-price strategy. Given $p_a \ge p_{bb}^h$, assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e_{bb}^h = \frac{(1-\lambda)m_o}{2a}$ in accordance with Lemma A.2. Thus, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e_{bb}^h + (1-e_{bb}^h)\lambda] - p_a = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right] - p_a. \tag{56}$$

However, given $p_a \ge p_{bb}^h$, when $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, according to Lemma A.3, the firm's optimal effort level $e_{bw}(p_a)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.4, $p_{bw}^h > p_{bb}^h > \max(p_{bw}^l, p_{bb}^l)$ holds when $k \to \infty$. Thus, with the high-price strategy, the firm might set $p_a \ge p_{bw}^h$, or $p_a \in [p_{bb}^h, p_{bw}^h)$. Next, we consider these two pricing intervals respectively.

Scenario 1. If the firm sets

$$p_a \ge p_{bw}^h, \tag{57}$$

then given p_a, assuming that $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, the firm would put an effort of $e_{bw}^h = \frac{(1-\lambda)(m_o+\frac{1}{k})}{2a}$ according to Lemma A.3. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e_{bw}^h + (1-e_{bw}^h)\lambda\right](v-p_r) = \beta\left[\lambda + \frac{(1-\lambda)^2(m_o+\frac{1}{k})}{2a}\right](v-1), \tag{58}$$

where $p_r = 1$ for similar reason as analyzed in the proof of Proposition 2. Anticipating this, from the necessary condition (55) of inducing all k consumers to purchase in advance, $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$ should be satisfied. By substituting $\mathbb{E}[u_{bb}]$ in (56) and $\mathbb{E}[u_{bw}]$ in (58) into this necessary condition gives

$$p_a \leq \beta\left[v\left(e^h_{bb} - e^h_{bw}\right)(1-\lambda) + e^h_{bw}(1-\lambda) + \lambda\right]. \tag{59}$$

When $k \to \infty$, we have $p^h_{bw} \to p^h_{bb}$ and $e^h_{bw} \to e^h_{bb}$. Thus, the constraints of (57) and (59) are transformed into the following:

$$\begin{cases} p_a \geq p^h_{bb} & (60) \\ p_a \leq \beta\left[e^h_{bb}(1-\lambda) + \lambda\right] & (61) \end{cases}$$

Next, we find optimal p_a bounded by (60)-(61) to maximize the firm's profit π^h_{bb}. Note that the feasible region constrained by (60)-(61) is nonempty only if $\beta\left[e^h_{bb}(1-\lambda)+\lambda\right] - p^h_{bb} \geq 0$, or equivalently, $I \leq \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o}{2a}\right]$. When the above condition is met, the firm would set p_a at the highest level that the customer is willing to advance buy. Thus, from Eq. (61), we have $p^*_a = \beta\left[e^h_{bb}(1-\lambda)+\lambda\right]$. Accordingly, the firm's expected profit is

$$\pi^h_{bb} = \beta\left[\frac{\left[(1-\lambda)m_o\right]^2}{4a} + \lambda m_o\right] + p^*_a - I = \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2(m_o^2+2m_o)}{4a}\right] - I.$$

Let $\zeta(I) = \pi^*_B - \pi^h_{bb}$ be the difference between the firm's profit under pure bank financing and that under the above advance selling strategy. We can obtain that $\zeta'(I) < 0$. Also, $\zeta(I) \geq 0$ at $I = \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o}{2a}\right]$. Thus, we have $\zeta(I) \geq 0$ for $I_l < I \leq \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o}{2a}\right]$. That is, the firm is unwilling to advance sell even though she is able to in this scenario.

Scenario 2. If the firm sets

$$p^h_{bb} \leq p_a < p^h_{bw}, \tag{62}$$

then given p_a, assuming that $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, the firm would put an effort of $e^l_{bw}(p_a)$, according to Lemma A.3. Accordingly, the expected surplus of the segment-i consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e^l_{bw}(p_a) + (1-e^l_{bw}(p_a))\lambda\right](v - p_r), \tag{63}$$

where $p_r = 1$ for the reason similar to that in the proof of Proposition 2. Anticipating this, from the necessary condition (55) of inducing all k consumers to purchase in advance, $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$ should be satisfied. By substituting $\mathbb{E}[u_{bb}]$ in (56) and $\mathbb{E}[u_{bw}]$ in (63) into this equivalent condition gives

$$p_a \leq \beta\left[v\left(e^h_{bb} - e^l_{bw}(p_a)\right)(1-\lambda) + e^l_{bw}(p_a)(1-\lambda) + \lambda\right]. \tag{64}$$

As $k \to \infty$, we have $p^h_{bw} \to p^h_{bb}$ and $e^l_{bw}(p_a) \to e^l_{bb}(p_a)$. Thus, the constraints of (62) and (64) are transformed into

$$\begin{cases} p_a = p^h_{bb} & (65) \\ p_a \leq \beta\left[e^h_{bb}(1-\lambda) + \lambda\right]. & (66) \end{cases}$$

Evidently, the feasible region of p_a satisfying the above constraints (65)-(66) is a sub-region of that constrained by (60)-(61). Therefore, with reference to the previous case of $p_a \geq p^h_{bw}$, the firm is unwilling to advance sell even though there exists a feasible region of p_a that the firm is able to advance sell.

Low-price strategy. Given $p^l_{bb} \leq p_a < p^h_{bb}$, assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e^l_{bb}(p_a) = \frac{(1-\lambda)m_o + \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{4a}$ in accordance with Lemma A.2. Thus, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e^l_{bb}(p_a) + (1 - e^l_{bb}(p_a))\lambda] - p_a. \tag{67}$$

However, given $p^l_{bb} \leq p_a < p^h_{bb}$, when $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, according to Lemma A.3, the firm's optimal effort level $e_{bw}(p_a)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.4, we have the following two cases:

I. If $\hat{I}_l \leq I \leq I_h$, where $\hat{I}_l =: \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o(2+m_o)}{8a}\right]$, then we have $p^l_{bb} < p^l_{bw} < p^h_{bb} < p^h_{bw}$ and $p^l_{bw} \to p^l_{bb}$ as $k \to \infty$;

II. If $I_l < I < \hat{I}_l$, then we have $p^l_{bw} < p^l_{bb} < p^h_{bb} < p^h_{bw}$ and $p^l_{bw} \to p^l_{bb}$ as $k \to \infty$;

In what follows, we consider these two cases, respectively.

Case I. In this case, we have $\hat{I}_l \leq I \leq I_h$ and $p^l_{bb} < p^l_{bw} < p^h_{bb} < p^h_{bw}$. Thus, with low-price strategy, the firm can set either $p^l_{bw} \leq p_a < p^h_{bb}$, or $p^l_{bb} \leq p_a < p^l_{bw}$. Next, we consider these two pricing intervals in turn.

Scenario 1. If the firm sets

$$p^l_{bw} \leq p_a < p^h_{bb}, \tag{68}$$

then given p_a, assuming that $k-1$ consumers purchase in advance but one consumer deviates to wait, the firm would put an effort of $e^l_{bw}(p_a)$ according to Lemma A.3. Accordingly, the expected surplus of the segment-i consumer who deviates to wait is

$$\mathbb{E}[u_{bw}] = \beta\left[e^l_{bw}(p_a) + (1 - e^l_{bw}(p_a))\lambda\right](v - p_r) = \beta\left[e^l_{bw}(p_a) + (1 - e^l_{bw}(p_a))\lambda\right](v - 1), \tag{69}$$

where $p_r = 1$ for the reason similar to that in the proof of Proposition 2. Anticipating this, from the necessary condition (55) of inducing all k consumers to purchase in advance, the consumers will purchase in advance if and only if $\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{bw}]$. By substituting $\mathbb{E}[u_{bb}]$ in (67) and $\mathbb{E}[u_{bw}]$ in (69) into this necessary condition gives

$$p_a \leq \beta\left[v\left(e^l_{bb}(p_a) - e^l_{bw}(p_a)\right)(1-\lambda) + e^l_{bw}(p_a)(1-\lambda) + \lambda\right]. \tag{70}$$

As $k \to \infty$, we have $p^l_{bw} \to p^l_{bb}$ and $e^l_{bw}(p_a) \to e^l_{bb}(p_a)$. Thus, the constraints of (68) and (70) are transformed into

$$\begin{cases} p^l_{bb} \leq p_a < p^h_{bb} \\ p_a \leq \beta\left[e^l_{bb}(p_a)(1-\lambda) + \lambda\right]. \end{cases} \tag{71}$$

The condition (71) can be rewritten as

$$2a\left(\frac{p_a}{\beta} - \lambda\right)^2 - (1-\lambda)^2(1+m_o)\left(\frac{p_a}{\beta} - \lambda\right) + (1-\lambda)^2\left[\frac{I}{\beta} - \lambda(1+m_o)\right] \leq 0,$$

which is quadratic in p_a. Solving this inequality leads to

$$\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)-(1-\lambda)\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{4a}\right]\leq p_a$$

$$\leq\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)+(1-\lambda)\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{4a}\right].$$

Thus, the optimal advance selling price satisfies

$$p_a^*\leq\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)+(1-\lambda)\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{4a}\right].$$

Accordingly, the firm's expected profit satisfies

$$\begin{aligned}\pi_{bb}^l(p_a^*)&=\beta a[e_{bb}^l(p_a^*)]^2\\&\leq\beta a\left[\frac{(1-\lambda)(1+m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{4a}\right]^2\leq\beta a(e_B^*)^2=\pi_B^*.\end{aligned}$$

Thus, the firm is unwilling to advance sell even though she is able to.

Scenario 2. If the firm sets

$$p_{bb}^l\leq p_a<p_{bw}^l, \tag{72}$$

then given p_a, assuming that $k-1$ segment-i consumers purchase in advance but one segment-i consumer deviates to wait, the firm fails to secure bank financing and has to declare bankruptcy, according to Lemma A.3. Accordingly, the expected surplus of the consumer who deviates to wait is

$$\mathbb{E}[u_{bw}]=0. \tag{73}$$

Anticipating this, by substituting $\mathbb{E}[u_{bb}]$ in (67) and $\mathbb{E}[u_{bw}]$ in (73) into the necessary condition (55) of inducing all k consumers to purchase in advance, one necessary condition for the segment-i consumers to purchase in advance is $p_a\leq\beta v[e_{bb}^l(p_a)+(1-e_{bb}^l(p_a))\lambda]$, or equivalently,

$$v\geq\frac{p_a}{\beta[e_{bb}^l(p_a)+(1-e_{bb}^l(p_a))\lambda]}. \tag{74}$$

On the other aspect, as $k\to\infty$, we have $p_{bw}^l\to p_{bb}^l$, and thus the constraint (72) is transformed into

$$p_a=p_{bb}^l=:I-\beta\left[\lambda m_o+\frac{(1-\lambda)^2m_o^2}{8a}\right]. \tag{75}$$

The constraints of (74) and (75) jointly imply that

$$v\geq\frac{p_{bb}^l}{\beta[e_{bb}^l(p_{bb}^l)+(1-e_{bb}^l(p_{bb}^l))\lambda]}=\frac{I-\beta\left[\lambda m_o+\frac{(1-\lambda)^2m_o^2}{8a}\right]}{\beta\left[\lambda+\frac{(1-\lambda)^2m_o}{4a}\right]}.$$

That is, the firm is able to advance sell only when $v \geq \frac{I-\beta\left[\lambda m_o+\frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda+\frac{(1-\lambda)^2 m_o}{4a}\right]}$. On condition that the firm is able to advance sell, the optimal effort level is $e_{bb}^l = \frac{(1-\lambda)m_o}{4a}$ and thus the expected profit satisfies

$$\pi_{bb}^l = \beta a(e_{bb}^l)^2 < \beta a(e_B^*)^2 = \pi_B^*.$$

Consequently, the firm is unwilling to advance sell in this case.

Case II. In this case, we have $I_l < I < \hat{I}_l$, then we have $p_{bw}^l < p_{bb}^l < p_{bb}^h < p_{bw}^h$. Thus, with low-price strategy, the firm will set $p_{bb}^l \leq p_a < p_{bb}^h$. With the same analysis for $p_{bw}^l \leq p_a < p_{bb}^h$ in Case I, it can be shown that the firm is unwilling to advance sell even if she is able to in this case. □

Proof of Proposition 8. For Part (i), which is related to advance selling with coupons ("C"), the result follows directly from Proposition A.5.

For Part (ii) (advance selling with an all-or-nothing clause, "A"), given $m_o \leq v-1$, according to Proposition A.3 and Proposition 1, the difference of profits between advance selling with an all-or-nothing clause and pure bank financing is

$$\pi_A^{s*} - \pi_B^* = \beta\left[-\frac{(1-\lambda)^2v^2 + (1-\lambda)v\sqrt{(1-\lambda)^2v^2 - 8a\left(\frac{I}{\beta}-\lambda v\right)}}{8a} - \frac{1}{2}\left(\frac{I}{\beta}-\lambda v\right) + \frac{(1-\lambda)^2 m_o v}{2a} + \frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right].$$

The right-side of the expression of $\pi_A^{s*} - \pi_B^*$ is quadratic in m_o, which indicates that $\pi_A^{s*} - \pi_B^* > 0$ if

$$m_o > \frac{-\left[\lambda + \frac{(1-\lambda)^2 v}{2a}\right] + \sqrt{\left[\lambda + \frac{(1-\lambda)^2 v}{2a}\right]^2 + \frac{(1-\lambda)^2}{a}\left[\frac{(1-\lambda)^2v^2+(1-\lambda)v\sqrt{(1-\lambda)^2v^2-8a(\frac{I}{\beta}-\lambda v)}}{8a} + \frac{1}{2}(\frac{I}{\beta}-\lambda v)\right]}}{\frac{(1-\lambda)^2}{2a}} =: \chi(v). \tag{76}$$

and $\pi_A^{s*} - \pi_B^* \leq 0$ if otherwise. Moreover, we have

$$\begin{aligned}\chi(v) &> \frac{-\left[\lambda + \frac{(1-\lambda)^2 v}{2a}\right] + \sqrt{\left[\lambda + \frac{(1-\lambda)^2 v}{2a}\right]^2 + \frac{(1-\lambda)^4 v^2}{8a^2}}}{\frac{(1-\lambda)^2}{2a}} \\ &> \frac{-\left[\lambda + \frac{(1-\lambda)^2}{2a}\right] + \sqrt{\left[\lambda + \frac{(1-\lambda)^2}{2a}\right]^2 + \frac{(1-\lambda)^4}{8a^2}}}{\frac{(1-\lambda)^2}{2a}} =: \overline{M}_A > 0\end{aligned} \tag{77}$$

where the first ">" holds because $I > I_l$ or equivalently $\frac{I}{\beta} > \lambda v$; the second ">" holds because $\frac{-\left[\lambda+\frac{(1-\lambda)^2 v}{2a}\right]+\sqrt{\left[\lambda+\frac{(1-\lambda)^2 v}{2a}\right]^2+\frac{(1-\lambda)^4 v^2}{8a^2}}}{\frac{(1-\lambda)^2}{2a}}$ is an increasing function of v and $v \geq 1$. (76) and (77) indicate that $\pi_A^{s*} - \pi_B^* < 0$ when $m_o \leq \overline{M}_A$. Therefore, Statement (a) of Part (ii) in Proposition 8 is proved.

We continue to prove Statement (b) of Part (ii). For $m_o > \overline{M}_A$, there are two relevant cases:

1. If $m_o > 1$ and $I_m^A := \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2 m_o}{2a}\right] < I \leq I_h =: \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$, there are two scenarios:
   (a) When $v \leq 1+m_o$, advance selling with all-or-nothing clause dominates pure bank financing only if $v \in [\underline{v}_A^l, 1+m_o]$ according to Proposition A.4, where $\underline{v}_A^l$ is defined in Proposition A.4;

(b) When $v > 1 + m_o$, according to Proposition A.3, the firm is able to advance sell if $v \geq \max\{1 + m_o, \overline{v}_A^s, v^0\} = 1 + m_o$, where $\overline{v}_A^s$ is defined in Proposition A.3; v^0 is uniquely determined by $I = I_h$, i.e., the positive solution to the equation of $\beta\left[\lambda v + \frac{(1-\lambda)^2 v^2}{8a}\right] = I$; and $\max\{1 + m_o, \overline{v}_A^s, v^0\} = 1 + m_o$ holds because $m_o > 1$ and $I_m^A < I \leq I_h$. Moreover, according to Proposition A.3, $\pi_A^{s*} - \pi_B^*$ decreases in v. This indicates that there exists a unique $\overline{v}_A^0$ such that $\pi_A^{s*}(\overline{v}_A^0) - \pi_B^*(\overline{v}_A^0) = 0$. We have $\pi_A^{s*} - \pi_B^* > 0$ if $v < \overline{v}_A^0$ and $\pi_A^{s*} - \pi_B^* \leq 0$ if otherwise. These conclusions indicate that advance selling with an all-or-nothing clause dominates pure bank financing only if $v \in (1 + m_o, \overline{v}_A^0)$

Based on the discussions in the above two scenarios, we can define two thresholds of v: $\underline{w}_A = \underline{v}_A^l$ and $\overline{w}_A = \overline{v}_A^0$, and advance selling with an all-or-nothing clause dominates pure bank financing only if $v \in (\underline{w}_A, \overline{w}_A)$.

2. Otherwise, there are also two scenarios:

(a) When $v \leq 1 + m_o$, advance selling with an all-or-nothing clause is dominated by pure bank financing according to Proposition A.4;

(b) When $v > 1 + m_o$, similarly, according to Proposition A.3, the firm is able to advance sell if $v \geq \max\{1 + m_o, \overline{v}_A^s, v^0\}$. Moreover, according to Proposition A.3, $\pi_A^{s*} - \pi_B^*$ decreases in v. This indicates that there exists a unique $\overline{v}_A^0$ such that $\pi_A^{s*}(\overline{v}_A^0) - \pi_B^*(\overline{v}_A^0) = 0$. We have $\pi_A^{s*} - \pi_B^* > 0$ if $v < \overline{v}_A^0$ and $\pi_A^{s*} - \pi_B^* \leq 0$ if otherwise. These conclusions indicate that advance selling with an all-or-nothing clause dominates pure bank financing only if $v \in (\max\{1 + m_o, \overline{v}_A^s, v^0\}, \overline{v}_A^0)$.

Based on the discussions in the above two scenarios, we can define two thresholds of v: $\underline{w}_A = \max\{1 + m_o, \overline{v}_A^s, v^0\}$ and $\overline{w}_A = \overline{v}_A^0$, and advance selling with an all-or-nothing clause dominates pure bank financing only if $v \in (\underline{w}_A, \overline{w}_A)$.

Combining the above cases, Statement (b) in Part (ii) of Proposition 8 holds. □

Proof of Proposition 9. This result follows directly from Proposition A.6. □

Proof of Proposition 10. This result follows from Proposition A.7 by substituting $m_o = 1$. Note that at $m_o = 1$, $\overline{w}_H^l = 2$, and thus $v > \overline{w}_H^l$ is irrelevant when $v \in (1, 2)$. Therefore, Scenario (ii)(b) in Proposition A.7 becomes irrelevant. □

Proof of Proposition 11. The result follows directly from Proposition A.8. □

Appendix C: Proofs of Supplemental Results

Proof of Lemma A.1. The first condition in Lemma A.1 ensures the existence of buying equilibrium (i.e., no consumer will deviate to wait if all other consumers purchase in advance. The second condition ensures that buying equilibrium is a dominant Nash strategy, which holds if at least one of the following conditions hold:

1. $\{\mathbb{E}[u_{wb}] \le \mathbb{E}[u_{ww}]\} \cap \{\mathbb{E}[u_{ww}] \le \mathbb{E}[u_{bb}]\}$, which ensures that even if the waiting equilibrium exists, it is dominated by buying equilibrium;
2. $\mathbb{E}[u_{wb}] > \mathbb{E}[u_{ww}]$, which represents the case that all k consumers wait is not an equilibrium.

Combining these two conditions lead to $\mathbb{E}[u_{wb}] > \mathbb{E}[u_{ww}]$ or $\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{ww}]$, the second condition in the Lemma. □

Proof of Proposition A.1. Similar to the analysis in the proof of Proposition 2, we have $p^*_{rH} = 1$. Under advance selling with an all-or-nothing clause, the firm would cancel advance selling and thus it degenerates to the benchmark case of pure bank financing, unless all consumers purchase in advance. Moreover, according to Proposition 1, when $I > I_h$, the firm fails to secure a bank loan and thus falls into bankruptcy under pure bank financing. Therefore, we have $\mathbb{E}[u_{bw}] = \mathbb{E}[u_{ww}] = 0$, which implies that the equivalent condition of inducing all segment-i consumers to purchase in advance degenerate to

$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{ww}] = 0 \tag{78}$$

for $I > I_h$, according to Lemma A.1.

According to Lemma A.5, the firm could advance sell by either high-price (i.e., $p_a \ge p^h_{bb}(p_s)$) strategy or low-price (i.e., $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$) strategy. In what follows, we first consider these two pricing strategies, respectively, and then compare the optimal results under these two strategies to derive the equilibrium results.

High-price strategy (i.e., $p_a \ge p^h_{bb}(p_s)$). Given $p_s \in [0,1]$ and $p_a \ge p^h_{bb}(p_s)$, assuming that all k segment-i consumers purchase coupons in advance, the firm would exert an effort of $e^h_{bb}(p_s) = \frac{(1-\lambda)(p_s+m_o)}{2a}$ in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta[e^h_{bb}(p_s) + (1 - e^h_{bb}(p_s))\lambda](v - p_s) - p_a. \tag{79}$$

Anticipating this, the consumers would purchase in advance if and only if the equivalent condition (78) is met. Substituting $\mathbb{E}[u_{bb}]$ in (79) into the equivalent condition (78) gives $p_a \le \beta[e^h_{bb}(p_s) + (1 - e^h_{bb}(p_s))\lambda](v - p_s)$, or equivalently

$$v \ge \frac{p_a}{\beta[e^h_{bb}(p_s) + (1 - e^h_{bb}(p_s))\lambda]} + p_s := \overline{\phi}_H(p_a, p_s).$$

Summarizing the above constraints, the firm's optimization problem can be formulated as follows:

$$\max\ \pi^h_{bb}(p_a, p_s) = \beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a} + \lambda(p_s+m_o)\right] + p_a - I \tag{80}$$

$$s.t. \quad \begin{cases} p_s \in [0,1] \\ p_a \geq p^h_{bb}(p_s) \\ v \geq \frac{p_a}{\beta[e^h_{bb}(p_s)+(1-e^h_{bb}(p_s))\lambda]} + p_s =: \overline{\phi}_H(p_a,p_s) \end{cases} \tag{81}$$

In what follows, we solve the above optimization problem in two steps.

Step 1. Note that the constraints in Eq. (81) imply that the firm is able to advance sell as long as v is sufficiently large. Let $\overline{\Delta}_H$ be the feasible region of (p_a,p_s), which is bounded by the first and second constraints of (81). Further, define $\underline{v}_H = \min_{(p_a,p_s)\in\overline{\Delta}_H} \overline{\phi}_H(p_a,p_s)$. Then, the firm is able to advance sell if and only if $v \geq \underline{v}_H$. In **Step 1, we will solve the value $\underline{v}_H$.**

Given $p_s \in [0,1]$, $\underline{\phi}_H(p_a,p_s)$ increases in $p_a \in [p^h_{bb}(p_s),\infty)$ and thus $\overline{\phi}_H(p_a,p_s)$ achieves its minimum at $p_a = p^h_{bb}(p_s)$, i.e.,

$$\overline{\phi}_H(p^h_{bb}(p_s),p_s) = \frac{I-\beta\lambda(p_s+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]} + p_s.$$

The first-order derivative of $\overline{\phi}_H(p^h_{bb}(p_s),p_s)$ with respect to p_s is

$$\frac{d\overline{\phi}_H(p^h_{bb}(p_s),p_s)}{dp_s} = \frac{(1-\lambda)^2}{2a}\cdot\frac{\vartheta(p_s)}{\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]^2},$$

where

$$\vartheta(p_s) =: \frac{\beta(1-\lambda)^2(p_s+m_o)^2}{2a} + 2\beta\lambda(p_s+m_o) - I.$$

Evidently, $\vartheta(p_s)$ increases in p_s. Moreover, $\vartheta(0) = \beta\lambda\cdot 2m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a} - I$, and $\vartheta(1) =: \beta\lambda\cdot 2(1+m_o) + \frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a} - I$. Thus, we have the following three cases:

1. If $\vartheta(0) \geq 0$, i.e., $I \leq \beta\lambda\cdot 2m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$ or equivalently $m_o \geq M_A$, then $\frac{d\overline{\phi}_H(p^h_{bb}(p_s),p_s)}{dp_s} \geq 0$ for $p_s \in [0,1]$. Therefore, $\overline{\phi}_H(p^h_{bb}(p_s),p_s)$ values minimum at $p_s = 0$, and thus $\underline{v}_H = \frac{I-\beta\lambda m_o}{\beta\left[\lambda+\frac{(1-\lambda)^2 m_o}{2a}\right]} = \overline{v}_A$;
2. If $\vartheta(0) < 0 \leq \vartheta(1)$, i.e., $\beta\lambda\cdot 2m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a} < I \leq \beta\lambda\cdot 2(1+m_o) + \frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, then there exists a unique p^0_s such that $\vartheta(p^0_s) = 0$, which gives $p^0_s = \frac{-2\beta a\lambda + \sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o$. $\overline{\phi}_H(p^h_{bb}(p_s),p_s)$ values minimum at $p_s = p^0_s$, and thus
$$\begin{aligned}\underline{v}_H &= \frac{I - \frac{-2\beta a\lambda^2+\lambda\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2 I}}{(1-\lambda)^2}}{\frac{\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2 I}}{2a}} + \frac{-2\beta a\lambda + \sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o \\ &= \frac{-4\beta a\lambda + 2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o = \underline{v}_A < \overline{v}_A.\end{aligned}$$
3. If $\vartheta(1) < 0$, i.e., $I > \beta\lambda\cdot 2(1+m_o) + \frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, then $\frac{d\overline{\phi}_H(p^h_{bb}(p_s),p_s)}{dp_s} < 0$ for $p_s \in [0,1]$. Therefore, $\overline{\phi}_H(p^h_{bb}(p_s),p_s)$ values minimum at $p_s = 1$, and thus $\underline{v}_H = \frac{I-\beta\lambda(1+m_o)}{\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1 < \overline{v}_A$.

Step 2. Given $v \geq \underline{v}_H$, the firm is able to advance sell with high-price strategy. **In Step 2, we find the optimal (p_a,p_s), denoted as (p^h_{aH},p^h_{sH}), to maximize the firm's expected profit,**

$$\pi^h_{bb}(p_a,p_s) = \beta\left[a(e^h_{bb}(p_s))^2 + \lambda(p_s+m_o)\right] + p_a - I.$$

Given $v \geq \underline{v}_H$ and p_s in the feasible region, the firm will set optimal coupon price, denoted as $p^o_a(p_s)$, as: $p^o_a(p_s) = \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda](v-p_s)$, and the corresponding expected profit is

$$\pi^h_{bb}(p^o_a(p_s),p_s) = \beta\left[a(e^h_{bb}(p_s))^2 + \lambda(p_s+m_o)\right] + \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda](v-p_s) - I.$$

Taking the first-order derivative of $\pi_{bb}^h(p_a^o(p_s),p_s)$ with respect to p_s leads to: $\frac{d\pi_{bb}^h(p_a^o(p_s),p_s)}{dp_s}=\frac{\beta(1-\lambda)^2(v-p_s)}{2a}\geq 0$. That is, the firm's expected profit increases in p_s and the firm should set p_s as the maximum value in the feasible region of p_s.

Moreover, for given $v\geq \underline{v}_H$, the constraints in (81) imply that the feasible region of p_s is bounded by

$$\begin{cases} p_s\in[0,1] \\ \overline{\phi}_H(p_{bb}^h(p_s),p_s)\leq v. \end{cases} \tag{82}$$

Since $\overline{\phi}_H(p_{bb}^h(p_s),p_s)$ is convex from the above analyses, the constraint condition (82) is equivalent to

$$p_s\in\left[\max\{0,[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_l^{-1}(v)\},\min\{1,[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_r^{-1}(v)\}\right],$$

where $[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_l^{-1}(v)$ and $[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_r^{-1}(v)$ represent the smaller and larger roots to the equation $\overline{\phi}_H(p_{bb}^h(p_s),p_s)=v$, respectively. Therefore, the optimal spot selling price is

$$p_{sH}^h=\min\{1,[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_r^{-1}(v)\},$$

with $[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_r^{-1}(v)=\frac{(1-\lambda)(v-m_o)+\sqrt{(1-\lambda)^2(v+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(v+m_o)\right]}}{2(1-\lambda)}$. Depending on wether $[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_r^{-1}(v)\leq 1$ or not, we have the following two cases:

1. If $v>\overline{\phi}_H(p_{bb}^h(1),1)=\frac{I-\beta\lambda(1+m_o)}{\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{2a}\right]}+1=:\overline{v}_H$, we have $p_{sH}^h=1$. Accordingly,

$$p_{aH}^h=p_a^o(1)=\beta[e_{bb}^h(1)+(1-e_{bb}^h(1))\lambda](v-1)=\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{2a}\right](v-1).$$

   The firm's optimal effort is $e_H^h=e_{bb}^h(p_{sH}^h)=\frac{(1-\lambda)(1+m_o)}{2a}$ and the expected profit is

$$\pi_H^h=\beta\left[\frac{(1-\lambda)^2(m_o^2-1)}{4a}+\lambda m_o+v\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{2a}\right]-\frac{I}{\beta}\right].$$

2. If $\underline{v}_H\leq v\leq\overline{v}_H$, we have

$$p_{sH}^h=[\overline{\phi}_H(p_{bb}^h(p_s),p_s)]_r^{-1}(v)]=\frac{(1-\lambda)(v-m_o)+\sqrt{(1-\lambda)^2(v+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(v+m_o)\right]}}{2(1-\lambda)}.$$

   Accordingly, the optimal advance selling price is

$$p_{aH}^h=p_a^o(p_{sH}^h)=I-\frac{\beta\lambda\left[(1-\lambda)(v+m_o)+\sqrt{(1-\lambda)^2(v+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(v+m_o)\right]}\right]}{2(1-\lambda)}.$$

   The optimal effort is

$$e_H^h=e_{bb}^h(p_{sH}^h)=\frac{(1-\lambda)(v+m_o)+\sqrt{(1-\lambda)^2(v+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(v+m_o)\right]}}{4a}$$

   and the expected profit is

$$\pi_H^h=\frac{\beta\left[(1-\lambda)(v+m_o)+\sqrt{(1-\lambda)^2(v+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(v+m_o)\right]}\right]^2}{16a}.$$

Low-price strategy (i.e., $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$). Given $p_s \in [0,1]$ and $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$, assuming that all k consumers purchase in advance, the firm would exert an effort of $e^l_{bb}(p_a,p_s) = \frac{(1-\lambda)(p_s+m_o)+\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}{4a}$ in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta(v-p_s)[e^l_{bb}(p_a,p_s) + (1-e^l_{bb}(p_a,p_s))\lambda] - p_a. \tag{83}$$

Anticipating this, the consumers would purchase in advance if and only if the equivalent condition (78) is met. Substituting $\mathbb{E}[u_{bb}]$ in (83) into the equivalent condition (78) gives $p_a \le \beta(v-p_s)[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda]$, or equivalently

$$v \ge \frac{p_a}{\beta[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda]} + p_s =: \underline{\phi}^h_H(p_a,p_s).$$

Summarizing the above conditions, the firm's optimization problem can be formulated as follows:

$$\max\ \pi^l_{bb}(p_a,p_s) = \beta a\left[e^l_{bb}(p_a,p_s)\right]^2 \tag{84}$$

$$s.t.\quad \begin{cases} p_s \in [0,1] \\ p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s) \\ v \ge \frac{p_a}{\beta[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda]} + p_s =: \underline{\phi}_H(p_a,p_s). \end{cases} \tag{85}$$

Step 1. The constraints (85) imply that the firm is able to advance sell as long as v is sufficiently large. Let $\underline{\Delta}_H$ be the feasible region of (p_a,p_s), which is bounded by the first two constraints of (85), and define $\underline{v}_H = \min_{(p_a,p_s)\in\underline{\Delta}_H} \underline{\phi}_H(p_a,p_s)$. Then, the firm is able to advance sell if and only if $v \ge \underline{v}_H$. **In Step 1, we will solve the threshold value $\underline{v}_H$.**

Given p_s, the first-order partial derivative of $\underline{\phi}_H(p_a,p_s)$ with respect to p_a is

$$\frac{\partial\underline{\phi}_H(p_a,p_s)}{\partial p_a} = \frac{\zeta_6(p_a,p_s)}{\beta^2\left[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda\right]^2\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}},$$

where

$$\begin{aligned}\zeta_6(p_a,p_s) :=& \beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{4a}\right]\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]} \\ &+ (1-\lambda)p_a + \frac{\beta(1-\lambda)^3(p_s+m_o)^2}{4a} - 2(1-\lambda)\left[I-\beta\lambda(p_s+m_o)\right],\end{aligned}$$

which increases in p_a. Moreover, at $p_a = p^l_{bb}(p_s)$, we have

$$\zeta_6(p^l_{bb}(p_s),p_s) = (1-\lambda)\left[\omega^l(p_s) - I\right] \le 0$$

as $I > I_h$, and at $p_a = p^h_{bb}(p_s)$, we have

$$\zeta_6(p^h_{bb}(p_s),p_s) = (1-\lambda)\left[\omega^h(p_s) - I\right],$$

where $\omega^l(p_s) := \beta\lambda(p_s+m_o) + \frac{\beta(1-\lambda)^2(p_s+m_o)^2}{8a}$, and $\omega^h(p_s) := 2\beta\lambda(p_s+m_o) + \frac{\beta(1-\lambda)^2(p_s+m_o)^2}{2a}$.

We discuss the above problem in the following three cases:

1. If $I \le \omega^h(0) = 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, we have $\zeta_6(p^l_{bb}(p_s),p_s) < 0$ and $\zeta_6(p^h_{bb}(p_s),p_s) > 0$. Thus, there exists a unique $p^0_a(p_s)$ such that $\zeta_6(p^0_a(p_s),p_s) = 0$. That is,

$$p^0_a(p_s) = \frac{8a\beta\lambda^2 + 4(1-\lambda)^2 I - [4a\lambda + (1-\lambda)^2(p_s+m_o)]\sqrt{4\beta^2\lambda^2 + \frac{2\beta(1-\lambda)^2 I}{a}}}{2(1-\lambda)^2}.$$

$\underline{\phi}_H(p_a,p_s)$ achieves its minimum at $p_a = p^0_a(p_s)$, and its value is

$$\begin{aligned}\underline{\phi}_H(p^0_a(p_s),p_s) =& \frac{p^0_a(p_s)}{\beta[e^l_{bb}(p^0_a(p_s),p_s) + (1-e^l_{bb}(p^0_a(p_s),p_s))\lambda]} + p_s \\ =& \frac{\frac{2\sqrt{4a^2\beta^2\lambda^2 + 2a\beta(1-\lambda)^2 I}}{\beta} - 4a\lambda - (1-\lambda)^2(p_s+m_o)}{(1-\lambda)^2} + p_s \\ =& \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o,\end{aligned}$$

which is a constant independent of p_s. Thus,

$$\underline{v}_H = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o = \underline{v}_A < \underline{v}_H = \overline{v}_A.$$

2. When $\omega^h(0) < I \le \omega^h(1) = 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2(1+m_o)^2}{2a}$, there exists a unique p^{h0}_s such that $\omega^h(p^{h0}_s) = I$, i.e., $p^{h0}_s = \frac{-2\beta a\lambda + \sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o$. We have $\zeta_6(p^h_{bb}(p_s),p_s) > 0$ for $p_s \in [p^{h0}_s, 1]$, and $\zeta_6(p^h_{bb}(p_s),p_s) < 0$ for $p_s \in [0, p^{h0}_s)$. Therefore, we have the following two sub-cases:
   (a) for given $p_s \in [p^{h0}_s, 1]$, similar to Case 1, it can be derived that

$$\underline{v}^1_H = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

   (b) for given $p_s \in [0, p^{h0}_s)$, $\underline{\phi}_H(p_a,p_s)$ values minimum at $p_a = p^h_{bb}(p_s)$, and its value is $\underline{\phi}_H(p^h_{bb}(p_s),p_s) = \frac{I - \beta\lambda(p_s+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]} + p_s$. Moreover, we have

$$\frac{d\underline{\phi}_H(p^h_{bb}(p_s),p_s)}{dp_s} = \frac{(1-\lambda)^2[\omega^h(p_s) - I]}{2a\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]^2} < 0$$

   for $p_s \in [0, p^{h0}_s)$. Thus, $\underline{\phi}_H(p^h_{bb}(p_s),p_s)$ values minimum at $p_s = p^{h0}_s$, which leads to

$$\underline{v}^2_H = \underline{\phi}_H(p^h_{bb}(p^{h0}_s), p^{h0}_s) = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o = \underline{v}^1_H.$$

   Summarizing the above Scenario (a)-(b), we have

$$\underline{v}_H = \min\{\underline{v}^1_H, \underline{v}^2_H\} = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o = \underline{v}_H.$$

3. When $I > \omega^h(1)$, we have $\zeta_6(p^l_{bb}(p_s),p_s) < \zeta_6(p^h_{bb}(p_s),p_s) < 0$. Thus, $\zeta_6(p_a,p_s) < 0$ holds for any $p_a \in [p^l_{bb}(p_s), p^h_{bb}(p_s))$ and $p_s \in [0,1]$. Accordingly, $\underline{\phi}_H(p_a,p_s)$ values minimum at $p_a = p^h_{bb}(p_s)$ for given p_s. Moreover, based on the above analyses, $\underline{\phi}_H(p^h_{bb}(p_s),p_s)$ decreases in $p_s \in [0,1]$ for $I > \omega^h(1)$. Thus, $\underline{\phi}_H(p^h_{bb}(p_s),p_s)$ values minimum at $p_s = 1$, and accordingly we have

$$\underline{v}_H = \underline{\phi}_H(p^h_{bb}(1),1) = \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1 = \underline{v}_H.$$

Step 2. In this step, we will show that low-price strategy is dominated by high-price strategy when both strategies are feasible. Under the low-price strategy, from the constraints in (85) of the optimization problem (84), we observe that given p_s^f in the feasible region, the optimal advance selling price is either bounded by $p_{bb}^l(p_s^f) \le p_a < p_{bb}^h(p_s^f)$ or $v \ge \underline{\phi}_H(p_a, p_s^f)$. Since the firm should set the advance selling price as high as possible to achieve the highest profit for given p_s^f. Thus, in the former case ($p_{bb}^l(p_s^f) \le p_a < p_{bb}^h(p_s^f)$), the optimal advance selling price is $p_a^o(p_s^f) = p_{bb}^h(p_s^f)$, which degenerates to the high-price strategy. In the latter case ($v \ge \underline{\phi}_H(p_a, p_s^f)$), the optimal advance selling price is

$$p_a^o(p_s^f) = \frac{\beta(v-p_s)\left[(1-\lambda)^2(v+m_o)+4a\lambda\right] + \beta(1-\lambda)(v-p_s)\sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}.$$

Accordingly, the firm's expected profit is

$$\pi_{bb}^l(p_a^o(p_s^f), p_s^f) = \beta a\left[e_{bb}^l(p_a^o(p_s^f), p_s^f)\right]^2 = \frac{\beta\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]^2}{16a}.$$

On the other hand, the optimization results derived under high-price strategy is divided into the following two cases:

1. If $v > \overline{v}_H$, we have $p_{sH}^h = 1$ and the firm's expected profit is

$$\pi_H^h = \beta\left[a(e_{bb}^h(1))^2 + \lambda(1+m_o)\right] + p_{aH}^{lh} - I > \beta a(e_{bb}^h(1))^2 > \beta a\left[e_{bb}^l(p_a^o(p_s^f), p_s^f)\right]^2 = \pi_{bb}^l(p_a^o(p_s^f), p_s^f).$$

   This indicates that low-price strategy is dominated by high-price strategy in this case.

2. If $\underline{v}_H \le v \le \overline{v}_H$, the firm's expected profit is

$$\pi_H^h = \frac{\beta\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]^2}{16a} =: \pi_{bb}^l(p_a^o(p_s^f), p_s^f).$$

   Again, the low-price strategy is also dominated by high-price strategy in this case.

To sum up, in either case, low-price strategy is dominated by high-price strategy if both strategies are feasible. This together with the results derived in Step 1 indicate the following conclusions:

1. If $I \ge 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, we have $\underline{\underline{v}}_H = \underline{v}_H$. This indicates that the feasible regions for high-price strategy and low-price strategy are the same. Therefore, the firm will advance sell with high-price strategy.
2. If $I < 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, we have $\underline{\underline{v}}_H < \underline{v}_H$. This means the firm would advance sell with high-price strategy if $v \ge \underline{v}_H$, with low-price strategy if $\underline{\underline{v}}_H \le v < \underline{v}_H$, and fail to advance sell if otherwise.

Step 3. In this step, we continue to solve the firm's optimal low-price strategies if $I < 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$ and $\underline{\underline{v}}_H \le v < \underline{v}_H$. From the above analysis, when $I < 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, $\underline{\phi}_H(p_a, p_s)$ first decreases and then increases in $p_a \in (p_{bb}^l(p_s), p_{bb}^h(p_s)]$ for given p_s. Therefore, given $\underline{\underline{v}}_H \le v < \underline{v}_H = \underline{\phi}_H(p_{bb}^h(0), 0) \le \underline{\phi}_H(p_{bb}^h(p_s), p_s)$, $v \ge \underline{\phi}_H(p_a, p_s)$ with $p_a \in (p_{bb}^l(p_s), p_{bb}^h(p_s)]$ is equivalent to $\max\{p_{bb}^l(p_s), [\underline{\phi}_H(p_a, p_s)]_l^{-1}(v)\} \le p_a \le [\underline{\phi}_H(p_a, p_s)]_r^{-1}(v)$, where $[\underline{\phi}_H(p_a, p_s)]_l^{-1}(v)$ and $[\underline{\phi}_H(p_a, p_s)]_r^{-1}(v)$ denote the two roots of p_a to the equation of $v = \underline{\phi}_H(p_a, p_s)$ for given p_s. Since the firm's expected profit in (84) increases in p_a, the optimal coupon price is

$$p_a^o(p_s) = [\underline{\phi}_H(p_a,p_s)]_r^{-1}(v) = \frac{\beta(v-p_s)\left[4a\lambda + (1-\lambda)^2(v+m_o) + (1-\lambda)\sqrt{(1-\lambda)^2(v+m_o)^2 + 8a\lambda(v+m_o) - \frac{8aI}{\beta}}\right]}{4a}.$$

Accordingly, the firm's optimal effort and profit are

$$e_{bb}^l(p_a^o(p_s), p_s) = \frac{(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}$$

and

$$\pi_{bb}^l(p_a^o(p_s), p_s) = \beta a\left[e_{bb}^l(p_a^o(p_s), p_s)\right]^2 = \frac{\beta\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]^2}{16a} = \pi_A^*,$$

which is independent of p_s. This means the optimal solution of (p_a, p_s) is not unique.

Summarizing the above results lead to the conclusions in Proposition A.1 . □

Proof of Proposition A.2. According to Proposition 3 and Proposition A.1, based on whether $m_o \le M_A$ or not, there are two relevant cases:

1. When $m_o \le M_A$ or equivalently $I \ge 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, the minimum valuation required for a successful advance selling is $\underline{v}_H$ under scheme "H" and is $\overline{v}_A$ under scheme "A". We have $\underline{v}_H < \overline{v}_A$, which has been shown in the proof of Proposition A.1. That is, the firm is able to advance sell under "H" over a larger region than under "A". Moreover, given $v \ge \overline{v}_A$, the firm can advance sell with both "A" and "H". In this case, it can be verified that $e_H^* > e_A^*$ and $\pi_H^* > \pi_A^*$ due to the greater flexibility of "H" than "A".
2. When $m_o > M_A$ or equivalently $I < 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, the minimum valuation required for a successful advance selling is $\underline{\underline{v}}_H$ under scheme "H" and is $\underline{v}_A$ under scheme "A". We have $\underline{\underline{v}}_H = \underline{v}_A$, which has been shown in the proof of Proposition A.1. That is, the region over which the firm could advance sell under "H" is identical to that under "A". Further, the following conclusions can be verified by comparing the optimal results under "H" and "A": (i) The firm's effort and profit under "H" and "A" are identical for $v \in [\underline{v}_A, \overline{v}_A]$; (ii) For $v > \overline{v}_A$, the firm's effort and profit is higher under "H" than under "A".

□

Proof of Proposition A.3. Similar to the proof of Proposition 2, we have that in this scenario $p_{rA}^{s*} = 1$. Under advance selling with an all-or-nothing clause, the firm would cancel advance selling unless all segment-i consumers purchase in advance. Thus, we have $\mathbb{E}[u_{bw}] = \mathbb{E}[u_{ww}]$. Therefore, by Lemma A.1, the firm can induce all consumers to purchase in advance if and only if

$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{ww}]. \tag{86}$$

Moreover, as $I_l < I \le I_h$, if all k segment-i consumers wait, it degenerates to the benchmark case of pure bank financing. In the case of $v > 1 + m_o$, the firm would set the regular price equal to $p_{rB}^* = v$ as discussed in the proof of Proposition 1. Thus, the expected surplus of a consumer when all k segment-i consumers wait is $\mathbb{E}[u_{ww}] = \beta[e^*_B + (1-e^*_B)\lambda](v - p^*_{rB}) = 0$, and thus Eq. (86) becomes

$$\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{ww}] = 0. \tag{87}$$

Next, note that according to Lemma A.2, the firm can advance sell with either high-price strategy (i.e., $p_a \geq p^h_{bb}$) or low-price strategy ($p^l_{bb} \leq p_a < p^h_{bb}$). In what follows, we consider these two pricing strategies respectively.

High-price strategy. Given

$$p_a \geq p^h_{bb}, \tag{88}$$

assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e^h_{bb} = \frac{(1-\lambda)m_o}{2a}$ in accordance with Lemma A.2. Accordingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta[e^h_{bb} + (1-e^h_{bb})\lambda]v - p_a. \tag{89}$$

Substituting $\mathbb{E}[u_{bb}]$ in (89) into Eq. (87), the segment-i consumers will purchase in advance if and only if

$$p_a \leq \beta\left[\lambda + (1-\lambda)e^h_{bb}\right]v. \tag{90}$$

To ensure the feasible region of p_a which satisfies (88) and (90) to be non-empty, we must have

$$v \geq \frac{p^h_{bb}}{\beta\left[\lambda + (1-\lambda)e^h_{bb}\right]} = \frac{I - \beta\lambda m_o}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]} =: \overline{v}^s_A.$$

When this condition is satisfied (i.e., $v \geq \overline{v}^s_A$), the optimal advance selling price is $p^{s*}_{aA} = \beta v\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]$, and the firm's expected profit follows

$$\pi^{s*}_A = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o + v\left(\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right)\right] - I.$$

Recall the definition of π^*_B in Proposition 1, we have

$$\pi^{s*}_A - \pi^*_B = \beta\left[-\frac{(1-\lambda)^2 v^2 + (1-\lambda)v\sqrt{(1-\lambda)^2 v^2 - 8a(\frac{I}{\beta} - \lambda v)}}{8a} - \frac{1}{2}(\frac{I}{\beta} - \lambda v) + \frac{(1-\lambda)^2 m_o v}{2a} + \frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right].$$

The first-order derivative of $\pi^{s*}_A - \pi^*_B$ with respect to v is:

$$\begin{aligned}
\frac{d(\pi^{s*}_A - \pi^*_B)}{dv} =& -\frac{(1-\lambda)^2 v}{4a} - \frac{(1-\lambda)\sqrt{(1-\lambda)^2 v^2 - 8a(\frac{I}{\beta} - \lambda v)}}{8a} - \frac{(1-\lambda)^3 v^2}{8a\sqrt{(1-\lambda)^2 v^2 - 8a(\frac{I}{\beta} - \lambda v)}} + \frac{(1-\lambda)^2 m_o}{2a} \\
& - \frac{\lambda(1-\lambda)v}{2\sqrt{(1-\lambda)^2 v^2 - 8a(\frac{I}{\beta} - \lambda v)}} + \frac{\lambda}{2} \\
<& -\frac{(1-\lambda)^2 v}{4a} - \frac{(1-\lambda)\sqrt{(1-\lambda)^2 v^2 - 8a(\frac{I}{\beta} - \lambda v)}}{8a} - \frac{(1-\lambda)^3 v^2}{8a\sqrt{(1-\lambda)^2 v^2 - 8a(\frac{I}{\beta} - \lambda v)}} + \frac{(1-\lambda)^2 m_o}{2a} \\
<& -\frac{(1-\lambda)^2 v}{2a} + \frac{(1-\lambda)^2 m_o}{2a} < 0.
\end{aligned}$$

where the first inequality holds because $-\frac{\lambda(1-\lambda)v}{2\sqrt{(1-\lambda)^2v^2-8a(\frac{I}{\beta}-\lambda v)}}+\frac{\lambda}{2}<0$; the second inequality holds because

$$\frac{(1-\lambda)\sqrt{(1-\lambda)^2v^2-8a(\frac{I}{\beta}-\lambda v)}}{8a}+\frac{(1-\lambda)^3v^2}{8a\sqrt{(1-\lambda)^2v^2-8a(\frac{I}{\beta}-\lambda v)}}\geq\frac{(1-\lambda)^2v}{4a};$$

and the third inequality holds because $v>1+m_o$. Thus, $\pi_A^{s*}-\pi_B^*$ decreases in v.

Low-price strategy. When the firm fails to advance sell with high-price strategy, she can further consider low-price strategy. Given

$$p_{bb}^l\leq p_a<p_{bb}^h, \tag{91}$$

assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e_{bb}^l(p_a)=\frac{(1-\lambda)m_o+\sqrt{(1-\lambda)^2m_o^2-8a\left(\frac{I-p_a}{\beta}-\lambda m_o\right)}}{4a}$ in accordance with Lemma A.2. Accordingly, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}]=\beta v[e_{bb}^l(p_a)+(1-e_{bb}^l(p_a))\lambda]-p_a. \tag{92}$$

From (87) and (92), the segment-i consumers would advance buy if and only if $p_a\leq\beta v[e_{bb}^l(p_a)+(1-e_{bb}^l(p_a))\lambda]$, or equivalently

$$v\geq\frac{p_a}{\beta[e_{bb}^l(p_a)+(1-e_{bb}^l(p_a))\lambda]}=:\underline{\phi}_A^s(p_a). \tag{93}$$

The above constraints of (91) and (93) together imply that the firm is able to advance sell with low-price strategy as long as v is sufficiently large. We define $\underline{v}_A^s=\min_{p_a\in[p_{bb}^l,p_{bb}^h)}\underline{\phi}_A^s(p_a)$, then the firm is able to advance sell with low-price strategy as long as $v\geq\underline{v}_A^s$. In what follows, we solve the threshold $\underline{v}_A^s$.

The first-order derivative of $\underline{\phi}_A^s(p_a)$ with respect to p_a is

$$\frac{d\underline{\phi}_A^s(p_a)}{dp_a}=\frac{\eta(p_a)}{\beta^2\sqrt{(1-\lambda)^2m_o^2-8a\left(\frac{I-p_a}{\beta}-\lambda m_o\right)}[\lambda+(1-\lambda)e_{bb}^l(p_a)]^2}$$

where

$$\begin{aligned}\eta(p_a):=&\beta\left[\lambda+\frac{(1-\lambda)^2m_o}{4a}\right]\sqrt{(1-\lambda)^2m_o^2-8a\left(\frac{I-p_a}{\beta}-\lambda m_o\right)}+(1-\lambda)p_a\\&+\frac{\beta(1-\lambda)^3m_o^2}{4a}-2(1-\lambda)(I-\beta\lambda m_o),\end{aligned}$$

which increases in p_a. At $p_a=p_{bb}^l$, we have

$$\eta(p_{bb}^l)=-(1-\lambda)\left[I-\beta\lambda m_o-\beta\frac{(1-\lambda)^2m_o^2}{8a}\right].$$

At $p_a=p_{bb}^h$, we have

$$\eta(p_{bb}^h)=-(1-\lambda)\left[I-\beta\lambda\cdot 2m_o-\beta\frac{(1-\lambda)^2(2m_o)^2}{8a}\right].$$

Evidently, the signs of $\eta(p_{bb}^l)$ and $\eta(p_{bb}^h)$ can be positive or negative depending on the value of I, and in what follows we discuss it in three cases:

1. If $\eta(p_{bb}^h) \le 0$, i.e., $I \ge \beta\lambda \cdot 2m_o + \beta\frac{(1-\lambda)^2(2m_o)^2}{8a}$, then $\eta(p_a) \le 0$ for $p_a \in [p_{bb}^l, p_{bb}^h)$ since $\eta(p_a)$ increases in p_a. This further implies that $\frac{d\underline{\phi}_A^s(p_a)}{dp_a} \le 0$ for $p_a \in [p_{bb}^l, p_{bb}^h)$ since the denominator of $\frac{d\underline{\phi}_A^s(p_a)}{dp_a}$ is positive. Therefore, $\underline{\phi}_A^s(p_a)$ values minimum at $p_a = p_{bb}^h$ and thus
$$\underline{v}_A^s = \underline{\phi}_A^s(p_{bb}^h) = \frac{I - \beta\lambda m_o}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{2a}\right]} =: \overline{v}_A^s.$$
That is, the lower bound of v beyond which the firm can advance sell with low-price strategy is identical to that with high-price strategy. Thus, in this case, the firm will always prefer to advance sell with high-price strategy.

2. If $\eta(p_{bb}^h) > 0$ and $\eta(p_{bb}^l) \le 0$, i.e., $\beta\lambda m_o + \beta\frac{(1-\lambda)^2 m_o^2}{8a} \le I < \beta\lambda \cdot 2m_o + \beta\frac{(1-\lambda)^2(2m_o)^2}{8a}$, then there exists a unique root, denoted as $p_a^0 \in (p_{bb}^l, p_{bb}^h)$, such that $\eta(p_a^0) = 0$, since $\eta(p_a)$ increases in p_a and $\eta(p_{bb}^l) \le 0$. That is,
$$\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I - p_a^0}{\beta} - \lambda m_o\right)} + (1-\lambda)p_a^0 + \frac{\beta(1-\lambda)^3 m_o^2}{4a} - 2(1-\lambda)(I - \beta\lambda m_o) = 0,$$
solving which gives
$$p_a^0 = \frac{8a\beta\lambda^2 + 4(1-\lambda)^2 I - [4a\lambda + (1-\lambda)^2 m_o]\sqrt{4\beta^2\lambda^2 + \frac{2\beta(1-\lambda)^2 I}{a}}}{2(1-\lambda)^2}.$$
$\eta(p_a) \le 0$ for $p_a \in [p_{bb}^l, p_a^0]$ and $\eta(p_a) > 0$ for $p_a \in (p_a^0, p_{bb}^h)$. Accordingly, $\frac{d\underline{\phi}_A^s(p_a)}{dp_a} \le 0$ for $p_a \in [p_{bb}^l, p_a^0]$ and $\frac{d\underline{\phi}_A^s(p_a)}{dp_a} > 0$ for $p_a \in (p_a^0, p_{bb}^h)$ since the denominator of $\frac{d\underline{\phi}_A^s(p_a)}{dp_a}$ is positive. This implies that $\underline{\phi}_A^s(p_a)$ values minimum at $p_a = p_a^0$. Thus,
$$\begin{aligned}\underline{v}_A^s = \underline{\phi}_A^s(p_a^0) &= \frac{p_a^0}{\beta[e_{bb}^l(p_a^0) + (1 - e_{bb}^l(p_a^0))\lambda]} \\ &= \frac{2\sqrt{4a^2\beta^2\lambda^2 + 2a\beta(1-\lambda)^2 I} - 4\beta a\lambda}{\beta(1-\lambda)^2} - m_o \\ &< \overline{v}_A^s.\end{aligned}$$

3. If $\eta(p_{bb}^l) > 0$, i.e., $I < \beta\lambda m_o + \beta\frac{(1-\lambda)^2 m_o^2}{8a}$, then $\eta(p_a) > 0$ for $p_a \in [p_{bb}^l, p_{bb}^h)$ since $\eta(p_a)$ increases in p_a. This further implies that $\frac{d\underline{\phi}_A^s(p_a)}{dp_a} > 0$ for $p_a \in [p_{bb}^l, p_{bb}^h)$ since the denominator of $\frac{d\underline{\phi}_A^s(p_a)}{dp_a}$ is positive. Therefore, $\underline{\phi}_A^s(p_a)$ values minimum at $p_a = p_{bb}^l$ and thus
$$\underline{v}_A^s = \underline{\phi}_A^s(p_{bb}^l) = \frac{I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]} < 0 < \overline{v}_A^s.$$

Based on the above analyses, the following conclusions can be drawn:

1. When $I \ge \beta\lambda \cdot 2m_o + \beta\frac{(1-\lambda)^2(2m_o)^2}{8a}$, the firm would advance sell with high-price strategy as long as $v \ge \overline{v}_A^s$, provided that the firm is willing to advance sell;
2. When $I < \beta\lambda \cdot 2m_o + \beta\frac{(1-\lambda)^2(2m_o)^2}{8a}$, the firm will advance sell with high-price strategy if $v \ge \overline{v}_A^s$, and low-price strategy if $\underline{v}_A^s \le v < \overline{v}_A^s$, provided that the firm is willing to advance sell.

Moreover, it can be derived that $\overline{v}_A^s < 1 + m_o$ when $I < \beta\left[2\lambda m_o + \frac{(1-\lambda)^2(2m_o)^2}{8a}\right]$. Thus, the firm would always advance sell with high-price strategy given $v > 1 + m_o$ provided that advance selling is more beneficial than pure bank financing. □

Proof of Proposition A.4. Similar to the analysis in the proof of Proposition 2, we have $p_{rA}^{l*} = 1$ in this scenario. Under advance selling with an all-or-nothing clause, the firm would cancel advance selling unless all consumers purchase in advance. Therefore, we have $\mathbb{E}[u_{bw}] = \mathbb{E}[u_{ww}]$, which indicates that the firm can induce all consumers to purchase in advance if and only if

$$\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{ww}] \tag{94}$$

under advance selling with all-or-nothing clause for $I_l < I \leq I_h$, in accordance with Lemma A.1. Moreover, if all k segment-i consumers wait, it degenerates to the benchmark case of pure bank financing for $v \leq 1 + m_o$. Since $v \leq 1 + m_o$, the firm would set the regular selling price equal to $p_{rB}^* = 1$ as discussed in the proof of Proposition 1, the expected surplus of a consumer when all segment-i consumers wait is

$$\mathbb{E}[u_{ww}] = \beta[e_B^* + (1 - e_B^*)\lambda](v - p_{rB}^*) = \beta[e_B^* + (1 - e_B^*)\lambda](v - 1),$$

where $e_B^* = \frac{(1-\lambda)(1+m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a}$. Thus, Eq. (94) can be further specified as

$$\mathbb{E}[u_{bb}] \geq \mathbb{E}[u_{ww}] = \beta[e_B^* + (1 - e_B^*)\lambda](v - 1). \tag{95}$$

Next, note that according to Lemma A.2, the firm can advance sell with either high-price strategy (i.e., $p_a \geq p_{bb}^h$) or low-price strategy ($p_{bb}^l \leq p_a < p_{bb}^h$). In what follows, we consider these two pricing strategies in turn.

High-price strategy. Given

$$p_a \geq p_{bb}^h, \tag{96}$$

assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e_{bb}^h = \frac{(1-\lambda)m_o}{2a}$ in accordance with Lemma A.2. Accordingly, when all k segment-i consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta[e_{bb}^h + (1 - e_{bb}^h)\lambda]v - p_a. \tag{97}$$

Substituting $\mathbb{E}[u_{bb}]$ in (97) into (95), the segment-i consumers will advance buy if and only if

$$p_a \leq \beta\left[(e_{bb}^h - e_B^*)(1 - \lambda)v + [e_B^* + (1 - e_B^*)\lambda]\right]. \tag{98}$$

Next, we will show that the firm is willing to advance sell (i.e., $\pi_{bb}^h > \pi_B^*$) if and only if $e_{bb}^h > e_B^*$. First, according to Proposition 1 and Lemma A.2, it is evident that when $I_l < I \leq I_h$, we have

$$\pi_{bb}^h = \beta a(e_{bb}^h)^2 + p_a - (I - \beta\lambda m_o) > \pi_B^* = \beta a(e_B^*)^2$$

if $e_{bb}^h > e_B^*$ since $p_a \geq p_{bb}^h =: I - \beta\lambda m_o$. Next, we continue to prove that $\pi_{bb}^h \leq \pi_B^*$ if $e_{bb}^h \leq e_B^*$. Note that $e_{bb}^h \leq e_B^*$ if and only if either of the following two conditions holds:

(i) $m_o \leq 1$ and $\beta\left[\lambda(1 + m_o) + \frac{(1-\lambda)^2 m_o}{2a}\right] =: I_m^A < I \leq I_h$;

(ii) $I_l < I \leq I_m^A$.

In what follows, we consider these two scenarios, respectively. First, note that since $e_{bb}^h \le e_B^*$ and $v > 1$, the inequality (98) leads to

$$p_a \le \beta \left[\lambda + \frac{(1-\lambda)^2 m_o}{2a} \right] \tag{99}$$

The conditions of (96) and (99) together implies that the firm fails to advance sell when $I > I_m^A$. Thus, in Scenario (i), the firm fails to advance sell.

For Scenario (ii), from (99) we have

$$\pi_{bb}^h - \pi_B^* \le \beta a (e_{bb}^h)^2 + \beta \left[\lambda + \frac{(1-\lambda)^2 m_o}{2a} \right] - (I - \beta \lambda m_o) - \beta a (e_B^*)^2 =: \zeta_2(I).$$

At $I = I_m^A$, we have $\zeta_2(I_m^A) = 0$ and also $\zeta_2'(I) > 0$, which gives $\zeta_2(I) < 0$. Thus, $\pi_{bb}^h < \pi_B^*$ in Scenario (ii).

Based on the above discussion, we only need to consider the case of $e_{bb}^h > e_B^*$, which holds if and only if $m_o > 1$ and $I_m^A < I \le I_h$ are simultaneously met. In this case, the firm will advance sell as long as she is able to. As $e_{bb}^h > e_B^*$, the advance-buy condition (98) can be rewritten as

$$v \ge \frac{\frac{p_a}{\beta} - [e_B^* + (1-e_B^*)\lambda]}{(e_{bb}^h - e_B^*)(1-\lambda)} =: \overline{\phi}_A^l(p_a).$$

Evidently, $\overline{\phi}_A^l(p_a)$ increases in p_a, and thus achieves its minimum value at $p_a = p_{bb}^h$. We denote the minimum value of $\overline{\phi}_A^l(p_a)$ as

$$\begin{aligned}
\overline{v}_A^l &= \frac{\frac{I}{\beta} - \lambda m_o - [e_B^* + (1-e_B^*)\lambda]}{(1-\lambda)(e_{bb}^h - e_B^*)} \\
&= \frac{\frac{I}{\beta} - \lambda(1+m_o) - (1-\lambda)\frac{(1-\lambda)(1+m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a}}{\left[\frac{(1-\lambda)m_o}{2a} - \frac{(1-\lambda)(1+m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a}\right](1-\lambda)} \\
&= \frac{4a\left[\frac{I}{\beta} - \lambda(1+m_o)\right] - (1-\lambda)\left[(1-\lambda)(1+m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}{(1-\lambda)\left[(1-\lambda)(m_o - 1) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}.
\end{aligned}$$

Next, we analyze the monotonicity of $\overline{v}_A^l$ regarding $I \in [I_m^A, I_h]$. The first-order derivative of $\overline{v}_A^l$ regarding I is

$$\frac{d\overline{v}_A^l}{dI} = \frac{(e_{bb}^h - e_B^*)\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]} - \left[\frac{I}{\beta} - \lambda(1+m_o) - (1-\lambda)e_{bb}^h\right]}{\beta(1-\lambda)(e_{bb}^h - e_B^*)^2\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}$$

For notational convenience, we define the numerator of $\frac{d\overline{v}_A^l}{dI}$ to be $\zeta_3(I)$. Then,

$$\frac{d\zeta_3(I)}{dI} = -\frac{4a(e_{bb}^h - e_B^*)}{\beta\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}} < 0.$$

That is, $\zeta_3(I)$ decreases in I. Moreover, we have $\zeta_3(I_m^A) = 0$. Thus, $\zeta_3(I) < 0$ for $I_m^A < I \le I_h$, which leads to $\frac{d\overline{v}_A^l}{dI} < 0$ for $I_m^A < I \le I_h$. That is, $\overline{v}_A^l$ decreases in I. Therefore, $\overline{v}_A^l$ values maximum at $I = I_m^A$ and minimum at $I = I_h$, and their respective values are denoted as follows: $\overline{\overline{v}}_A^l = m_o$ and $\underline{\overline{v}}_A^l = \frac{m_o+1}{2}$. It can be observed that both $\overline{\overline{v}}^l_A < 1+m_o$ and $\underline{\overline{v}}^l_A < 1+m_o$. Thus, the feasible region of v for the firm to conduct high-price advance selling is non-empty.

Based on the above discussion, the firm should adopt high-price advance selling if and only if $\overline{v}^l_A \le v \le 1+m_o$ for $I^A_m < I \le I_h$ and $m_o > 1$ in this case. According to Lemma A.2, under advance selling with high-price strategy, the firm's expected profit is $\pi^h_{bb}(p_a) = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right] + p_a - I$, which increases in p_a. Therefore, the optimal advance selling price, denoted as p^{l*}_{aA}, should be the maximum value in the feasible region of p_a. That is, $p^{l*}_{aA} = \beta\left[(e^h_{bb} - e^*_B)(1-\lambda)v + [e^*_B + (1-e^*_B)\lambda]\right]$. Correspondingly, the effort level is $e^{l*}_A = \frac{(1-\lambda)m_o}{2a}$ and the firm's expected profit is

$$\pi^{l*}_A = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o + \left[\frac{(1-\lambda)m_o}{2a} - e^*_B\right](1-\lambda)v + [e^*_B + (1-e^*_B)\lambda]\right] - I.$$

Low-price strategy. When the firm fails to advance sell with high-price strategy, she can further consider low-price strategy. Given

$$p^l_{bb} \le p_a < p^h_{bb}, \tag{100}$$

assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e^l_{bb}(p_a) = \frac{(1-\lambda)m_o + \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{4a}$ in accordance with Lemma A.2. Consequently, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}] = \beta v[e^l_{bb}(p_a) + (1-e^l_{bb}(p_a))\lambda] - p_a. \tag{101}$$

From (95) and (101), the segment-i consumers would advance buy if and only if

$$p_a \le \beta\left[(e^l_{bb}(p_a) - e^*_B)(1-\lambda)v + [e^*_B + (1-e^*_B)\lambda]\right]. \tag{102}$$

Similarly, by comparing the firm's expected profits under pure bank financing and low-price advance selling in Proposition 1 and Lemma A.2, respectively, the firm is willing to advance sell if and only if $e^l_{bb}(p_a) > e^*_B$. Thus, we only need to consider the case of $e^l_{bb}(p_a) > e^*_B$. For $e^l_{bb}(p_a) > e^*_B$ to be true, the following conditions should be satisfied simultaneously: $m_o > 1$, and $I^A_m < I \le I_h$, and $p_a \in (\bar{p}^1_a, p^h_{bb})$ where

$$\bar{p}^1_a =: \beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o) + (1-\lambda)\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a}\right].$$

In this region, the condition (102) can be rewritten as

$$v \ge \frac{\frac{p_a}{\beta} - [e^*_B + (1-e^*_B)\lambda]}{[e^l_{bb}(p_a) - e^*_B](1-\lambda)} =: \underline{\phi}^l_A(p_a).$$

Define $\underline{v}^l_A = \min_{p_a \in (\bar{p}^1_a, p^h_{bb})} \underline{\phi}^l_A(p_a)$, then the firm is able to advance sell with low-price strategy as long as $v \ge \underline{v}^l_A$. In what follows, we solve the threshold $\underline{v}^l_A$.

The first-order derivative of $\underline{\phi}^l_A(p_a)$ with respect to p_a is

$$\frac{d\underline{\phi}^l_A(p_a)}{dp_a} = \frac{\zeta_4(p_a)}{(1-\lambda)\beta[e^l_{bb}(p_a) - e^*_B]^2\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}$$

where

$$\zeta_4(p_a) =: \frac{p_a}{\beta} - \left[e_B^* - \frac{(1-\lambda)m_o}{4a}\right]\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}$$
$$+ \frac{(1-\lambda)^2 m_o^2}{4a} - 2\left(\frac{I}{\beta} - \lambda m_o\right) + e_B^* + (1-e_B^*)\lambda.$$

The first-order derivative of $\zeta_4(p_a)$ regarding p_a is

$$\zeta_4'(p_a) = \frac{1}{\beta}\left[1 - \frac{4ae_B^* - (1-\lambda)m_o}{\sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}\right]$$

Note that $\zeta_4'(p_a)$ increases in p_a, and meanwhile $\zeta_4'(\bar{p}_a^1) = 0$. Thus, $\zeta_4'(p_a) > 0$ for $p_a \in (\bar{p}_a^1, p_{bb}^h)$. That is, $\zeta_4(p_a)$ increases in $p_a \in (\bar{p}_a^1, p_{bb}^h)$. Moreover, we have $\zeta_4(\bar{p}_a^1) = 0$. Thus, $\zeta_4(p_a) > 0$ and hence $\frac{d\underline{\phi}_A^l(p_a)}{dp_a} > 0$ for $p_a \in (\bar{p}_a^1, p_{bb}^h)$. That is, $\underline{\phi}_A^l(p_a)$ increases in $p_a \in (\bar{p}_a^1, p_{bb}^h)$. Thus, $\underline{\phi}_A^l(p_a)$ achieves its minimum value when $p_a \to \bar{p}_a^1$, i.e.,

$$\underline{v}_A^l = \lim_{p_a \to \bar{p}_a^1} \underline{\phi}_A^l(p_a) = \frac{4ae_B^*}{1-\lambda} - m_o,$$

where the second equality is obtained according to the L' Hospital rule. Evidently, $\underline{v}_A^l$ decreases in I. Therefore, $\underline{v}_A^l$ values maximum at $I = I_m^A$ and minimum at $I = I_h$, and their respective values are denoted as follows: $\overline{\underline{v}}_A^l = m_o$ and $\underline{\underline{v}}_A^l = 1$. Since $\overline{\overline{v}}_A^l < 1 + m_o$ and $\underline{\underline{v}}_A^l < 1 + m_o$, the feasible region of v for the firm to conduct advance selling, i.e., $\underline{v}_A^l < v \le 1 + m_o$, exists.

Based on the above discussion, the firm should adopt advance selling with low-price strategy if and only if $\underline{v}_A^l \le v < \overline{v}_A^l$ for $I_m^A < I \le I_h$ and $m_o > 1$. According to Lemma A.2, under advance selling with low-price strategy, the firm's expected profit is $\pi_{bb}^l(p_a) = \beta a\left[e_{bb}^l(p_a)\right]^2$, which increases in p_a. Therefore, the optimal advance selling price, denoted as p_{aA}^{l*}, should be the maximum value in the feasible region of p_a. That is, $p_{aA}^{l*} = [\underline{\phi}_A^l]^{-1}(v)$, which denotes the large root to the equation of $\underline{\phi}_A^l(p_a) = v$. Correspondingly, the effort level is $e_A^{l*} = e_{bb}^l(p_{aA}^{l*})$ and the firm's expected profit is $\pi_A^{l*} = \beta a[e_{bb}^l(p_{aA}^{l*})]^2$. ☐

Proof of Proposition A.5. For $I_l < I \le I_h$, the firm is able to secure a bank loan even without advance selling. As the two conditions in Lemma A.1 for the firm to induce purchase in advance are too complicated to analyze simultaneously, in the subsequent analysis, we first consider the first condition, i.e., $\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{bw}]$, and then show that the firm is either unable or unwilling to advance sell even though the pricing constraints are relaxed, that is, the second condition in the Lemma is not even taken into consideration.

According to Lemma A.5, the firm can advance sell either with high-price strategy (i.e., $p_a \ge p_{bb}^h(p_s)$) or with low-price strategy (i.e., $p_{bb}^l(p_s) \le p_a < p_{bb}^h(p_s)$) . Subsequently, we consider these two pricing strategies, respectively. Moreover, although the coupon price p_a and spot price p_s are determined simultaneously by the firm, in what follows, we first characterize the optimal coupon price p_a for given p_s, and then derive the optimal spot price p_s.

High-price strategy. Given $p_a \ge p_{bb}^h(p_s)$, assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of $e_{bb}^h(p_s) = \frac{(1-\lambda)(p_s+m_o)}{2a}$ in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta(v-p_s)[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda] - p_a. \tag{103}$$

However, given $p_a \geq p_{bb}^h(p_s)$, when $k-1$ consumers purchase in advance but one consumer deviates to wait, according to Lemma A.6, the firm's optimal effort level $e_{bw}(p_a, p_s)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.7, we have $p_{bb}^l(p_s)/p_{bw}^l(p_s) < p_{bb}^h(p_s) < p_{bw}^h(p_s)$ as $k \to \infty$ for $I_l < I \leq I_h$. Thus, with high-price strategy, that is, $p_a \geq p_{bb}^h(p_s)$, the firm can set either $p_a \geq p_{bw}^h(p_s)$ or $p_{bb}^h(p_s) \leq p_a < p_{bw}^h(p_s)$. Next, we consider these two pricing intervals in turn.

1. Given $p_s \in [0,1]$ and $p_a \geq p_{bw}^h(p_s)$, with the same analysis as in the proof of Proposition 4 for the case of $p_a \geq p_{bw}^h(p_s)$, we derive that one necessary condition for the firm to be able to advance sell with high-price strategy is as follows:

$$\begin{cases} p_s \in [0,1] & (104) \\ p_a \geq p_{bb}^h(p_s) & (105) \\ p_a \leq \beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda](1-p_s) & (106) \end{cases}$$

as $k \to \infty$. We define the region of (p_a, p_s) bounded by (104)-(106) as $\overline{\Delta}_C^m$. In what follows we find optimal $(p_a, p_s) \in \overline{\Delta}_C^m$ to maximize the firm's expected profit

$$\pi_{bb}^h(p_a, p_s) = \beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a} + \lambda(p_s+m_o)\right] + p_a - I. \quad (107)$$

It can be observed from (104)-(106) that $\overline{\Delta}_C^m$ is nonempty if and only if

$$\beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda](1-p_s) - p_{bb}^h(p_s) = \frac{\beta(1-\lambda)^2(p_s+m_o)(1-p_s)}{2a} + \beta\lambda(1+m_o) - I =: \delta(p_s) \geq 0.$$

Since $\delta(p_s)$ is quadratic on $p_s \in [0,1]$, there exist two roots to $\delta(p_s) = 0$, which we denote as p_s^d and p_s^u, respectively, where

$$p_s^d = \frac{(1-\lambda)(1-m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)},$$

$$p_s^u = \frac{(1-\lambda)(1-m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)} \in [0,1].$$

$\delta(p_s) \geq 0$ if and only if $p_s \in [\max\{p_s^d, 0\}, p_s^u]$. Given $p_s \in [\max\{p_s^d, 0\}, p_s^u]$, it can be observed from (107) that the optimal coupon price, denoted as $p_a^o(p_s)$, should be the highest in the region constrained by (104)–(106), and thus

$$p_a^o(p_s) = \beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda](1-p_s)$$

Plugging $p_a^o(p_s)$ above into (107) leads to the firm's expected profit as follows:

$$\begin{aligned} \pi_{bb}^h(p_a^o(p_s), p_s) &= \beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a} + \lambda(p_s+m_o)\right] + p_a^o(p_s) - I \\ &= \beta\left[\frac{(1-\lambda)^2(p_s+m_o)\,(2-p_s+m_o)}{4a} + \lambda(1+m_o)\right] - I. \end{aligned}$$

Note that the firm's profit $\pi_{bb}^h(p_a^o(p_s),p_s)$ increases in $p_s \in [\max\{p_s^d,0\},p_s^u]$. Thus, the optimal coupon price is equal to p_s^u. Accordingly, the firm's expected profit is:

$$\pi_{bb}^h(p_a^o(p_s^u),p_s^u) = \beta a[e_{bb}^h(p_s^u)]^2 = \beta a \left[\frac{(1-\lambda)(1+m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{4a}\right]^2.$$

However, note that $\pi_{bb}^h(p_a^o(p_s^u),p_s^u) \leq \pi_B^*$ for $I_l < I \leq I_h$. That is, the firm is unwilling to advance sell even though the pricing conditions are relaxed. Therefore, the firm will not advance sell in this case even though she is able to.

2. Given $p_s \in [0,1]$, and $p_{bb}^h(p_s) \leq p_a < p_{bw}^h(p_s)$, with the same analysis as in the proof of Proposition 4 for the case of $p_{bb}^h(p_s) \leq p_a < p_{bw}^h(p_s)$, we derive that one necessary condition for the firm to induce all k consumers to purchase in advance is

$$\begin{cases} p_s \in [0,1] & (108) \\ p_a = p_{bb}^h(p_s) & (109) \\ p_a \leq \beta[e_{bb}^h(p_s)+(1-e_{bb}^h(p_s))\lambda](1-p_s) & (110) \end{cases}$$

as $k \to \infty$. Evidently, the region of (p_a,p_s) constrained by (108)-(110) is a subset of $\overline{\Delta}_C^m$. Therefore, similar to the previous case of $p_a \geq p_{bw}^h(p_s)$, the firm will not advance sell in this case.

Low-price strategy. Given $p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s)$, assuming that all segment-i consumers purchase in advance, according to Lemma A.5, the firm's effort level is:

$$e_{bb}^l(p_a,p_s) = \frac{(1-\lambda)(p_s+m_o)+\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}{4a}.$$

Accordingly, a consumer's expected surplus is:

$$\mathbb{E}[u_{bb}] = \beta(v-p_s)[e_{bb}^l(p_a,p_s)+(1-e_{bb}^l(p_s))\lambda]-p_a.$$

However, given $p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s)$, when $k-1$ consumers purchase in advance and one chooses to wait, according to Lemma A.6, the firm's optimal effort level $e_{bw}(p_a,p_s)$ and the associated expected surplus of the consumer who deviates to wait, $\mathbb{E}[u_{bw}]$, depend on the specific pricing interval p_a locates in. In accordance with Lemma A.7, for $I_l < I \leq I_h$, there exist two relevant scenarios regarding the relationship between $p_{bb}^l(p_s)$ and $p_{bw}^l(p_s)$, specified as follows:

1. If (i) $I > \hat{I}_h$, or (ii) $\hat{I}_l \leq I \leq \hat{I}_h$ and $p_s \in (0,\ddot{p}_s)$, where $\ddot{p}_s$ uniquely solves $I_t(\ddot{p}_s) = I$, then we have $p_{bb}^l(p_s) < p_{bw}^l(p_s) < p_{bb}^h(p_s) < p_{bw}^h(p_s)$ and $p_{bw}^l(p_s) \to p_{bb}^l(p_s)$ as $k \to \infty$;
2. If (i) $\hat{I}_l \leq I \leq \hat{I}_h$ and $p_s \in [\ddot{p}_s,1]$, or (ii) $I < \hat{I}_l$, we have $p_{bw}^l(p_s) < p_{bb}^l(p_s) < p_{bb}^h(p_s) < p_{bw}^h(p_s)$ and $p_{bw}^l(p_s) \to p_{bb}^l(p_s)$ as $k \to \infty$.

In what follows, we analyze these two scenarios, respectively.

Scenario 1 [(i) $I > \hat{I}_h$, or (ii) $\hat{I}_l \leq I \leq \hat{I}_h$ and $p_s \in (0,\ddot{p}_s)$]: In this scenario, we have $p_{bb}^l(p_s) < p_{bw}^l(p_s) < p_{bb}^h(p_s) < p_{bw}^h(p_s)$ as $k \to \infty$. Therefore, with low-price strategy, that is, $p_{bb}^l(p_s) \leq p_a < p_{bb}^h(p_s)$, the firm can set either $p_a \in [p_{bw}^l(p_s),\, p_{bb}^h(p_s))$ or $p_a \in [p_{bb}^l(p_s),\, p_{bw}^l(p_s))$. Next, we consider these two pricing intervals in turn.

1. Given (i) $I > \hat{I}_h$, or (ii) $\hat{I}_l \le I \le \hat{I}_h$, $p_s \in (0, \ddot{p}_s)$, and $p^l_{bw}(p_s) \le p_a < p^h_{bb}(p_s)$, with the same analysis as in the proof of Proposition 4 for the case of $p^l_{bw}(p_s) \le p_a < p^h_{bb}(p_s)$, we derive that one necessary condition for the firm to induce all k consumers to purchase in advance is

$$\begin{cases} (i) I > \hat{I}_h, \quad \text{or} \quad (ii) \hat{I}_l \le I \le \hat{I}_h \quad \text{and} \quad p_s \in (0, \ddot{p}_s) & (111) \\ p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s), & (112) \\ p_a \le \beta[e^l_{bb}(p_a, p_s) + (1 - e^l_{bb}(p_a, p_s))\lambda](1 - p_s). & (113) \end{cases}$$

as $k \to \infty$. In what follows, we find the optimal (p_a, p_s) in their feasible region bounded by the above constraints (111)–(113) to maximize the firm's expected profit $\pi^l_{bb}(p_a, p_s) = \beta a[e^l_{bb}(p_a, p_s)]^2$.

By some algebra, the constraint (113) is reformulated as

$$2ap_a^2 - \beta(1 - p_s)[4a\lambda + (1 - \lambda)^2(1 + m_o)]p_a + [2a\beta^2\lambda^2 + \beta(1 - \lambda)^2 I](1 - p_s)^2 \le 0. \tag{114}$$

Note that the left hand side of (114) is quadratic in p_a, so it has no feasible solution if $I > \hat{I}_h$. Therefore, the firm fails to advance sell for the case (i) $I > \hat{I}_h$. For case (ii) $\hat{I}_l \le I \le \hat{I}_h$ and $p_s \in (0, \ddot{p}_s)$, solving the inequality (114) leads to

$$\beta(1 - p_s)\left[\lambda + \frac{(1-\lambda)^2(1+m_o) - (1-\lambda)\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a}\right]$$
$$\le p_a \le$$
$$\beta(1 - p_s)\left[\lambda + \frac{(1-\lambda)^2(1+m_o) + (1-\lambda)\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a}\right]. \tag{115}$$

The above inequality (115) implies that

$$e^l_{bb}(p_a, p_s) = \frac{(1-\lambda)(p_s + m_o) + \sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]}}{4a}$$
$$\le \frac{(1-\lambda)(1 + m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{4a} \le e^*_B.$$

Accordingly,

$$\pi^l_{bb}(p_a, p_s) = \beta a\left[e^l_{bb}(p_a, p_s)\right]^2 \le \beta a(e^*_B)^2 = \pi^*_B,$$

which indicates that the firm is unwilling to advance sell in this case even though she is able to.

2. Given (i) $I > \hat{I}_h$, or (ii) $\hat{I}_l \le I \le \hat{I}_h$, $p_s \in (0, \ddot{p}_s)$, and $p^l_{bb}(p_s) \le p_a < p^l_{bw}(p_s)$, with the same analysis as in the proof of Proposition 4 for the case of $p^l_{bb}(p_s) \le p_a < p^l_{bw}(p_s)$, we derive that one necessary condition for the firm to induce all k segment-i consumers to purchase in advance is

$$\begin{cases} (i) I > \hat{I}_h, \quad \text{or} \quad (ii) \hat{I}_l \le I \le \hat{I}_h \quad \text{and} \quad p_s \in (0, \ddot{p}_s) & (116) \\ p_a = p^l_{bb}(p_s) & (117) \\ v \ge \dfrac{p^l_{bb}(p_s)}{\beta\left[(1-\lambda)e^l_{bb}(p^l_{bb}(p_s), p_s) + \lambda\right]} + p_s = \dfrac{I - \beta\left[\lambda(p_s + m_o) + \frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]}{\beta\left[\frac{(1-\lambda)^2(p_s+m_o)}{4a} + \lambda\right]} + p_s. & (118) \end{cases}$$

as $k \to \infty$. On condition that the firm is able to advance sell in this case, the above constraints of (117)-(118) indicates that the optimal effort level satisfies

$$e^l_{bb}(p_a,p_s) = e^l_{bb}(p^l_{bb}(p_s),p_s) = \frac{(1-\lambda)(p_s+m_o)}{4a} < \frac{(1-\lambda)(1+m_o)}{4a} < e^*_B.$$

Accordingly, the firm's profit satisfies $\pi^l_{bb}(p_a,p_s) < \pi^*_B$. Thus, the firm is unwilling to advance sell even though she is able to in this case.

Scenario 2 [(i) $\hat{I}_l \le I \le \hat{I}_h$ and $p_s \in [\ddot{p}_s, 1]$, or (ii) $I < \hat{I}_l$]: In this scenario, we have $p^l_{bw}(p_s) < p^l_{bb}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ as $k \to \infty$. In this case, with low-price strategy, the firm can only set $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$. With the same analysis for $p^l_{bw}(p_s) \le p_a < p^h_{bb}(p_s)$ in Scenario 1, it can be shown that the firm is unwilling to advance sell even if she is able to.

To summarize, the firm will not advance sell whether with high-price strategy or low-price strategy under advance selling with discount coupons. Thus, advance selling with discount coupons (i.e., scheme "C") is dominated by bank financing (i.e., scheme "B"). □

Proof of Proposition A.6. Similar to the analysis in the proof of Proposition 2, we can show that $p^{s*}_{rH} = 1$ in this case. Under advance selling with an all-or-nothing clause, the firm would cancel advance selling unless all segment-i consumers purchase in advance. Thus, we have $\mathbb{E}[u_{bw}] = \mathbb{E}[u_{ww}]$. Therefore, by Lemma A.1, the firm can induce all consumers to purchase in advance if and only if

$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{ww}]. \tag{119}$$

Moreover, as $I_l < I \le I_h$, if all k segment-i consumers wait, it degenerates to the benchmark case of pure bank financing. In the case of $v > 1 + m_o$, the firm would set the regular price equal to $p^*_{rB} = v$ as discussed in the proof of Proposition 1. Thus, the expected surplus of a segment-i consumer when all consumers wait is $\mathbb{E}[u_{ww}] = \beta[e^*_B + (1-e^*_B)\lambda](v - p^*_{rB}) = 0$, and thus Eq. (119) becomes

$$\mathbb{E}[u_{bb}] \ge \mathbb{E}[u_{ww}] = 0. \tag{120}$$

According to Lemma A.5, the firm could advance sell by either high-price (i.e., $p_a \ge p^h_{bb}(p_s)$) strategy or low-price (i.e., $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$) strategy. In what follows, we first consider these two pricing strategies, respectively, and then compare the optimal results under these two strategies to derive the equilibrium results.

High-price strategy (i.e., $p_a \ge p^h_{bb}(p_s)$). Given $p_s \in [0,1]$ and $p_a \ge p^h_{bb}(p_s)$, assuming that all k consumers purchase coupons in advance, the firm would exert an effort of $e^h_{bb}(p_s) = \frac{(1-\lambda)(p_s+m_o)}{2a}$ in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda](v-p_s) - p_a. \tag{121}$$

Anticipating this, the consumers would purchase in advance if and only if the equivalent condition (120) is met. Substituting $\mathbb{E}[u_{bb}]$ in (121) into the equivalent condition (120) gives $p_a \le \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda](v - p_s)$, or equivalently

$$v \ge \frac{p_a}{\beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda]} + p_s := \overline{\phi}^s_H(p_a,p_s).$$

Summarizing the above constraints, the firm's optimization problem can be formulated as follows:

$$\max\ \pi_{bb}^h(p_a,p_s)=\beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a}+\lambda(p_s+m_o)\right]+p_a-I \tag{122}$$

$$s.t.\quad \begin{cases} p_s\in[0,1]\\ p_a\geq p_{bb}^h(p_s)\\ v\geq \frac{p_a}{\beta[e_{bb}^h(p_s)+(1-e_{bb}^h(p_s))\lambda]}+p_s=:\overline{\phi}_H^s(p_a,p_s)\end{cases} \tag{123}$$

In what follows, we solve the above optimization problem in two steps.

Step 1. Note that the constraints in Eq. (123) imply that the firm is able to advance sell as long as v is sufficiently large. Let $\overline{\Delta}_H^s$ be the feasible region of (p_a,p_s), which is bounded by the first and second constraints of (123). Further, define $\overline{v}_H^s=\min_{(p_a,p_s)\in\overline{\Delta}_H^s}\overline{\phi}_H^s(p_a,p_s)$. Then the firm is able to advance sell if and only if $v\geq\overline{v}_H^s$. In **Step 1, we will solve the value $\overline{v}_H^s$.**

Given $p_s\in[0,1]$, $\overline{\phi}_H^s(p_a,p_s)$ increases in $p_a\in[p_{bb}^h(p_s),\infty)$ and thus $\overline{\phi}_H^s(p_a,p_s)$ achieves its minimum at $p_a=p_{bb}^h(p_s)$, i.e.,

$$\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)=\frac{I-\beta\lambda(p_s+m_o)}{\beta\left[\lambda+\frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]}+p_s$$

The first-order derivative of $\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)$ with respect to p_s is

$$\frac{d\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)}{dp_s}=\frac{(1-\lambda)^2}{2a}\cdot\frac{\vartheta(p_s)}{\beta\left[\lambda+\frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]^2}$$

where

$$\vartheta(p_s)=:\frac{\beta(1-\lambda)^2(p_s+m_o)^2}{2a}+2\beta\lambda(p_s+m_o)-I.$$

Evidently, $\vartheta(p_s)$ increases in p_s. Moreover, $\vartheta(0)=\beta\lambda\cdot 2m_o+\frac{\beta(1-\lambda)^2(2m_o)^2}{8a}-I$, and $\vartheta(1)=:\beta\lambda\cdot 2(1+m_o)+\frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}-I$. Thus, we have the following three cases:

1. If $\vartheta(0)\geq 0$, i.e., $I\leq\beta\lambda\cdot 2m_o+\frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, then $\frac{d\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)}{dp_s}\geq 0$ for $p_s\in[0,1]$. Therefore, $\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)$ values minimum at $p_s=0$, and thus $\overline{v}_H^s=\frac{I-\beta\lambda m_o}{\beta\left[\lambda+\frac{(1-\lambda)^2m_o}{2a}\right]}$;
2. If $\vartheta(0)<0\leq\vartheta(1)$, i.e., $\beta\lambda\cdot 2m_o+\frac{\beta(1-\lambda)^2(2m_o)^2}{8a}<I\leq\beta\lambda\cdot 2(1+m_o)+\frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, then there exists a unique p_s^0 such that $\vartheta(p_s^0)=0$, which gives

$$p_s^0=\frac{-2\beta a\lambda+\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o.$$

   $\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)$ values minimum at $p_s=p_s^0$, and thus

$$\begin{aligned}\overline{v}_H^s&=\frac{I-\frac{-2\beta a\lambda^2+\lambda\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{(1-\lambda)^2}}{\frac{\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{2a}}+\frac{-2\beta a\lambda+\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o\\&=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o\end{aligned}$$

3. If $\vartheta(1)<0$, i.e., $I>\beta\lambda\cdot 2(1+m_o)+\frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, then $\frac{d\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)}{dp_s}<0$ for $p_s\in[0,1]$. Therefore, $\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)$ values minimum at $p_s=1$, and thus $\overline{v}_H^s=\frac{I-\beta\lambda(1+m_o)}{\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{2a}\right]}+1$.

Step 2. Given $v \geq \overline{v}_H^s$, the firm is able to advance sell with high-price strategy. **In Step 2, we find the optimal** (p_a, p_s)**, denoted as** $(p_{aH}^{sh}, p_{sH}^{sh})$**, to maximize the firm's expected profit**

$$\pi_{bb}^h(p_a, p_s) = \beta\left[a(e_{bb}^h(p_s))^2 + \lambda(p_s + m_o)\right] + p_a - I.$$

Given $v \geq \overline{v}_H^s$ and p_s in the feasible region, the firm will set optimal coupon price, denoted as $p_a^o(p_s)$, as follows:

$$p_a^o(p_s) = \beta[e_{bb}^h(p_s) + (1 - e_{bb}^h(p_s))\lambda](v - p_s),$$

and the corresponding expected profit is:

$$\pi_{bb}^h(p_a^o(p_s), p_s) = \beta\left[a(e_{bb}^h(p_s))^2 + \lambda(p_s + m_o)\right] + \beta[e_{bb}^h(p_s) + (1 - e_{bb}^h(p_s))\lambda](v - p_s) - I.$$

Taking the first-order derivative of $\pi_{bb}^h(p_a^o(p_s), p_s)$ with respect to p_s leads to:

$$\frac{d\pi_{bb}^h(p_a^o(p_s), p_s)}{dp_s} = \frac{\beta(1-\lambda)^2(v - p_s)}{2a} \geq 0.$$

That is, the firm's expected profit increases in p_s and the firm should set p_s as the maximum value in the feasible region of p_s. Moreover, for given $v \geq \overline{v}_H^s$, the constraints in (123) imply that the feasible region of p_s is bounded by

$$\begin{cases} p_s \in [0, 1] \\ \overline{\phi}_H^s(p_{bb}^h(p_s), p_s) \leq v, \end{cases}$$

which is divided into the following three relevant cases:

1. When $I \leq \beta\lambda(1 + 2m_o) + \frac{\beta(1-\lambda)^2 \cdot 4m_o(1+m_o)}{8a}$, we have $\overline{\phi}_H^s(p_{bb}^h(p_s), p_s) < 1 + m_o < v$ for $p_s \in [0, 1]$. Thus, the feasible region of p_s is $[0, 1]$ and thereby we have $p_{sH}^{sh} = 1$. Accordingly,

$$p_{aH}^{sh} = p_a^o(1) = \beta[e_{bb}^h(1) + (1 - e_{bb}^h(1))\lambda](v - 1) = \beta\left[\lambda + \frac{(1-\lambda)^2(1 + m_o)}{2a}\right](v - 1).$$

   The firm's equilibrium effort is

$$e_H^{sh} = e_{bb}^h(p_s) = \frac{(1-\lambda)(1 + m_o)}{2a},$$

   and the expected profit is

$$\pi_H^{sh} = \beta\left[\frac{(1-\lambda)^2(m_o^2 - 1)}{4a} + \lambda m_o + v\left[\lambda + \frac{(1-\lambda)^2(1 + m_o)}{2a}\right] - \frac{I}{\beta}\right]$$

2. When $\beta\lambda(1 + 2m_o) + \frac{\beta(1-\lambda)^2 m_o(1+m_o)}{2a} < I \leq 2\beta\lambda(1 + m_o) + \frac{\beta(1-\lambda)^2(1+m_o)^2}{2a}$, then depending on v, there are two relevant cases:

   (a) If $\max\{\overline{v}_H^s, 1 + m_o\} \leq v < \overline{\phi}_H^s(p_{bb}^h(1), 1) = \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1$, then the feasible region of p_s is $\left[[\overline{\phi}_H^s(p_{bb}^h(p_s), p_s)]_l^{-1}(v), [\overline{\phi}_H^s(p_{bb}^h(p_s), p_s)]_r^{-1}(v)\right]$, where $[\overline{\phi}_H^s(p_{bb}^h(p_s), p_s)]_l^{-1}(v)$ and $[\overline{\phi}_H^s(p_{bb}^h(p_s), p_s)]_r^{-1}(v)$ represent the smaller and larger roots to the equation $\overline{\phi}_H^s(p_{bb}^h(p_s), p_s) = v$, respectively. Thus, we have

$$p_{sH}^{sh} = [\overline{\phi}_H^s(p_{bb}^h(p_s), p_s)]_r^{-1}(v) = \frac{(1-\lambda)(v - m_o) + \sqrt{(1-\lambda)^2(v + m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v + m_o)\right]}}{2(1-\lambda)}.$$

Accordingly, the optimal advance selling price is

$$p_{aH}^{sh} = p_a^o(p_{sH}^{sh}) = I - \beta\lambda \left(p_{sH}^{sh} + m_o\right).$$

The equilibrium effort is

$$e_H^{sh} = \frac{(1-\lambda)\left(p_{sH}^{sh} + m_o\right)}{2a}$$

and the expected profit is

$$\pi_H^{sh} = \frac{\beta\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]^2}{16a} = \left(e_H^{sh}\right)^2.$$

Evidently, we have $\pi_H^{sh} \geq \pi_B^*$ in this case, where "=" holds if and only if $m_o = 0$.

Finally, it should be noted that in this case $v > \overline{v}_H^s$ always holds since $I \leq I_h$. Thus, $v \geq \max\{\overline{v}_H^s, 1+m_o\}$ is equivalent to $v > 1+m_o$.

(b) If $v \geq \overline{\phi}_H^s(p_{bb}^h(1),1)$, then the feasible region of p_s is either $\left[[\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)]_l^{-1}(v), 1\right]$ or $[0,1]$. In either case, we have $p_{sH}^{sh} = 1$. Accordingly,

$$p_{aH}^{sh} = p_a^o(1) = \beta[e_{bb}^h(1) + (1-e_{bb}^h(1))\lambda](v-1) = \beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right](v-1)$$

and the firm's expected profit is

$$\pi_H^{sh} = \beta\left[\frac{(1-\lambda)^2(m_o^2-1)}{4a} + \lambda m_o + v\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right] - \frac{I}{\beta}\right]$$

3. When $I > \beta\lambda \cdot 2(1+m_o) + \frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, we have $\overline{v}_H^s > 1+m_o$. Then, given $v > \overline{v}_H^s$, the feasible region of p_s is either $\left[[\overline{\phi}_H^s(p_{bb}^h(p_s),p_s)]_l^{-1}(v), 1\right]$ or $[0,1]$. In either case, we have $p_{sH}^{sh} = 1$. Accordingly,

$$p_{aH}^{sh} = p_a^o(1) = \beta[e_{bb}^h(1) + (1-e_{bb}^h(1))\lambda](v-1) = \beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right](v-1)$$

and the firm's expected profit is

$$\pi_H^{sh} = \beta\left[\frac{(1-\lambda)^2(m_o^2-1)}{4a} + \lambda m_o + v\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right] - \frac{I}{\beta}\right].$$

Next, we show that $\pi_H^{sh} - \pi_B^*$ decreases in v in the case of $p_{sH}^{sh} = 1$. Note that the first-order derivative of $\pi_H^{sh} - \pi_B^*$ regarding v follows:

$$\begin{aligned}
\frac{d(\pi_H^{sh} - \pi_B^*)}{dv} =& -\frac{(1-\lambda)^2 v}{4a} - \frac{(1-\lambda)\sqrt{(1-\lambda)^2v^2 - 8a(\frac{I}{\beta} - \lambda v)}}{8a} - \frac{(1-\lambda)^3 v^2}{8a\sqrt{(1-\lambda)^2v^2 - 8a(\frac{I}{\beta} - \lambda v)}} \\
&+ \frac{(1-\lambda)^2(1+m_o)}{2a} - \frac{\lambda(1-\lambda)v}{2\sqrt{(1-\lambda)^2v^2 - 8a(\frac{I}{\beta} - \lambda v)}} + \frac{\lambda}{2} \\
<& -\frac{(1-\lambda)^2 v}{4a} - \frac{(1-\lambda)\sqrt{(1-\lambda)^2v^2 - 8a(\frac{I}{\beta} - \lambda v)}}{8a} - \frac{(1-\lambda)^3 v^2}{8a\sqrt{(1-\lambda)^2v^2 - 8a(\frac{I}{\beta} - \lambda v)}} \\
&+ \frac{(1-\lambda)^2(1+m_o)}{2a} \\
<& -\frac{(1-\lambda)^2 v}{2a} + \frac{(1-\lambda)^2(1+m_o)}{2a} \\
<& 0
\end{aligned}$$

That is, $\pi_H^{sh} - \pi_B^*$ decreases in v in the case of $p_{sH}^{sh} = 1$.

Low-price strategy (i.e., $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$). Given $p_s \in [0,1]$ and $p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s)$, assuming that all k segment-i consumers purchase in advance, the firm would exert an effort of

$$e^l_{bb}(p_a,p_s) = \frac{(1-\lambda)(p_s+m_o) + \sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]}}{4a}$$

in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta(v-p_s)[e^l_{bb}(p_a,p_s) + (1-e^l_{bb}(p_a,p_s))\lambda] - p_a. \tag{124}$$

Anticipating this, the consumers would purchase in advance if and only if the equivalent condition (120) is met. Substituting $\mathbb{E}[u_{bb}]$ in (124) into the equivalent condition (120) gives $p_a \le \beta(v-p_s)[e^l_{bb}(p_a,p_s) + (1-e^l_{bb}(p_a,p_s))\lambda]$, or equivalently

$$v \ge \frac{p_a}{\beta[e^l_{bb}(p_a,p_s) + (1-e^l_{bb}(p_a,p_s))\lambda]} + p_s =: \underline{\phi}^s_H(p_a,p_s).$$

Summarizing the above conditions, the firm's optimization problem can be formulated as follows:

$$\max\ \pi^l_{bb}(p_a,p_s) = \beta a\left[e^l_{bb}(p_a,p_s)\right]^2 \tag{125}$$

$$s.t.\quad \begin{cases} p_s \in [0,1] \\ p^l_{bb}(p_s) \le p_a < p^h_{bb}(p_s) \\ v \ge \frac{p_a}{\beta[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda]} + p_s =: \underline{\phi}^s_H(p_a,p_s). \end{cases} \tag{126}$$

In what follows, we will show that low-price strategy is dominated by high-price strategy, which is proved in two steps.

Step 1. The constraints (126) imply that the firm is able to advance sell as long as v is sufficiently large. Let $\underline{\Delta}^s_H$ be the feasible region of (p_a,p_s), which is bounded by the first two constraints of (126), and define $\underline{v}^s_H = \min_{(p_a,p_s)\in\underline{\Delta}^s_H} \underline{\phi}^s_H(p_a,p_s)$. Then, the firm is able to advance sell if and only if $v \ge \underline{v}^s_H$. **In Step 1, we will first solve the threshold value $\underline{v}^s_H$ and then show that** $\max\{1+m_o, \overline{v}^s_H\} = \max\{1+m_o, \underline{v}^s_H\}$.

Given p_s, the first-order partial derivative of $\underline{\phi}^s_H(p_a,p_s)$ with respect to p_a is

$$\frac{\partial \underline{\phi}^s_H(p_a,p_s)}{\partial p_a} = \frac{\zeta_6(p_a,p_s)}{\beta^2\left[e^l_{bb}(p_a,p_s) + (1-e^l_{bb}(p_a,p_s))\lambda\right]^2\sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]}},$$

where

$$\begin{aligned}\zeta_6(p_a,p_s) :=& \beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{4a}\right]\sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]} \\ &+ (1-\lambda)p_a + \frac{\beta(1-\lambda)^3(p_s+m_o)^2}{4a} - 2(1-\lambda)\left[I - \beta\lambda(p_s+m_o)\right],\end{aligned}$$

which increases in p_a. Moreover, at $p_a = p^l_{bb}(p_s)$, we have

$$\zeta_6(p^l_{bb}(p_s),p_s) = (1-\lambda)\left[\omega^l(p_s) - I\right],$$

and at $p_a = p^h_{bb}(p_s)$, we have

$$\zeta_6(p^h_{bb}(p_s),p_s) = (1-\lambda)\left[\omega^h(p_s) - I\right],$$

where $\omega^l(p_s) := \beta\lambda(p_s+m_o) + \frac{\beta(1-\lambda)^2(p_s+m_o)^2}{8a}$, and $\omega^h(p_s) := 2\beta\lambda(p_s+m_o) + \frac{\beta(1-\lambda)^2(p_s+m_o)^2}{2a}$.

We discuss the above problem in the following two cases:

1. $m_o \le 1$. In this case, we have $\omega^l(1) > \omega^h(0)$. Depending on I, we have the following five relevant sub-cases.

   (a) When $I \le \omega^l(0) = \beta\lambda m_o + \frac{\beta(1-\lambda)^2 m_o^2}{8a}$, we have $\zeta_6(p_{bb}^h(p_s), p_s) > \zeta_6(p_{bb}^l(p_s), p_s) > 0$. Thus, $\zeta_6(p_a, p_s) > 0$ holds for any $p_a \in [p_{bb}^l(p_s), p_{bb}^h(p_s))$ and $p_s \in [0,1]$. Accordingly, $\underline{\phi}_H^s(p_a, p_s)$ values minimum at $p_a = p_{bb}^l(p_s)$ for given p_s, and its value is

$$\underline{\phi}_H^s(p_{bb}^l(p_s), p_s) = \frac{I - \beta\left[\lambda(p_s + m_o) + \frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{4a}\right]} + p_s.$$

   Moreover, we have

$$\frac{d\underline{\phi}_H^s(p_{bb}^l(p_s), p_s)}{dp_s} = \frac{(1-\lambda)^2[\omega^l(p_s) - I]}{4a\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{4a}\right]^2}.$$

   Observe that $\underline{\phi}_H^s(p_{bb}^l(p_s), p_s)$ increases in $p_s \in [0,1]$ for $I \le \omega^l(0)$. Thus,

$$\underline{v}_H^s = \min_{p_s \in [0,1]} \underline{\phi}_H^s(p_{bb}^l(p_s), p_s) = \underline{\phi}_H^s(p_{bb}^l(0), 0) = \frac{I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]} < 0.$$

   (b) When $\omega^l(0) < I \le \omega^h(0) = 2\beta\lambda m_o + \frac{\beta(1-\lambda)^2(2m_o)^2}{8a}$, there exists a unique p_s^{l0} such that $\omega^l(p_s^{l0}) = I$, i.e.,

$$p_s^{l0} = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

   We have $\zeta_6(p_{bb}^l(p_s), p_s) > 0$ for $p_s \in [p_s^{l0}, 1]$ and $\zeta_6(p_{bb}^l(p_s), p_s) < 0$ for $p_s \in [0, p_s^{l0})$. Moreover, we have $\zeta_6(p_{bb}^h(p_s), p_s) > 0$ for any $p_s \in [0,1]$. Therefore, we have the following two cases:

      i. for given $p_s \in [p_s^{l0}, 1]$, $\underline{\phi}_H^s(p_a, p_s)$ values minimum at $p_a = p_{bb}^l(p_s)$, and its value is

$$\underline{\phi}_H^s(p_{bb}^l(p_s), p_s) = \frac{I - \beta\left[\lambda(p_s + m_o) + \frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2(p_s+m_o)}{4a}\right]} + p_s.$$

      Note that $\underline{\phi}_H^s(p_{bb}^l(p_s), p_s)$ increases in $p_s \in [p_s^{l0}, 1]$, and thus $\underline{\phi}_H^s(p_{bb}^l(p_s), p_s)$ achieves its minimum at $p_s = p_s^{l0}$. Therefore,

$$\begin{aligned}\underline{v}_H^{s1} &= \min_{p_s \in [p_s^{l0}, 1]} \underline{\phi}_H^s(p_{bb}^l(p_s), p_s) = \underline{\phi}_H^s(p_{bb}^l(p_s^{l0}), p_s^{s0}) \\ &= p_s^{l0} = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.\end{aligned}$$

      ii. for given $p_s \in [0, p_s^{l0}]$, we have $\zeta_6(p_{bb}^l(p_s), p_s) < 0$ and $\zeta_6(p_{bb}^h(p_s), p_s) > 0$. Thus, there exists a unique $p_a^0(p_s)$ such that $\zeta_6(p_a^0(p_s), p_s) = 0$. That is,

$$p_a^0(p_s) = \frac{8a\beta\lambda^2 + 4(1-\lambda)^2 I - [4a\lambda + (1-\lambda)^2(p_s + m_o)]\sqrt{4\beta^2\lambda^2 + \frac{2\beta(1-\lambda)^2 I}{a}}}{2(1-\lambda)^2}.$$

      $\underline{\phi}_H^s(p_a, p_s)$ achieves its minimum at $p_a = p_a^0(p_s)$, and its value is

$$\underline{\phi}_H^s(p_a^0(p_s), p_s) = \frac{p_a^0(p_s)}{\beta[e_{bb}^l(p_a^0(p_s), p_s) + (1 - e_{bb}^l(p_a^0(p_s), p_s))\lambda]} + p_s$$

$$=\frac{\frac{2\sqrt{4a^2\beta^2\lambda^2+2a\beta(1-\lambda)^2I}}{\beta}-4a\lambda-(1-\lambda)^2(p_s+m_o)}{(1-\lambda)^2}+p_s$$
$$=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o,$$

which is a constant independent of p_s. Thus,

$$\underline{v}_H^{s2}=\min_{p_s\in[0,p_s^{s0}]}\underline{\phi}_H^s(p_a^0(p_s),p_s)=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o=\underline{v}_H^{l1}.$$

Summarizing the above Cases i and ii, we have

$$\underline{v}_H^s=\min\{\underline{v}_H^{s1},\underline{v}_H^{s2}\}=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o.$$

(c) When $\omega^h(0)<I\le\omega^l(1)=\beta\lambda(1+m_o)+\frac{\beta(1-\lambda)^2(1+m_o)^2}{8a}$, there exists a unique p_s^{h0} such that $\omega^h(p_s^{h0})=I$, i.e., $p_s^{h0}=\frac{-2\beta a\lambda+\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o$. We have $\zeta_6(p_{bb}^l(p_s),p_s)>0$ and $\zeta_6(p_{bb}^h(p_s),p_s)>0$ for $p_s\in[p_s^{l0},1]$, $\zeta_6(p_{bb}^l(p_s),p_s)<0$ and $\zeta_6(p_{bb}^h(p_s),p_s)>0$ for $p_s\in[p_s^{h0},p_s^{l0})$, and $\zeta_6(p_{bb}^l(p_s),p_s)<0$ and $\zeta_6(p_{bb}^h(p_s),p_s)<0$ for $p_s\in[0,p_s^{h0})$. Therefore, we have the following three cases:

i. for given $p_s\in[p_s^{l0},1]$, similar to Case i in the above Scenario (b), it can be derived that

$$\underline{v}_H^{s1}=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o.$$

ii. for given $p_s\in[p_s^{h0},p_s^{l0})$, similar to Case ii in the above Scenario (b), it can be derived that

$$\underline{v}_H^{s2}=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o=\underline{v}_H^{s1}.$$

iii. for given $p_s\in[0,p_s^{h0})$, $\underline{\phi}_H^s(p_a,p_s)$ values minimum at $p_a=p_{bb}^h(p_s)$, and its value is

$$\underline{\phi}_H^s(p_{bb}^h(p_s),p_s)=\frac{I-\beta\lambda(p_s+m_o)}{\beta\left[\lambda+\frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]}+p_s.$$

Moreover, we have

$$\frac{d\underline{\phi}_H^s(p_{bb}^h(p_s),p_s)}{dp_s}=\frac{(1-\lambda)^2[\omega^h(p_s)-I]}{2a\beta\left[\lambda+\frac{(1-\lambda)^2(p_s+m_o)}{2a}\right]^2}<0$$

for $p_s\in[0,p_s^{h0})$. Thus, $\underline{\phi}_H^s(p_{bb}^h(p_s),p_s)$ values minimum at $p_s=p_s^{h0}$, which leads to

$$\underline{v}_H^{s3}=\underline{\phi}_H^s(p_{bb}^h(p_s^{h0}),p_s^{h0})=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o=\underline{v}_H^{s1}.$$

Summarizing the above Cases i–iii, we have:

$$\underline{v}_H^s=\min\{\underline{v}_H^{s1},\underline{v}_H^{s2},\underline{v}_H^{s3}\}=\frac{-4\beta a\lambda+2\sqrt{4\beta^2a^2\lambda^2+2\beta a(1-\lambda)^2I}}{\beta(1-\lambda)^2}-m_o.$$

(d) When $\omega^l(1)<I\le\omega^h(1)=2\beta\lambda(1+m_o)+\frac{\beta(1-\lambda)^2(1+m_o)^2}{2a}$, we have $\zeta_6(p_{bb}^h(p_s),p_s)>0$ for $p_s\in[p_s^{h0},1]$ and $\zeta_6(p_{bb}^h(p_s),p_s)<0$ for $p_s\in[0,p_s^{h0})$. Moreover, we have $\zeta_6(p_{bb}^l(p_s),p_s)<0$ for any $p_s\in[0,1]$. Therefore, we have the following two cases:

i. for given $p_s \in [p_s^{h0}, 1]$, similar to Case ii in Scenario (b), it can be derived that

$$\underline{v}_H^{s1} = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

ii. for given $p_s \in [0, p_s^{h0})$, similar to Case iii in Scenario (c), it can be derived that

$$\underline{v}_H^{s2} = \underline{\phi}_H^s(p_{bb}^h(p_s^{h0}), p_s^{h0}) = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o = \underline{v}_H^{s1}.$$

Summarizing the above Case i-ii, we have

$$\underline{v}_H^s = \min\{\underline{v}_H^{s1}, \underline{v}_H^{s2}\} = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

(e) When $I > \omega^h(1)$, we have $\zeta_6(p_{bb}^l(p_s), p_s) < \zeta_6(p_{bb}^h(p_s), p_s) < 0$. Thus, $\zeta_6(p_a, p_s) < 0$ holds for any $p_a \in [p_{bb}^l(p_s), p_{bb}^h(p_s))$ and $p_s \in [0,1]$. Accordingly, $\underline{\phi}_H^s(p_a, p_s)$ values minimum at $p_a = p_{bb}^h(p_s)$ for given p_s. Moreover, based on the above analysis, $\underline{\phi}_H^s(p_{bb}^h(p_s), p_s)$ decreases in $p_s \in [0,1]$. Thus, $\underline{\phi}_H^s(p_{bb}^h(p_s), p_s)$ values minimum at $p_s = 1$, and accordingly we have

$$\underline{v}_H^s = \underline{\phi}_H^s(p_{bb}^h(1), 1) = \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1.$$

2. $m_o > 1$. In this case, we have $\omega^l(1) < \omega^h(0)$. Also, there are five relevant subcases, depending on I, as follows:

(a) When $I \le \omega^l(0)$, similar to Scenario (a) in Case 1, we have

$$\underline{v}_H^s = \min_{p_s \in [0,1]} \underline{\phi}_H^s(p_{bb}^l(p_s), p_s) = \underline{\phi}_H^s(p_{bb}^l(0), 0) = \frac{I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]} < 0.$$

(b) When $\omega^l(0) < I \le \omega^l(1)$, similar to Scenario (b) in Case 1, we have

$$\underline{v}_H^s = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

(c) When $\omega^l(1) < I \le \omega^h(0)$, we have $\zeta_6(p_{bb}^l(p_s), p_s) < 0$ and $\zeta_6(p_{bb}^h(p_s), p_s) > 0$ for $p_s \in [0,1]$. Similarly, we have

$$\underline{v}_H^s = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

(d) When $\omega^h(0) < I \le \omega^h(1)$, similar to Scenario (d) in Case 1, we have

$$\underline{v}_H^s = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$

(e) When $I > \omega^h(1)$, similar to Scenario (e) in Case 1, we have

$$\underline{v}_H^s = \underline{\phi}_H^s(p_{bb}^h(1), 1) = \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1.$$

Summarizing Cases 1 and 2, we could obtain the following results in either case ($m_o \le 1$ or $m_o > 1$):

1. When $I \le \beta\lambda m_o + \frac{\beta(1-\lambda)^2 m_o^2}{8a}$, we have

$$\underline{v}_H^s = \underline{\phi}_H^s(p_{bb}^l(0), 0) = \frac{I - \beta\left[\lambda m_o + \frac{(1-\lambda)^2 m_o^2}{8a}\right]}{\beta\left[\lambda + \frac{(1-\lambda)^2 m_o}{4a}\right]} < 0.$$

2. When $\beta\lambda m_o + \frac{\beta(1-\lambda)^2 m_o^2}{8a} < I \le 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, we have
$$\underline{v}_H^s = \frac{-4\beta a\lambda + 2\sqrt{4\beta^2 a^2\lambda^2 + 2\beta a(1-\lambda)^2 I}}{\beta(1-\lambda)^2} - m_o.$$
3. When $I > 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2[2(1+m_o)]^2}{8a}$, we have
$$\underline{v}_H^s = \underline{\phi}_H^s(p_{bb}^h(1), 1) = \frac{I - \beta\lambda(1+m_o)}{\beta\left[\lambda + \frac{(1-\lambda)^2(1+m_o)}{2a}\right]} + 1.$$

Based on the above results, it can be concluded that $\max\{1+m_o, \overline{v}_H^s\} = \max\{1+m_o, \underline{v}_H^s\}$. This indicates that given v, if the firm is able to advance sell with low-price strategy, she is also able to advance sell with high-price strategy.

Step 2. In this step, we will show that low-price strategy is dominated by high-price strategy. Under the low-price strategy, from the constraints in (126) of the optimization problem (125), we observe that given p_s^f in the feasible region, the optimal advance selling price is either bounded by $p_{bb}^l(p_s^f) \le p_a < p_{bb}^h(p_s^f)$ or $v \ge \underline{\phi}_H^s(p_a, p_s^f)$. Since the firm should set the advance selling price as high as possible to achieve the highest profit for given p_s^f. Thus, in the former case ($p_{bb}^l(p_s^f) \le p_a < p_{bb}^h(p_s^f)$), the optimal advance selling price is
$$p_a^o(p_s^f) = p_{bb}^h(p_s^f),$$
which degenerates to the high-price strategy. In the latter case ($v \ge \underline{\phi}_H^s(p_a, p_s^f)$), the optimal advance selling price is
$$p_a^o(p_s^f) = \frac{\beta(v-p_s)\left[(1-\lambda)^2(v+m_o)+4a\lambda\right] + \beta(1-\lambda)(v-p_s)\sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}}{4a}$$
Accordingly, the firm's expected profit is
$$\begin{aligned}\pi_{bb}^l(p_a^o(p_s^f), p_s^f) &= \beta a\left[e_{bb}^l(p_a^o(p_s^f), p_s^f)\right]^2 \\ &= \frac{\beta\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]^2}{16a}.\end{aligned}$$

On the other hand, the optimization results derived under high-price strategy is divided into the following two cases:

1. When $\beta\lambda(1+2m_o) + \frac{\beta(1-\lambda)^2 m_o(1+m_o)}{2a} < I \le 2\beta\lambda(1+m_o) + \frac{\beta(1-\lambda)^2(1+m_o)^2}{2a}$ and meanwhile $\max\{\underline{v}_H^s, 1+m_o\} \le v < \overline{\phi}_H^s(p_{bb}^h(1), 1)$, the firm's expected profit is
$$\pi_H^{sh} = \frac{\beta\left[(1-\lambda)(v+m_o) + \sqrt{(1-\lambda)^2(v+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(v+m_o)\right]}\right]^2}{16a} =: \pi_{bb}^l(p_a^o(p_s^f), p_s^f).$$
This indicates that low-price strategy is dominated by high-price strategy in this case.
2. In the remaining cases, we have $p_{sH}^{sh} = 1$ and the firm's expected profit is
$$\begin{aligned}\pi_H^{sh} &= \beta\left[a(e_{bb}^h(1))^2 + \lambda(1+m_o)\right] + p_{aH}^{lh} - I \\ &> \beta a(e_{bb}^h(1))^2 \\ &> \beta a\left[e_{bb}^l(p_a^o(p_s^f), p_s^f)\right]^2 = \pi_{bb}^l(p_a^o(p_s^f), p_s^f).\end{aligned}$$
Again, the low-price strategy is also dominated by high-price strategy in this case.

To sum up, in either case, low-price strategy is dominated by high-price strategy, and thus the equilibrium results are identical to that under high-price strategy. That is, the optimal prices are $p^{s*}_{rH}=1$, $p^{s*}_{sH}=p^{sh}_{sH}$, $p^{s*}_{aH}=p^{sh}_{aH}$, and the firm's expected profit is $\pi^{s*}_H=\pi^{sh}_H$. Moreover, we define $\overline{w}^s_H=\overline{\phi}^s_H(p^h_{bb}(1),1)=\frac{I-\beta\lambda(1+m_o)}{\beta\left[\lambda+\frac{(1-\lambda)^2(1+m_o)}{2a}\right]}+1$, and the conclusions in Proposition A.6 can be drawn by an organization of the results derived above. □

Proof of Proposition A.7. Similar to the analysis in the proof of Proposition 2, we have $p^{l*}_{rH}=1$ in this scenario. Under advance selling with an all-or-nothing clause, the firm would cancel advance selling unless all segment-i consumers purchase in advance. Therefore, we have $\mathbb{E}[u_{bw}]=\mathbb{E}[u_{ww}]$, which indicates that the firm can induce all consumers to purchase in advance if and only if

$$\mathbb{E}[u_{bb}]\geq\mathbb{E}[u_{ww}] \tag{127}$$

under advance selling with all-or-nothing clause for $I_l<I\leq I_h$, in accordance with Lemma A.1. Moreover, if all k segment-i consumers wait, it degenerates to the benchmark case of pure bank financing for $v\leq 1+m_o$. Since $v\leq 1+m_o$, the firm would set the regular selling price equal to $p^*_{rB}=1$ as discussed in the proof of Proposition 1, the expected surplus of a segment-i consumer when all consumers wait is

$$\mathbb{E}[u_{ww}]=\beta[e^*_B+(1-e^*_B)\lambda](v-p^*_{rB})=\beta[e^*_B+(1-e^*_B)\lambda](v-1),$$

where $e^*_B=\frac{(1-\lambda)(1+m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{4a}$. Thus, Eq. (127) can be further specified as

$$\mathbb{E}[u_{bb}]\geq\mathbb{E}[u_{ww}]=\beta[e^*_B+(1-e^*_B)\lambda](v-1). \tag{128}$$

Further, according to Lemma A.5, the firm could advance sell by either high-price (i.e., $p_a\geq p^h_{bb}(p_s)$) strategy or low-price (i.e., $p^l_{bb}(p_s)\leq p_a<p^h_{bb}(p_s)$) strategy. In what follows, we first consider these two pricing strategies, respectively, and then compare the optimal results under these two strategies to derive the equilibrium results.

High-price strategy (i.e., $p_a\geq p^h_{bb}(p_s)$). Given $p_s\in[0,1]$ and $p_a\geq p^h_{bb}(p_s)$, assuming that all k consumers purchase coupons in advance, the firm would exert an effort of $e^h_{bb}(p_s)$ in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a consumer is

$$\mathbb{E}[u_{bb}]=\beta[e^h_{bb}(p_s)+(1-e^h_{bb}(p_s))\lambda](v-p_s)-p_a. \tag{129}$$

Anticipating this, the consumers would purchase in advance if and only if the equivalent condition (128) is met. Substituting $\mathbb{E}[u_{bb}]$ in (129) into the equivalent condition (128) gives

$$p_a\leq\beta\left[e^h_{bb}(p_s)-e^*_B\right](1-\lambda)v+\beta[e^*_B+(1-e^*_B)\lambda]-\beta[e^h_{bb}(p_s)+(1-e^h_{bb}(p_s))\lambda]p_s.$$

Summarizing the above constraints, the firm's optimization problem can be formulated as follows:

$$\max\ \pi^h_{bb}(p_a,p_s)=\beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a}+\lambda(p_s+m_o)\right]+p_a-I \tag{130}$$

$$s.t.\quad\begin{cases}p_s\in[0,1]\\ p_a\geq p^h_{bb}(p_s)\\ p_a\leq\beta\left[e^h_{bb}(p_s)-e^*_B\right](1-\lambda)v+\beta[e^*_B+(1-e^*_B)\lambda]-\beta[e^h_{bb}(p_s)+(1-e^h_{bb}(p_s))\lambda]p_s\end{cases} \tag{131}$$

In what follows, we solve the above optimization problem in three steps.

Step 1. At $p_s = p_s^u$, we have $e_{bb}^h(p_s^u) = e_B^*$, where $p_s^u = \frac{(1-\lambda)(1-m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2-8a[\frac{I}{\beta}-\lambda(1+m_o)]}}{2(1-\lambda)} \in [0,1]$. Then, $e_{bb}^h(p_s) < e_B^*$ for $p_s \in [0, p_s^u)$ while $e_{bb}^h(p_s) > e_B^*$ for $p_s \in (p_s^u, 1]$ since $e_{bb}^h(p_s)$ increases in p_s. **In Step 1, we show that it is not optimal for the firm to set $p_s \in [0, p_s^u]$.**

By contradiction, suppose that the firm sets $p_s \in [0, p_s^u]$, then $e_{bb}^h(p_s) \le e_B^*$. Together with the third constraint in (131), it indicates

$$
\begin{aligned}
p_a \le & \beta\left[e_{bb}^h(p_s) - e_B^*\right](1-\lambda) + \beta[e_B^* + (1-e_B^*)\lambda] - \beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda]p_s \\
= & \beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda](1-p_s) \\
= & \beta\left[-\frac{(1-\lambda)^2 p_s^2}{2a} + \frac{(1-\lambda)^2(1-m_o)p_s}{2a} + \lambda(1-p_s) + \frac{(1-\lambda)^2 m_o}{2a}\right] =: \mu(p_s).
\end{aligned} \tag{132}
$$

The above condition (132) and the second constraint in (131) jointly imply

$$
I - \beta\lambda(p_s + m_o) \le \beta\left[-\frac{(1-\lambda)^2 p_s^2}{2a} + \frac{(1-\lambda)^2(1-m_o)p_s}{2a} + \lambda(1-p_s) + \frac{(1-\lambda)^2 m_o}{2a}\right]. \tag{133}
$$

Solving the inequality (133) gives $p_s \in [p_s^d, p_s^u]$, where

$$
p_s^d = \frac{(1-\lambda)(1-m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)}. \tag{134}
$$

Thus, the firm fails to advance sell for $p_s \in [0, p_s^d)$.

Next, we show that the firm is unwilling to advance sell for $p_s \in [p_s^d, p_s^u]$ even if she is able to. Given $p_s \in [p_s^d, p_s^u]$, the inequality (132) implies

$$
\pi_{bb}^h(p_a, p_s) \le \pi_{bb}^h(\mu(p_s), p_s)
$$

since $\pi_{bb}^h(p_a, p_s)$ increases in p_a for given p_s. For $p_s \in [p_s^d, p_s^u]$, $\pi_{bb}^h(\mu(p_s), p_s)$ values maximum at $p_s = p_s^u$, and $\pi_{bb}^h(\mu(p_s^u), p_s^u) = \pi_B^* = \beta a(e_B^*)^2$. Therefore, $\pi_{bb}^h(p_a, p_s) \le \pi_{bb}^h(\mu(p_s), p_s) \le \pi_B^*$. Thus, the firm is unwilling to advance sell in this case.

To sum up, the firm is either unable or unwilling to advance sell by choosing $p_s \in [0, p_s^u]$. Thus, the firm will never set $p_s \in [0, p_s^u]$ for optimality.

Step 2. Based on the conclusion from Step 1, the firm will always set $p_s \in (p_s^u, 1]$. As such, we have $e_{bb}^h(p_s) > e_B^*$. Accordingly, the firm's optimization problem in (130)–(131) can be reformulated as follows:

$$
\max\ \pi_{bb}^h(p_a, p_s) = \beta\left[a(e_{bb}^h(p_s))^2 + \lambda(p_s + m_o)\right] + p_a - I \tag{135}
$$

$$
s.t. \quad \begin{cases} p_s \in (p_s^u, 1] \\ p_a \ge p_{bb}^h(p_s) \\ v \ge \frac{p_a + \beta[e_{bb}^h(p_s) + (1-e_{bb}^h(p_s))\lambda]p_s - \beta[e_B^* + (1-e_B^*)\lambda]}{\beta\left[e_{bb}^h(p_s) - e_B^*\right](1-\lambda)} =: \overline{\phi}_H^l(p_a, p_s) \end{cases} \tag{136}
$$

The constraints (136) imply that the firm is able to advance sell if v is sufficiently large. Let $\overline{\Delta}_H^l$ be the feasible region of (p_a, p_s), which is bounded by the first and the second constraints of (136). Further, define $\overline{v}_H^l = \min\limits_{(p_a, p_s) \in \overline{\Delta}_H^l} \overline{\phi}_H^l(p_a, p_s)$. Then the firm is able to advance sell if and only if $v \ge \overline{v}_H^l$. **In Step 2, we will solve the value of $\overline{v}_H^l$.**

Given $p_s \in (p_s^u, 1]$, $\overline{\phi}_H^l(p_a, p_s)$ increases in $p_a \in [p_{bb}^h(p_s), \infty)$ and thus $\overline{\phi}_H^l(p_a, p_s)$ achieves its minimum at $p_a = p_{bb}^h(p_s)$, that is,

$$\begin{aligned}\overline{\phi}_H^l(p_{bb}^h(p_s), p_s) &= \frac{I - \beta\lambda(p_s + m_o) + \beta[e_{bb}^h(p_s) + (1 - e_{bb}^h(p_s))\lambda]p_s - \beta[e_B^* + (1 - e_B^*)\lambda]}{\beta\left[e_{bb}^h(p_s) - e_B^*\right](1 - \lambda)} \\ &= \frac{I - \beta\lambda(1 + m_o) + \frac{\beta(1-\lambda)^2 p_s(p_s + m_o)}{2a} - \beta(1-\lambda)e_B^*}{\frac{\beta(1-\lambda)^2(p_s + m_o)}{2a} - \beta(1-\lambda)e_B^*}\end{aligned}$$

The first-order derivative of $\overline{\phi}_H^l(p_{bb}^h(p_s), p_s)$ with respect to p_s is

$$\begin{aligned}\frac{d\overline{\phi}_H^l(p_{bb}^h(p_s), p_s)}{dp_s} &= \frac{\beta(1-\lambda)^2}{2a} \cdot \frac{\frac{\beta(1-\lambda)^2(p_s+m_o)^2}{2a} - 2\beta(1-\lambda)e_B^*(p_s + m_o) + \beta(1-\lambda)e_B^*(1 + m_o) + \beta\lambda(1 + m_o) - I}{\left[\frac{\beta(1-\lambda)^2(p_s+m_o)}{2a} - \beta(1-\lambda)e_B^*\right]^2} \\ &\geq 0\end{aligned}$$

since $[2\beta(1-\lambda)e_B^*]^2 - \frac{2\beta(1-\lambda)^2\left[\beta(1-\lambda)e_B^*(1+m_o)+\beta\lambda(1+m_o)-I\right]}{a} = 0$. That is, $\overline{\phi}_H^l(p_{bb}^h(p_s), p_s)$ increases in $p_s \in (p_s^u, 1]$. Therefore,

$$\begin{aligned}\overline{v}_H^l &= \lim_{p_s \to p_s^u} \overline{\phi}_H^l(p_{bb}^h(p_s), p_s) \\ &= \lim_{p_s \to p_s^u} \frac{I - \beta\lambda(1 + m_o) + \frac{\beta(1-\lambda)^2 p_s(p_s+m_o)}{2a} - \beta(1-\lambda)e_B^*}{\frac{\beta(1-\lambda)^2(p_s+m_o)}{2a} - \beta(1-\lambda)e_B^*} \\ &= 1 + \frac{\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{1-\lambda}.\end{aligned}$$

Evidently, $\overline{v}_H^l$ decreases in I. Further, we have $\overline{v}_H^l = 1 < 1 + m_o$ at $I = I_h$, and $\overline{v}_H^l = 2 + m_o > 1 + m_o$ at $I = I_l$. Therefore, there exists a unique $I_m := \beta\left[\lambda(1+m_o) + \frac{(1-\lambda)^2(2m_o+1)}{8a}\right] \in (I_l, I_h)$, such that $\overline{v}_H^l(I_m) = 1 + m_o$. The firm can advance sell for $I \in (I_m, I_h]$ with the high-price strategy if $v \geq \overline{v}_H^l$.

Step 3. For $I \in (I_m, I_h]$, if $v \geq \overline{v}_H^l$, the firm is able to advance sell with the high-price strategy. Moreover, we have

$$\pi_{bb}^h(p_a, p_s) = \beta\left[a(e_{bb}^h(p_s))^2 + \lambda(p_s + m_o)\right] + p_a - I > \pi_B^* = \beta a(e_B^*)^2,$$

as $e_{bb}^h(p_s) > e_B^*$ and $p_a \geq p_{bb}^h(p_s) = I - \beta\lambda(p_s + m_o)$, which indicates that the firm is also willing to advance sell provided that she is able to. **In Step 3, we find the optimal (p_a, p_s), denoted as $(p_{aH}^{lh}, p_{sH}^{lh})$, to maximize the firm's expected profit $\pi_{bb}^h(p_a, p_s)$, given $v \geq \overline{v}_H^l$.**

For given $\overline{v}_H^l \leq v \leq 1 + m_o$, the constraints (136) imply that the feasible region of p_s is $\overline{\phi}_H^l(p_{bb}^h(p_s), p_s) \leq v$. As previously mentioned, $\overline{\phi}_H^l(p_{bb}^h(p_s), p_s)$ increases in $p_s \in (p_s^u, 1]$, the feasible region of p_s is divided into the following two cases:

(1) If $\overline{v}_H^l \leq v \leq \min\{1 + m_o, \overline{\phi}_H^l(p_{bb}^h(1), 1)\}$, where

$$\overline{\phi}_H^l(p_{bb}^h(1), 1) := 1 + \frac{(1-\lambda)(1+m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)},$$

then the feasible region of p_s is $\left(p_s^u, [\overline{\phi}_H^l(p_{bb}^h(p_s), p_s)]^{-1}(v)\right]$ with

$$[\overline{\phi}_H^l(p_{bb}^h(p_s), p_s)]^{-1}(v) = \frac{(1-\lambda)(2v - 1 - m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)};$$

(2) If $\min\{1+m_o, \overline{\phi}^l_H(p^h_{bb}(1),1)\} < v \le 1+m_o$, then the feasible region of p_s is $(p^u_s, 1]$.

Given $v \ge \overline{v}^l_H$ and p_s in the feasible region, the firm will set optimal coupon price, denoted as $p^o_a(p_s)$, as follows:

$$p^o_a(p_s) = \beta\left[e^h_{bb}(p_s) - e^*_B\right](1-\lambda)v + \beta[e^*_B + (1-e^*_B)\lambda] - \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda]p_s,$$

and accordingly generate an expected profit of

$$\begin{aligned}\pi^h_{bb}(p^o_a(p_s), p_s) =& \beta\left[a(e^h_{bb}(p_s))^2 + \lambda(p_s+m_o)\right] + \beta\left[e^h_{bb}(p_s) - e^*_B\right](1-\lambda)v + \beta[e^*_B + (1-e^*_B)\lambda] \\ & - \beta[e^h_{bb}(p_s) + (1-e^h_{bb}(p_s))\lambda]p_s - I\end{aligned}$$

The first-order derivative of $\pi^h_{bb}(p^o_a(p_s), p_s)$ with respect to p_s is

$$\frac{d\pi^h_{bb}(p^o_a(p_s), p_s)}{dp_s} = \frac{\beta(1-\lambda)^2(v-p_s)}{2a} \ge 0.$$

Thus, depending on v, we have the following two relevant cases:

(1) If $\overline{v}^l_H \le v \le \min\{1+m_o, \overline{\phi}^l_H(p^h_{bb}(1),1)\}$, the optimal spot price is

$$p^{lh}_{sH} = \frac{(1-\lambda)(2v-1-m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)},$$

and accordingly the optimal coupon price is

$$p^{lh}_{aH} = p^o_a(p^{lh}_{sH}) = I - \beta\lambda\left(p^{lh}_{sH} + m_o\right).$$

Correspondingly, the optimal effort is

$$e^{lh}_H = \frac{(1-\lambda)\left(p^{lh}_{sH} + m_o\right)}{2a},$$

and the firm earns an expected profit of

$$\pi^{lh}_H = \frac{\beta\left[(1-\lambda)(2v-1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]^2}{16a} = \beta a\left(e^{lh}_H\right)^2. \tag{137}$$

(2) If $\min\{1+m_o, \overline{\phi}^l_H(p^h_{bb}(1),1)\} < v \le 1+m_o$, the optimal spot price is $p^{lh}_{sH} = 1$, and accordingly the optimal coupon price is

$$\begin{aligned}p^{lh}_{aH} &= p^o_a(p^{lh}_{sH}) = \beta(1-\lambda)(v-1)[e^h_{bb}(1) - e^*_B] \\ &= \frac{\beta(1-\lambda)(v-1)\left[(1-\lambda)(1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}{4a}.\end{aligned}$$

Correspondingly, the optimal effort is $e^{lh}_H = \frac{(1-\lambda)(1+m_o)}{2a}$, and the firm's expected profit is

$$\begin{aligned}\pi^{lh}_H &= \beta\left[a(e^{lh}_H)^2 + \lambda(1+m_o)\right] + p^{lh}_{aH} - I \\ &= \beta\left[\frac{(1-\lambda)^2(1+m_o)^2}{4a} + \lambda(1+m_o)\right] \\ &+ \frac{\beta(1-\lambda)(v-1)\left[(1-\lambda)(1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}{4a} - I.\end{aligned} \tag{138}$$

Low-price strategy (i.e., $p_{bb}^l(p_s) \le p_a < p_{bb}^h(p_s)$). Given $p_s \in [0,1]$ and $p_{bb}^l(p_s) \le p_a < p_{bb}^h(p_s)$, assuming that all k consumers purchase in advance, the firm would exert an effort of

$$e_{bb}^l(p_a, p_s) = \frac{(1-\lambda)(p_s + m_o) + \sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a}{\beta} - \lambda(p_s+m_o)\right]}}{4a}$$

in accordance with Lemma A.5. Accordingly, when all k consumers purchase in advance, the expected surplus of a segment-i consumer is

$$\mathbb{E}[u_{bb}] = \beta(v - p_s)[e_{bb}^l(p_a,p_s) + (1 - e_{bb}^l(p_a,p_s))\lambda] - p_a. \tag{139}$$

Anticipating this, the consumers would purchase in advance if and only if the equivalent condition (128) is met. Substituting $\mathbb{E}[u_{bb}]$ in (139) into the equivalent condition (128) gives

$$p_a \le \beta\left[e_{bb}^l(p_a,p_s) - e_B^*\right](1-\lambda)v + \beta[e_B^* + (1-e_B^*)\lambda] - \beta[e_{bb}^l(p_a,p_s) + (1-e_{bb}^l(p_a,p_s))\lambda]p_s. \tag{140}$$

Moreover, from the firm's perspective, the expected profit is $\pi_{bb}^l(p_a,p_s) = \beta a\left[e_{bb}^l(p_a,p_s)\right]^2$ in the case of a successful advance selling with low-price strategy according to Lemma A.5, while the expected profit is $\pi_B^* = \beta a\left[e_B^*\right]^2$ without advance selling according to Proposition 1. Thus, the firm is willing to advance sell if and only if $\pi_{bb}^l(p_a,p_s) > \pi_B^*$, or equivalently

$$e_{bb}^l(p_a,p_s) > e_B^*. \tag{141}$$

From (141), Eq. (140) for the consumers to purchase in advance can be rewritten as

$$v \ge \frac{p_a + \beta[e_{bb}^l(p_a,p_s) + (1-e_{bb}^l(p_a,p_s))\lambda]p_s - \beta[e_B^* + (1-e_B^*)\lambda]}{\beta\left[e_{bb}^l(p_a,p_s) - e_B^*\right](1-\lambda)} =: \underline{\phi}_H^l(p_a,p_s).$$

Summarizing the above conditions, the firm's optimization problem can be formulated as follows:

$$\max\ \pi_{bb}^h(p_a,p_s) = \beta a\left[e_{bb}^l(p_a,p_s)\right]^2 \tag{142}$$

$$s.t.\quad \begin{cases} p_s \in [0,1] \\ p_{bb}^l(p_s) \le p_a < p_{bb}^h(p_s) \\ e_{bb}^l(p_a,p_s) > e_B^* \\ v \ge \frac{p_a + \beta[e_{bb}^l(p_a,p_s) + (1-e_{bb}^l(p_a,p_s))\lambda]p_s - \beta[e_B^* + (1-e_B^*)\lambda]}{\beta\left[e_{bb}^l(p_a,p_s) - e_B^*\right](1-\lambda)} =: \underline{\phi}_H^l(p_a,p_s). \end{cases} \tag{143}$$

In what follows, we will solve the above optimization problem in two steps.

Step 1. The constraints (143) imply that the firm is able to advance sell as long as v is sufficiently large. Let $\underline{\Delta}_H^l$ be the feasible region of (p_a,p_s), which is bounded by the first three constraints of (143), and define $\underline{v}_H^l = \min_{(p_a,p_s)\in\underline{\Delta}_H^l} \underline{\phi}_H^l(p_a,p_s)$. Then the firm is able to advance sell if and only if $v \ge \underline{v}_H^l$. **In Step 1, we solve the threshold value $\underline{v}_H^l$.**

Before solving $\underline{v}_H^l$, we first analyze the feasible region $\underline{\Delta}_H^l$. Since $e_{bb}^l(p_a,p_s)$ increases in p_a for given p_s, $e_{bb}^l(p_a,p_s)$ values maximum at $p_a = p_{bb}^h(p_s)$ and minimum at $p_a = p_{bb}^l(p_s)$. For ease of exposition, we define

$$e_{bb}^l(p_{bb}^h(p_s),p_s) = \frac{(1-\lambda)(p_s+m_o)}{2a}$$

and

$$e_{bb}^l(p_{bb}^l(p_s),p_s) = \frac{(1-\lambda)(p_s+m_o)}{4a},$$

and we have

$$e^l_{bb}(p^l_{bb}(p_s),p_s) = \frac{(1-\lambda)(p_s+m_o)}{4a} \leq e^l_{bb}(p_a,p_s) < \frac{(1-\lambda)(p_s+m_o)}{2a} = e^l_{bb}(p^h_{bb}(p_s),p_s).$$

Note that to ensure the set of $\{(p_a,p_s)|e^l_{bb}(p_a,p_s) > e^*_B\}$ be nonempty, $e^l_{bb}(p^h_{bb}(p_s),p_s) > e^*_B$, or equivalently, $p_s > p^u_s$, where p^u_s has been defined previously in the high-price strategy, should be satisfied. Moreover, $e^l_{bb}(p^l_{bb}(p_s),p_s) < e^*_B$ always hold for $p_s \in [0,1]$. Therefore, for given $p_s \in (p^u_s,1]$, there exist a unique $\overline{p}_a(p_s) \in (p^l_{bb}(p_s),p^h_{bb}(p_s))$ such that $e^l_{bb}(\overline{p}_a(p_s),p_s) = e^*_B$, i.e.,

$$\overline{p}_a(p_s) = \beta\left[\lambda + (1-\lambda)e^*_B\right](1-p_s).$$

We have $e^l_{bb}(p_a,p_s) > e^*_B$ for $p_a \in (\overline{p}_a(p_s),p^h_{bb}(p_s))$ and $e^l_{bb}(p_a,p_s) < e^*_B$ for $p_a \in [p^l_{bb}(p_s),\overline{p}_a(p_s))$. As a result, the feasible region $\underline{\Delta}^l_H$ can be formulated alternatively as

$$\underline{\Delta}^l_H = \left\{(p_a,p_s)|p^u_s < p_s \leq 1, \overline{p}_a(p_s) < p_a < p^h_{bb}(p_s)\right\}.$$

We continue to solve $\underline{v}^l_H$. Given p_s, the first-order partial derivative of $\underline{\phi}^l_H(p_a,p_s)$ with respect to p_a is

$$\frac{\partial\underline{\phi}^l_H(p_a,p_s)}{\partial p_a} = \frac{\zeta_5(p_a,p_s)}{\beta^2\left[e^l_{bb}(p_a,p_s)-e^*_B\right]^2(1-\lambda)^2},$$

where

$$\begin{aligned}\zeta_5(p_a,p_s) :=& \beta\left[e^l_{bb}(p_a,p_s)-e^*_B\right](1-\lambda)\left[1+\beta(1-\lambda)p_s\frac{de^l_{bb}(p_a,p_s)}{dp_a}\right]\\ &-\beta(1-\lambda)\frac{de^l_{bb}(p_a,p_s)}{dp_a}\left[p_a+\beta[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda]p_s-\beta[e^*_B+(1-e^*_B)\lambda]\right]\\ =&\beta(1-\lambda)[e^l_{bb}(p_a,p_s)-e^*_B] + \frac{(1-\lambda)\left[\beta(1-\lambda)(1-p_s)e^*_B+\beta\lambda(1-p_s)-p_a\right]}{\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}.\end{aligned}$$

The first-order partial derivative of $\zeta_5(p_a,p_s)$ with respect to p_a is

$$\frac{\partial\zeta_5(p_a,p_s)}{\partial p_a} = \frac{4a(1-\lambda)(p_a-\overline{p}_a(p_s))}{\beta\left[(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]\right]\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}.$$

Evidently, $\frac{\partial\zeta_5(p_a,p_s)}{\partial p_a} > 0$ for $\overline{p}_a(p_s) < p_a < p^h_{bb}(p_s)$. That is, $\zeta_5(p_a,p_s)$ increases in p_a for $p_a \in (\overline{p}_a(p_s),p^h_{bb}(p_s))$. Moreover, at $p_a = \overline{p}_a(p_s)$, we have $\zeta_5(\overline{p}_a(p_s),p_s) = 0$. As a result, $\zeta_5(p_a,p_s) > 0$ and thus $\frac{\partial\underline{\phi}^l_H(p_a,p_s)}{\partial p_a} > 0$ for $p_a \in (\overline{p}_a(p_s),p^h_{bb}(p_s))$. Therefore, $\underline{\phi}^l_H(p_a,p_s)$ reaches its minimum at $p_a = \overline{p}_a(p_s)$ for given p_s. At $p_a = \overline{p}_a(p_s)$, we have

$$\begin{aligned}\underline{\phi}^l_H(\overline{p}_a(p_s),p_s) =& \lim_{p_a\to\overline{p}_a(p_s)}\underline{\phi}^l_H(p_a,p_s)\\ =& \lim_{p_a\to\overline{p}_a(p_s)}\frac{p_a+\beta[e^l_{bb}(p_a,p_s)+(1-e^l_{bb}(p_a,p_s))\lambda]p_s-\beta[e^*_B+(1-e^*_B)\lambda]}{\beta\left[e^l_{bb}(p_a,p_s)-e^*_B\right](1-\lambda)}\\ =& 1+\frac{\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{1-\lambda},\end{aligned}$$

which is independent of p_s, and thus $\underline{v}^l_H = 1+\frac{\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{1-\lambda} = \overline{v}^l_H$. Therefore, the firm can advance sell for $I \in (I_m,I_h]$ with low-price strategy if $v \geq \underline{v}^l_H$, where $I_m := \beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(2m_o+1)}{8a}\right] \in (I_l,I_h)$.

Step 2. Given $v \geq \underline{v}_H^l$, the firm is able and willing to advance sell with low-price strategy for $I \in (I_m, I_h]$. **In Step 2, we will find the optimal (p_a, p_s), denoted as $(p_{aH}^{ll}, p_{sH}^{ll})$, to maximize the firm's expected profit $\pi_{bb}^l(p_a, p_s)$, given $v \geq \underline{v}_H^l$.**

Given p_s, it can be observed from (142) that the firm's expected profit $\pi_{bb}^l(p_a, p_s)$ increases in p_a, and thus the firm will choose maximum p_a in the feasible region bounded by (143). Depending on v, we have the following relevant cases.

(1) If $\underline{v}_H^l \leq v \leq \min\{1 + m_o, \underline{\phi}_H^l(p_{bb}^h(1), 1)\}$, where $\underline{\phi}_H^l(p_{bb}^h(1), 1) = 1 + \frac{(1-\lambda)(1+m_o)+\sqrt{(1-\lambda)^2(1+m_o)^2-8a\left[\frac{I}{\beta}-\lambda(1+m_o)\right]}}{2(1-\lambda)}$, given p_s, there exist a unique root, denoted as $\overline{p}_s$, to the equation $\underline{\phi}_H^l(p_{bb}^h(\overline{p}_s), \overline{p}_s) = v$. And it follows:

$$\overline{p}_s = \frac{(1-\lambda)(2v-1-m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)}.$$

According to whether $p_s \geq \overline{p}_s$ or not, there are two relevant cases as follows.

(i) If $p_s \in (\overline{p}_s, 1]$, the firm's optimal coupon price, denoted as $p_a^o(p_s)$, satisfies $\underline{\phi}_H^l(p_a^o, p_s) = v$, which gives

$$p_a^o(p_s) = \beta\lambda(1-p_s) + \frac{\beta(1-\lambda)^2(v-p_s)(v+m_o)}{2a} - \beta(1-\lambda)(2v-p_s-1)e_B^*.$$

Accordingly, the firm earns an expected profit

$$\begin{aligned}\pi_{bb}^l(p_a^o(p_s), p_s) &= \beta a\left[\frac{(1-\lambda)(p_s+m_o) + \sqrt{(1-\lambda)^2(p_s+m_o)^2 - 8a\left[\frac{I-p_a^o(p_s)}{\beta} - \lambda(p_s+m_o)\right]}}{4a}\right]^2 \\ &= \frac{\beta\left[(1-\lambda)(2v-1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]^2}{16a},\end{aligned}$$

which is independent of p_s. Thus, the firm's maximum profit in this case is

$$\pi_H^{ll1} = \frac{\beta\left[(1-\lambda)(2v-1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]^2}{16a}. \tag{144}$$

(ii) If $p_s \in (p_s^u, \overline{p}_s]$, given p_s, the firm's optimal coupon price is $p_a^o(p_s) = p_{bb}^h(p_s)$. Accordingly, the firm's expected profit is $\pi_{bb}^l(p_a^o(p_s), p_s) = \beta a\left[\frac{(1-\lambda)(p_s+m_o)}{2a}\right]^2$, which increases in p_s. Thus, the optimal spot price is $p_{sH}^{ll2} = \overline{p}_s$. Correspondingly, the optimal coupon price is

$$\begin{aligned}p_{aH}^{ll2} &= p_a^o(p_{sH}^{ll2}) = I - \beta\lambda(\overline{p}_s + m_o) \\ &= I - \frac{\beta\lambda\left[(1-\lambda)(2v-1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]}{2(1-\lambda)}\end{aligned}$$

and the firm's profit is

$$\pi_H^{ll2} = \frac{\beta\left[(1-\lambda)(2v-1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]^2}{16a}. \tag{145}$$

Combining Cases (i) and (ii), we observe from (144) and (145) that the firm earns identical profits in either case. Thus, if $\underline{v}_H^l \le v \le \min\{1+m_o, \underline{\phi}_H^l(p_{bb}^h(1),1)\}$, the firm's maximum profit with low-price strategy is

$$\pi_H^{ll} = \max(\pi_H^{ll1}, \pi_H^{ll2}) = \frac{\beta\left[(1-\lambda)(2v-1+m_o) - \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}\right]^2}{16a}. \quad (146)$$

(2) If $\min\{1+m_o, \underline{\phi}_H^l(p_{bb}^h(1),1)\} < v \le 1+m_o$, given p_s, the firm's optimal coupon price is $p_a^o(p_s) = p_{bb}^h(p_s)$. Accordingly, the firm's expected profit is $\pi_{bb}^l(p_a^o(p_s), p_s) = \beta a\left[\frac{(1-\lambda)(p_s+m_o)}{2a}\right]^2$, which increases in p_s. Thus, the optimal spot price is $p_{sH}^{ll} = 1$. Correspondingly, the optimal coupon price is $p_{aH}^{ll} = p_a^o(p_{sH}^{ll}) = I - \beta\lambda(1+m_o)$, and the firm's profit is

$$\pi_H^{ll} = \frac{\beta(1-\lambda)^2(1+m_o)^2}{4a}. \quad (147)$$

Equilibrium results. Finally, we derive the equilibrium results by a comparison of the high-price strategy and low-price strategy. Note that

$$\overline{v}_H^l = \underline{v}_H^l = 1 + \frac{\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{1-\lambda} =: \underline{w}_H^l,$$

and

$$\min\{1+m_o, \overline{\phi}_H^l(p_{bb}^h(1),1)\} = \min\{1+m_o, \underline{\phi}_H^l(p_{bb}^h(1),1)\} =: \overline{w}_H^l,$$

where

$$\overline{\phi}_H^l(p_{bb}^h(1),1) = \underline{\phi}_H^l(p_{bb}^h(1),1) =: 1 + \frac{(1-\lambda)(1+m_o) + \sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}}{2(1-\lambda)}.$$

Depending on v, there are two relevant cases below.

(1) If $\underline{w}_H^l \le v \le \overline{w}_H^l$, it can be observed from (137) and (146) that $\pi_H^{lh} = \pi_H^{ll}$.

(2) If $v > \overline{w}_H^l$, it follows from (138) and (147) that $\pi_H^{lh} > \pi_H^{ll}$.

To summarize, we have $\pi_H^{lh} \ge \pi_H^{ll}$ irrespective of v. Therefore, the firm will always adopt high-price strategy for $v \ge \underline{w}_H^l$, and the equilibrium results are equal to the optimal results under high-price strategy, i.e., $p_{aH}^{l*} = p_{aH}^{lh}$, $p_{sH}^{l*} = p_{sH}^{lh}$, and $\pi_H^{l*} = \pi_H^{lh}$. If $v < \underline{w}_H^l$, the firm will not advance sell and consequently adopts pure bank financing. □

Proof of Proposition A.8. The monotonicity is divided into the following two cases:

1. When $m_o < v-1$, $\pi_H^* = \max\{\pi_B^*, \pi_H^s\}$. It can be observed from Propositions 1 and A.6 that both π_H^s and π_B^* decrease in I. Thus, π_H^* decreases in I in this case.

2. When $m_o \ge v-1$, according to Proposition A.7,

   (a) if $\underline{w}_H^l < v \le \overline{w}_H^l$, then it can be observed that $\pi_H^* = \pi_H^{l*}$ increases in I;

(b) if $v > \overline{w}_H^l$, then the first-order derivative of π_H^* regarding I is

$$\frac{d\pi_H^*}{dI} = \frac{d\pi_H^{l*}}{dI} = \frac{(1-\lambda)(v-1)}{\sqrt{(1-\lambda)^2(1+m_o)^2 - 8a\left[\frac{I}{\beta} - \lambda(1+m_o)\right]}} - 1 > 0, \tag{148}$$

where the ">" holds because $v > \overline{w}_H^l$. That is, π_H^* increases in I.

Combining Scenarios (a) and (b) leads to that π_H^* increases in I in this case. □

Proof of Lemma A.2. Since all k segment-i consumers purchase in advance, the market demand in the financing period, D_f, is 1. Given that the firm succeeds in loan application and continues in the second period, the firm will always set $p_r^* = 1$ since only segment-o customers will buy in the repayment period. Accordingly, the market demand in the repayment period, D_r, is divided into the following two cases. First, if the firm's effort results in "success", all segment-o consumers will purchase and thus the demand is $D_r = m_o$. Second, if the effort results in "no success", only λ proportion of segment-o consumers will purchase and thus the market demand is $D_r = \lambda m_o$. Combining these two scenarios, let $S = D_r p_r^*$ represent the sales revenue of the firm in the repayment period, then S follows a binary distribution:

$$S = \begin{cases} m_o, & \text{with probability } e. \\ \lambda m_o, & \text{with probability } 1-e. \end{cases}$$

Under advance price p_a, if $p_a \geq I$, the firm has no need to borrow from the bank. Thus, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi_{bb}^s(e;p_a) = p_a - I + \mathbb{E}[S] - ae^2 = -ae^2 + (1-\lambda)m_o e + p_a + \lambda m_o - I.$$

By maximizing $\pi_{bb}^s(e;p_a)$, the optimal effort level is derived as: $e_{bb}^{ss} = \frac{(1-\lambda)m_o}{2a}$, and accordingly the firm's maximum expected profit is

$$\pi_{bb}^{ss}(p_a) = \frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o + p_a - I.$$

However, if the firm fails to continue to the second period, then the firm's expected profit is

$$\pi_{bb}^{sn}(p_a) = p_a - I.$$

Summarizing the above two cases, given $p_a \geq I$ and each segment-i customer advance buys, the firm's final expected profit is

$$\pi_{bb}^s(p_a) = \beta\pi_{bb}^{ss}(p_a) + (1-\beta)\pi_{bb}^{sn}(p_a) = \beta\left[\frac{(1-\lambda)^2 m_o^2}{4a} + \lambda m_o\right] + p_a - I.$$

By contrast, if $p_a < I$, the firm has to borrow $I - p_a$ from the bank. Given p_a and loan interest rate r, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi_{bb}^s(e;p_a,r) = \mathbb{E}[S - (I-p_a)(1+r)]^+ - ae^2.$$

We note that $(I-p_a)(1+r) < m_o$ should be satisfied, otherwise the bank loan will not be granted. Thus, the firm's profit can be rewritten as

$$\pi_{bb}^s(e;p_a,r) = -ae^2 + \left[m_o - (I-p_a)(1+r) - [\lambda m_o - (I-p_a)(1+r)]^+\right]e + [\lambda m_o - (I-p_a)(1+r)]^+.$$

By maximizing $\pi_{bb}^s(e;p_a,r)$, the optimal effort level is derived as:

$$e_{bb}(p_a,r)=\frac{m_o-(I-p_a)(1+r)-[\lambda m_o-(I-p_a)(1+r)]^+}{2a}, \tag{149}$$

and accordingly the firm's maximum expected profit is

$$\pi_{bb}^s(p_a,r)=[\lambda m_o-(I-p_a)(1+r)]^+ + \frac{[m_o-(I-p_a)(1+r)-[\lambda m_o-(I-p_a)(1+r)]^+]^2}{4a}$$

By contrast, given p_a and loan interest rate r, if the firm fails to continue to the second period, then the firm's expected profit is

$$\pi_{bb}^n(p_a,r)=0.$$

Summarizing the above two cases, given p_a, each segment-i customer advance buys, and interest rate r, the firm's final expected profit is

$$\pi_{bb}(p_a,r)=\beta\pi_{bb}^s(p_a,r)+(1-\beta)\pi_{bb}^n(p_a,r)=\beta\pi_{bb}^s(p_a,r).$$

Next, we consider the bank's pricing decision on interest rate r. By lending I to the firm, if the firm succeeds to continue to the second period, then the repayment collected from the firm, defined as Γ, would be $\min\{S,(I-p_a)(1+r)\}$; if the firm fails to continue into the second period, then the repayment Γ is 0. Thus, in the repayment period, Γ approximately follows the following distribution:

$$\Gamma=\begin{cases}(I-p_a)(1+r), & \text{with probability } \beta e_{bb}.\\ \min\{\lambda m_o,(I-p_a)(1+r)\}, & \text{with probability } \beta(1-e_{bb}).\\ 0, & \text{with probability } 1-\beta.\end{cases}$$

According to the fair pricing principle, the interest rate r is uniquely determined by the following equation:

$$I-p_a=\mathbb{E}[\Gamma]=\beta e_{bb}(I-p_a)(1+r)+\beta(1-e_{bb})\min\{\lambda m_o,(I-p_a)(1+r)\}. \tag{150}$$

Depending on the relationship between λm_o and $(I-p_a)(1+r)$, we solve the problem in the following two cases:

1. If $(I-p_a)(1+r)\le\lambda m_o$, then substituting (149) into (150) leads to $r_{bb}^h=\frac{1}{\beta}-1$. To ensure $(I-p_a)(1+r)\le\lambda m_o$ holds, it should be satisfied that $p_a\ge I-\beta\lambda m_o=:p_{bb}^h$. If the firm succeeds and continues to the second period, then the optimal effort is $e_{bb}^h=\frac{(1-\lambda)m_o}{2a}$. Accordingly, the firm's expected profit is
$$\pi_{bb}^h(p_a)=\beta\left[\frac{(1-\lambda)^2m_o^2}{4a}+\lambda m_o\right]+p_a-I.$$
2. If $m_o>(I-p_a)(1+r)>\lambda m_o$, then substituting (149) into (150) leads to
$$I-p_a=\beta(I-p_a)(1+r)\frac{m_o-(I-p_a)(1+r)}{2a}+\beta\left[1-\frac{m_o-(I-p_a)(1+r)}{2a}\right]\lambda m_o,$$
which can be rewritten as
$$\frac{\beta}{2a}[(I-p_a)(1+r)]^2-\frac{\beta}{2a}(1+\lambda)m_o(I-p_a)(1+r)+I-p_a-\beta\left(1-\frac{m_o}{2a}\right)\lambda m_o=0, \tag{151}$$
which is quadratic in r. Thus, the bank will lend to the firm if and only if there exists a solution r satisfying $m_o>(I-p_a)(1+r)>\lambda m_o$ to the equation (151), which is equivalent to
$$p_a\ge I-\beta\left[\lambda m_o+\frac{(1-\lambda)^2m_o^2}{8a}\right]=:p_{bb}^l. \tag{152}$$

As long as the above inequity holds, the condition that there exists a solution r satisfying $m_o > (I - p_a)(1+r) > \lambda m_o$ to the equation (151) is satisfied. When the condition (152) is met, solving equation (151) leads to the equilibrium interest rate, which is equal to the smaller root due to the competitiveness of the bank credit market, as follows

$$r^l_{bb}(p_a) = \frac{(1+\lambda)m_o - \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{2(I-p_a)} - 1.$$

If the firm succeeds to continue, then the optimal effort is

$$e^l_{bb}(p_a) = \frac{(1-\lambda)m_o + \sqrt{(1-\lambda)^2 m_o^2 - 8a\left(\frac{I-p_a}{\beta} - \lambda m_o\right)}}{4a}.$$

Accordingly, the firm's expected profit is $\pi^l_{bb}(p_a) = \beta a\left[e^l_{bb}(p_a)\right]^2$.

Otherwise, if $p_a < p^l_{bb}$, the firm fails to obtain bank finance through advance selling. □

Proof of Lemma A.3. In this case, $k-1$ segment-i consumers with a mass of $\left(1-\frac{1}{k}\right)$ purchase in advance while one consumer with a mass of $\frac{1}{k}$ does not. Thus, the demand in the financing period is $D_f = 1 - \frac{1}{k}$. Given that the firm succeeds in loan application and continues into the second period, the firm will always set $p_r^* = 1$ since we consider the case of $k \to \infty$ and thus the one segment-i customer who waits is negligible compared with the segment-o customers in the repayment period. Accordingly, the market demand in the repayment period, D_r, is divided into the following two cases. If the firm's effort results in a success, all outer consumers as well as the waiting segment-i customer will purchase and thus the demand is $D_r = m_o + \frac{1}{k}$. However, if the effort does not result in a success, the market demand is $D_r = \lambda(m_o + \frac{1}{k})$. Let $S = D_r p_r^*$ represent the sales revenue of the firm in the repayment period, then S follows a binary distribution:

$$S = \begin{cases} m_o + \frac{1}{k}, & \text{with probability } e. \\ \lambda\left(m_o + \frac{1}{k}\right), & \text{with probability } 1-e. \end{cases}$$

Given p_a, if $p_a\left(1-\frac{1}{k}\right) \geq I$, the firm has no need to borrow from the bank. Thus, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi^s_{bw}(e; p_a) = p_a\left(1-\frac{1}{k}\right) - I + \mathbb{E}[S] - ae^2 = -ae^2 + (1-\lambda)\left(m_o + \frac{1}{k}\right)e + \lambda\left(m_o + \frac{1}{k}\right) + p_a\left(1-\frac{1}{k}\right) - I.$$

By maximizing $\pi^s_{bw}(e; p_a)$, the optimal effort level is derived as: $e^{ss}_{bw} = \frac{(1-\lambda)\left(m_o + \frac{1}{k}\right)}{2a}$, and accordingly the firm's maximum expected profit is

$$\pi^{ss}_{bw}(p_a) = \frac{(1-\lambda)^2\left(m_o + \frac{1}{k}\right)^2}{4a} + \lambda\left(m_o + \frac{1}{k}\right) + p_a\left(1-\frac{1}{k}\right) - I.$$

However, if the firm fails to continue to the second period, then the firm's expected profit is

$$\pi^{sn}_{bw}(p_a) = p_a\left(1-\frac{1}{k}\right) - I.$$

Summarizing the above two cases, given $p_a\left(1-\frac{1}{k}\right) \geq I$ and each segment-i customer advance buys, the firm's final expected profit is

$$\pi^s_{bw}(p_a) = \beta\pi^{ss}_{bw}(p_a) + (1-\beta)\pi^{sn}_{bw}(p_a) = \beta\left[\frac{(1-\lambda)^2\left(m_o + \frac{1}{k}\right)^2}{4a} + \lambda\left(m_o + \frac{1}{k}\right)\right] + p_a\left(1-\frac{1}{k}\right) - I.$$

By contrast, if $p_a\left(1-\frac{1}{k}\right)<I$, the firm has to borrow $I-p_a\left(1-\frac{1}{k}\right)$ from the bank. Given p_a and loan interest rate r, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi^s_{bw}(e;p_a,r)=\mathbb{E}\left[S-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+-ae^2.$$

We note that $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)<m_o+\frac{1}{k}$ should be satisfied, otherwise the bank loan will not be granted. Thus, the firm's profit can be rewritten as

$$\begin{aligned}\pi^s_{bw}(e;p_a,r)=&-ae^2+\left[m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)-\left[\lambda\left(m_o+\frac{1}{k}\right)-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+\right]e\\&+\left[\lambda\left(m_o+\frac{1}{k}\right)-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+.\end{aligned}$$

By maximizing $\pi^s_{bw}(e;p_a,r)$, the optimal effort level is derived as:

$$e_{bw}(p_a,r)=\frac{m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)-\left[\lambda\left(m_o+\frac{1}{k}\right)-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+}{2a}. \tag{153}$$

Next, we consider the bank's pricing decision on interest rate r. By lending $I-p_a(1-\frac{1}{k})$ to the firm, if the firm succeeds to continue to the second period, then the repayment collected from the firm, defined as Γ, would be $\min\{S,\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\}$; if the firm fails to continue into the second period, then the repayment Γ is 0. Thus, in the repayment period, Γ approximately follows the following distribution:

$$\Gamma=\begin{cases}\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r), & \text{with probability } \beta e_{bw}.\\ \min\{\lambda\left(m_o+\frac{1}{k}\right),\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\}, & \text{with probability } \beta(1-e_{bw}).\\ 0, & \text{with probability } 1-\beta.\end{cases}$$

According to the fair pricing principle, the interest rate r is uniquely determined by the following equation:

$$\begin{aligned}I-p_a\left(1-\frac{1}{k}\right)&=\mathbb{E}[\Gamma]\\&=\beta e_{bw}\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)+\beta(1-e_{bw})\min\{\lambda\left(m_o+\frac{1}{k}\right),\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\}.\end{aligned} \tag{154}$$

Depending on the relationship between $\lambda\left(m_o+\frac{1}{k}\right)$ and $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)$, we solve the problem in the following two cases:

1. If $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\le\lambda\left(m_o+\frac{1}{k}\right)$, then substituting (153) into (154) leads to $r^h_{bw}=\frac{1}{\beta}-1$. To ensure $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\le\lambda\left(m_o+\frac{1}{k}\right)$ holds, it should be satisfied that $p_a\ge\frac{I-\beta\lambda\left(m_o+\frac{1}{k}\right)}{1-\frac{1}{k}}=:p^h_{bw}$. If the firm succeeds and continues to the second period, then the optimal effort is
$$e^h_{bw}=\frac{(1-\lambda)\left(m_o+\frac{1}{k}\right)}{2a}.$$
2. If $m_o+\frac{1}{k}>\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)>\lambda\left(m_o+\frac{1}{k}\right)$, then substituting (153) into (154) leads to
$$\begin{aligned}I-p_a\left(1-\frac{1}{k}\right)=&\beta\frac{m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)}{2a}\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\\&+\beta\left[1-\frac{m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)}{2a}\right]\lambda\left(m_o+\frac{1}{k}\right)\end{aligned}$$

which can be rewritten as

$$\frac{\beta}{2a}\left[\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^2-\frac{\beta}{2a}(1+\lambda)\left(m_o+\frac{1}{k}\right)\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)$$
$$+I-p_a\left(1-\frac{1}{k}\right)-\beta\left(1-\frac{m_o+\frac{1}{k}}{2a}\right)\lambda\left(m_o+\frac{1}{k}\right)=0 \tag{155}$$

which is quadratic in r. Thus, the bank will lend to the firm if and only if there exists a solution r satisfying $m_o+\frac{1}{k}>\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)>\lambda\left(m_o+\frac{1}{k}\right)$ to the equation (155), which is equivalent to

$$p_a\geq\frac{I-\beta\left[\lambda\left(m_o+\frac{1}{k}\right)+\frac{(1-\lambda)^2\left(m_o+\frac{1}{k}\right)^2}{8a}\right]}{1-\frac{1}{k}}=:p_{bw}^l. \tag{156}$$

As long as the above inequity holds, the condition that there exists a solution r satisfying $m_o+\frac{1}{k}>\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)>\lambda\left(m_o+\frac{1}{k}\right)$ to the equation (155) is satisfied. When the condition (156) is met, solving equation (155) leads to the equilibrium interest rate, which is equal to the smaller root due to the competitiveness of the bank credit market, as follows:

$$r_{bw}^l(p_a)=\frac{(1+\lambda)\left(m_o+\frac{1}{k}\right)-\sqrt{(1-\lambda)^2\left(m_o+\frac{1}{k}\right)^2-8a\left[\frac{I-p_a\left(1-\frac{1}{k}\right)}{\beta}-\lambda\left(m_o+\frac{1}{k}\right)\right]}}{2\left[I-p_a\left(1-\frac{1}{k}\right)\right]}-1.$$

If the firm succeeds, then the optimal effort is:

$$e_{bw}^l(p_a)=\frac{(1-\lambda)\left(m_o+\frac{1}{k}\right)+\sqrt{(1-\lambda)^2\left(m_o+\frac{1}{k}\right)^2-8a\left[\frac{I-p_a\left(1-\frac{1}{k}\right)}{\beta}-\lambda\left(m_o+\frac{1}{k}\right)\right]}}{4a}.$$

Otherwise, if $p_a<p_{bw}^l$, the firm fails to obtain bank finance via advance selling. □

Proof of Lemma A.4. We observe from the definitions of p_{bb}^h and p_{bw}^h that $p_{bb}^h<p_{bw}^h$ for $I>I_l$. Moreover, we can show that $p_{bw}^l<p_{bw}^h$. In order to establish the relationship between p_{bw}^l, which is dependent of k, and p_{bb}^h/p_{bb}^l, we solve the first-order derivative of p_{bw}^l with respect to k as follows:

$$\frac{dp_{bw}^l}{dk}=-\frac{1}{k^2}\left[\frac{I-\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]}{(1-\frac{1}{k})^2}+\frac{\beta(1-\lambda)^2}{8a}\right]. \tag{157}$$

Let $\hat{I}_h=:\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$ and $\hat{I}_l=:\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2m_o(2+m_o)}{8a}\right]$. Then, based on analyses of the first-order condition in (157), we have:

1. For $I\geq\hat{I}_h$, we have $\frac{dp_{bw}^l}{dk}<0$;
2. For $\hat{I}_l\leq I<\hat{I}_h$, we have $\frac{dp_{bw}^l}{dk}>0$ first and then $\frac{dp_{bw}^l}{dk}<0$;
3. For $I<\hat{I}_l$, we have $\frac{dp_{bw}^l}{dk}>0$.

The above results imply that as $k\to\infty$, $p_{bw}^l>p_{bb}^l$ for $I\geq\hat{I}_l$ and $p_{bw}^l<p_{bb}^l$ for $I<\hat{I}_l$. Moreover, it can be observed that $p_{bw}^l\to p_{bb}^l$ as $k\to\infty$. Lemma A.4 follows immediately from these results together with $p_{bb}^l<p_{bb}^h<p_{bw}^h$ for $I>I_l$. □

Proof of Lemma A.5. Since all k segment-i consumers purchase coupons in advance, the market demand in the financing period, D_f, is 1. Given that the firm succeeds in loan application and continues to the second period, the market demand in the repayment period, D_r, is comprised of two parts: demand from the segment-i customers, D_r^i, and demand from the segment-o customers, D_r^o.

Here, the magnitude of D_r^i follows two cases. If the firm's effort results in "success", all segment-i consumers will purchase and thus the demand is $D_r^i = 1$. However, if the effort results in "no success", the market demand D_r^i is random and satisfies

$$D_r^i = \frac{\sum_{j=1}^k D_r^{ij}}{k}, \tag{158}$$

where D_r^{ij} represents inner consumer i's purchasing behavior with $D_r^{ij} = 1$ denoting purchasing and $D_r^{ij} = 0$ not purchasing. Evidently, D_r^{ij} follows a Bernoulli distribution, i.e., $D_r^{ij} = 1$ with probability λ and $D_r^{ij} = 0$ with probability $1 - \lambda$. Thus, $\sum_{i=1}^k D_r^{ij}$ follows a binomial distribution: $\sum_{j=1}^k D_r^{ij} \sim B(k, \lambda)$. According to the Central Limit Theorem, the binomial distribution converges to a normal distribution, i.e.,

$$\sum_{j=1}^k D_r^{ij} \sim B(k, \lambda) \to N\left(k\lambda, k\lambda(1-\lambda)\right) \tag{159}$$

when $k \to \infty$. (158) and (159) jointly implies

$$D_r^i \to N\left(\lambda, \frac{\lambda(1-\lambda)}{k}\right) \tag{160}$$

for $k \to \infty$. From (160), the variance of D_r^i is $\frac{\lambda(1-\lambda)}{k}$, which approaches to zero when $k \to \infty$. Accordingly, D_r^i converges to its mean value λ. Based on the above analysis on demand D_r^i in the cases of success and no success, respectively, it can be concluded that when $k \to \infty$, D_r^i follows a binary distribution:

$$D_r^i = \begin{cases} 1, & \text{with probability } e. \\ \lambda, & \text{with probability } 1-e. \end{cases}$$

For D_r^o, it relies on both p_r and the effort outcome. Evidently, the firm will always set $p_r^* = 1$ since the segment-i and segment-o customers are priced separately in the repayment period. Thus, accordingly, D_r^o is divided into the following two cases. If the firm's effort results in "success", all outer consumers will purchase and thus the demand is $D_r^o = m_o$. However, if the effort results in "no success", the market demand is $D_r^o = \lambda m_o$. That is,

$$D_r^o = \begin{cases} m_o, & \text{with probability } e. \\ \lambda m_o, & \text{with probability } 1-e. \end{cases}$$

Let $S = D_r^i p_s + D_r^o p_r^*$ represent the sales revenue of the firm in the repayment period, then S follows a binary distribution:

$$S = \begin{cases} p_s + m_o, & \text{with probability } e. \\ \lambda(p_s + m_o), & \text{with probability } 1-e. \end{cases}$$

Given p_a and p_s, if $p_a \geq I$, the firm has no need to borrow from the bank. Thus, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi_{bb}^s(e; p_a, p_s) = p_a - I + \mathbb{E}[S] - ae^2 = -ae^2 + (1-\lambda)(p_s + m_o)e + p_a + \lambda(p_s + m_o) - I.$$

By maximizing $\pi^s_{bb}(e;p_a,p_s)$, the optimal effort level is derived as: $e^{ss}_{bb}(p_s)=\frac{(1-\lambda)(p_s+m_o)}{2a}$, and accordingly the firm's maximum expected profit is

$$\pi^{ss}_{bb}(p_a,p_s)=\frac{(1-\lambda)^2(p_s+m_o)^2}{4a}+\lambda(p_s+m_o)+p_a-I.$$

However, if the firm fails to continue to the second period, then the firm's expected profit is

$$\pi^{sn}_{bb}(p_a)=p_a-I.$$

Summarizing the above two cases, given $p_a\geq I$ and each segment-i customer advance buys, the firm's final expected profit is

$$\pi^s_{bb}(p_a,p_s)=\beta\pi^{ss}_{bb}(p_a,p_s)+(1-\beta)\pi^{sn}_{bb}(p_a)=\beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a}+\lambda(p_s+m_o)\right]+p_a-I.$$

By contrast, if $p_a<I$, the firm has to borrow $I-p_a$ from the bank. Given (p_a,p_s) and loan interest rate r, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi^s_{bb}(e;p_a,p_s,r)=\mathbb{E}[S-(I-p_a)(1+r)]^+-ae^2.$$

We note that $(I-p_a)(1+r)<p_s+m_o$ should be satisfied, otherwise the bank loan will not be granted. Thus, the firm's profit can be rewritten as

$$\begin{aligned}\pi^s_{bb}(e;p_a,p_s,r)=&-ae^2+\left[p_s+m_o-(I-p_a)(1+r)-[\lambda(p_s+m_o)-(I-p_a)(1+r)]^+\right]e\\&+[\lambda(p_s+m_o)-(I-p_a)(1+r)]^+.\end{aligned}$$

By maximizing $\pi^s_{bb}(e;p_a,p_s,r)$, the optimal effort level is derived as:

$$e_{bb}(p_a,p_s,r)=\frac{p_s+m_o-(I-p_a)(1+r)-[\lambda(p_s+m_o)-(I-p_a)(1+r)]^+}{2a},\tag{161}$$

and accordingly the firm's maximum expected profit is

$$\pi^s_{bb}(p_a,p_s,r)=[\lambda(p_s+m_o)-(I-p_a)(1+r)]^++\frac{\left[p_s+m_o-(I-p_a)(1+r)-[\lambda(p_s+m_o)-(I-p_a)(1+r)]^+\right]^2}{4a}.$$

By contrast, given (p_a,p_s) and loan interest rate r, if the firm fails to continue to the second period, then the firm's expected profit is

$$\pi^n_{bb}(p_a,p_s,r)=0.$$

Summarizing the above two cases, given (p_a,p_s), each segment-i customer advance buys, and interest rate r, the firm's final expected profit is

$$\pi_{bb}(p_a,p_s,r)=\beta\pi^s_{bb}(p_a,p_s,r)+(1-\beta)\pi^n_{bb}(p_a,p_s,r)=\beta\pi^s_{bb}(p_a,p_s,r).$$

Next, we consider the bank's pricing decision on interest rate r. By lending I to the firm, if the firm succeeds to continue to the second period, then the repayment collected from the firm, defined as Γ, would be $\min\{S,(I-p_a)(1+r)\}$; if the firm fails to continue into the second period, then the repayment Γ is 0. Thus, in the repayment period, Γ approximately follows the following distribution:

$$\Gamma=\begin{cases}(I-p_a)(1+r), & \text{with probability } \beta e_{bb}.\\ \min\{\lambda(p_s+m_o),(I-p_a)(1+r)\}, & \text{with probability } \beta(1-e_{bb}).\\ 0, & \text{with probability } 1-\beta.\end{cases}$$

According to the fair pricing principle, the interest rate r is uniquely determined by the following equation:

$$I-p_a=\mathbb{E}[\Gamma]=\beta e_{bb}(I-p_a)(1+r)+\beta(1-e_{bb})\min\{\lambda(p_s+m_o),(I-p_a)(1+r)\}. \tag{162}$$

Depending on the relationship between $\lambda(p_s+m_o)$ and $(I-p_a)(1+r)$, we solve the problem in the following two cases:

1. If $(I-p_a)(1+r)\leq\lambda(p_s+m_o)$, then substituting (161) into (162) leads to $r^h_{bb}=\frac{1}{\beta}-1$. To ensure $(I-p_a)(1+r)\leq\lambda(p_s+m_o)$ holds, it should be satisfied that $p_a\geq I-\beta\lambda(p_s+m_o)=:p^h_{bb}(p_s)$. If the firm succeeds and continues to the second period, then the optimal effort is $e^h_{bb}(p_s)=\frac{(1-\lambda)(p_s+m_o)}{2a}$. Accordingly, the firm's expected profit is

$$\pi^h_{bb}(p_a,p_s)=\beta\left[\frac{(1-\lambda)^2(p_s+m_o)^2}{4a}+\lambda(p_s+m_o)\right]+p_a-I.$$

2. If $p_s+m_o>(I-p_a)(1+r)>\lambda(p_s+m_o)$, then substituting (161) into (162) leads to

$$I-p_a=\beta(I-p_a)(1+r)\frac{p_s+m_o-(I-p_a)(1+r)}{2a}+\beta\left[1-\frac{p_s+m_o-(I-p_a)(1+r)}{2a}\right]\lambda(p_s+m_o)$$

which can be rewritten as

$$\frac{\beta}{2a}[(I-p_a)(1+r)]^2-\frac{\beta}{2a}(1+\lambda)(p_s+m_o)(I-p_a)(1+r)+I-p_a-\beta\left(1-\frac{p_s+m_o}{2a}\right)\lambda(p_s+m_o)=0 \tag{163}$$

which is quadratic in r. Thus, the bank will lend to the firm if and only if there exists a solution r satisfying $p_s+m_o>(I-p_a)(1+r)>\lambda(p_s+m_o)$ to the equation (163), which is equivalent to

$$p_a\geq I-\beta\left[\lambda(p_s+m_o)+\frac{(1-\lambda)^2(p_s+m_o)^2}{8a}\right]=:p^l_{bb}(p_s). \tag{164}$$

As long as the above inequity holds, the condition that there exists a solution r satisfying $p_s+m_o>(I-p_a)(1+r)>\lambda(p_s+m_o)$ to the equation (163) is satisfied. When the condition (164) is met, solving equation (163) leads to the equilibrium interest rate, which is equal to the smaller root due to the competitiveness of the bank credit market, as follows

$$r^l_{bb}(p_a,p_s)=\frac{(1+\lambda)(p_s+m_o)-\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}{2(I-p_a)}-1.$$

If the firm succeeds, then the optimal effort is:

$$e^l_{bb}(p_a,p_s)=\frac{(1-\lambda)(p_s+m_o)+\sqrt{(1-\lambda)^2(p_s+m_o)^2-8a\left[\frac{I-p_a}{\beta}-\lambda(p_s+m_o)\right]}}{4a}.$$

Accordingly, the firm's expected profit is $\pi^l_{bb}(p_a,p_s)=\beta a\left[e^l_{bb}(p_a,p_s)\right]^2$. Otherwise, if $p_a<p^l_{bb}(p_s)$, the firm fails to obtain bank finance via advance selling. □

Proof of Lemma A.6. In this case, $k-1$ segment-i consumers with a mass of $\left(1-\frac{1}{k}\right)$ purchase in advance while one consumer with a mass of $\frac{1}{k}$ does not. Thus, the demand in the financing period is $D_f = 1-\frac{1}{k}$. Given that the firm succeeds in loan application and continues into the second period, the market demand in the repayment period, D_r, is comprised of three parts: demand from the segment-i customers who advance buy, D_r^{ib}, demand from the segment-i customer who waits, D_r^{iw}, and demand from the segment-o customers, D_r^o. The firm will always set $p_r^* = 1$ since we consider the case of $k \to \infty$ and thus the one segment-i customer who waits is negligible compared with the segment-o customers in the repayment period. Accordingly, with similar analysis to that in the proof of Lemma A.5, we derive that D_r^{ib}, D_r^{iw}, and D_r^o follow the following distributions:

$$D_r^{ib} = \begin{cases} 1-\frac{1}{k}, & \text{with probability } e, \text{ if } z=s. \\ \lambda\left(1-\frac{1}{k}\right), & \text{with probability } 1-e, \text{ if } z=n. \end{cases}$$

$$D_r^{iw} = \begin{cases} \frac{1}{k}, & \text{with probability } e, \text{ if } z=s. \\ \frac{\lambda}{k}, & \text{with probability } 1-e, \text{ if } z=n. \end{cases}$$

$$D_r^{o} = \begin{cases} m_o, & \text{with probability } e, \text{ if } z=s. \\ \lambda m_o, & \text{with probability } 1-e, \text{ if } z=n. \end{cases}$$

Let $S = D_r^{ib} p_s + D_r^{iw} p_r^* + D_r^o p_r^*$ represent the sales revenue of the firm in the repayment period, then S follows a binary distribution:

$$S = \begin{cases} p_s\left(1-\frac{1}{k}\right) + m_o + \frac{1}{k}, & \text{with probability } e. \\ \lambda p_s\left(1-\frac{1}{k}\right) + \lambda\left(m_o + \frac{1}{k}\right), & \text{with probability } 1-e. \end{cases}$$

Given (p_a, p_s), if $p_a\left(1-\frac{1}{k}\right) \geq I$, the firm has no need to borrow from the bank. Thus, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\begin{aligned} \pi_{bw}^s(e; p_a, p_s) =& p_a\left(1-\frac{1}{k}\right) - I + \mathbb{E}[S] - ae^2 \\ =& -ae^2 + (1-\lambda)\left[p_s\left(1-\frac{1}{k}\right) + m_o + \frac{1}{k}\right]e + \lambda\left[p_s\left(1-\frac{1}{k}\right) + m_o + \frac{1}{k}\right] + p_a\left(1-\frac{1}{k}\right) - I. \end{aligned}$$

By maximizing $\pi_{bw}^s(e; p_a, p_s)$, the optimal effort level is derived as:

$$e_{bw}^{ss}(p_s) = \frac{(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right) + m_o + \frac{1}{k}\right]}{2a},$$

and accordingly the firm's maximum expected profit is

$$\pi_{bw}^{ss}(p_a, p_s) = \frac{\left[(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right) + m_o + \frac{1}{k}\right]\right]^2}{4a} + \lambda\left[p_s\left(1-\frac{1}{k}\right) + m_o + \frac{1}{k}\right] + p_a\left(1-\frac{1}{k}\right) - I.$$

However, if the firm fails to continue to the second period, then the firm's expected profit is

$$\pi_{bw}^{sn}(p_a) = p_a\left(1-\frac{1}{k}\right) - I.$$

Summarizing the above two cases, given $p_a\left(1-\frac{1}{k}\right) \geq I$ and each segment-i customer advance buys, the firm's final expected profit is

$$\pi_{bw}^s(p_a, p_s) = \beta \pi_{bw}^{ss}(p_a, p_s) + (1-\beta)\pi_{bw}^{sn}(p_a)$$

$$=\beta\left[\frac{\left[(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]\right]^2}{4a}+\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]\right]+p_a\left(1-\frac{1}{k}\right)-I.$$

By contrast, if $p_a\left(1-\frac{1}{k}\right)<I$, the firm has to borrow $I-p_a\left(1-\frac{1}{k}\right)$ from the bank. Given p_a and loan interest rate r, if the firm succeeds to continue to the second period, her expected profit by exerting effort e is:

$$\pi^s_{bw}(e;p_a,p_s,r)=\mathbb{E}\left[S-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+-ae^2.$$

We note that $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)<p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}$ should be satisfied, otherwise the bank loan will not be granted. Thus, the firm's profit can be rewritten as

$$\begin{aligned}\pi^s_{bw}(e;p_a,p_s,r)=&\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}-[I-p_a\left(1-\frac{1}{k}\right)](1+r)-\left[\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+\right]e\\&-ae^2+\left[\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+.\end{aligned}$$

By maximizing $\pi^s_{bw}(e;p_a,p_s,r)$, the optimal effort level is derived as:

$$e_{bw}(p_a,p_s,r)=\frac{p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)-\left[\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^+}{2a}. \tag{165}$$

Next, we consider the bank's pricing decision on interest rate r. By lending $I-p_a\left(1-\frac{1}{k}\right)$ to the firm, if the firm succeeds to continue to the second period, then the repayment collected from the firm, defined as Γ, would be $\min\{S,\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\}$; if the firm fails to continue into the second period, then the repayment Γ is 0. Thus, in the repayment period, Γ approximately follows the following distribution:

$$\Gamma=\begin{cases}\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r), & \text{with probability } \beta e_{bw}.\\ \min\{\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right],\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\}, & \text{with probability } \beta(1-e_{bw}).\\ 0, & \text{with probability } 1-\beta.\end{cases}$$

According to the fair pricing principle, the interest rate r is uniquely determined by the following equation:

$$\begin{aligned}I-p_a\left(1-\frac{1}{k}\right)&=\mathbb{E}[\Gamma]\\&=\beta e_{bw}\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)+\beta(1-e_{bw})\min\{\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right],\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\}.\end{aligned} \tag{166}$$

Depending on the relationship between $\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]$ and $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)$, we solve the problem in the following two cases:

1. If $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\le\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]$, then substituting (165) into (166) leads to $r^h_{bw}=\frac{1}{\beta}-1$. To ensure $\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\le\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]$ holds, it should be satisfied that $p_a\ge\frac{I-\beta\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]}{1-\frac{1}{k}}=:p^h_{bw}(p_s)$. If the firm succeeds and continues to the second period, then the optimal effort is

$$e^s_{bw}(p_s)=\frac{(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]}{2a}.$$

2. If $p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}>\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)>\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]$, then substituting (165) into (166) leads to

$$I-p_a\left(1-\frac{1}{k}\right)=\beta\frac{p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)}{2a}\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)$$

$$+\beta\left[1-\frac{p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}-\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)}{2a}\right]\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right],$$

which can be rewritten as

$$\frac{\beta}{2a}\left[\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)\right]^2-\frac{\beta}{2a}(1+\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)$$
$$+I-p_a\left(1-\frac{1}{k}\right)-\beta\left[1-\frac{p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}}{2a}\right]\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]=0, \tag{167}$$

which is quadratic in r. Thus, the bank will lend to the firm if and only if there exists a solution r satisfying $p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}>\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)>\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]$ to the equation (167), which is equivalent to

$$p_a \geq \frac{I-\beta\left[\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]+\frac{(1-\lambda)^2\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]^2}{8a}\right]}{1-\frac{1}{k}}=: p^l_{bw}(p_s). \tag{168}$$

As long as the above inequity holds, the condition that there exists a solution r satisfying

$$p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}>\left[I-p_a\left(1-\frac{1}{k}\right)\right](1+r)>\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]$$

to the equation (167) is satisfied. When the condition (168) is met, solving equation (167) leads to the equilibrium interest rate, which is equal to the smaller root due to the competitiveness of the bank credit market, as follows:

$$r^l_{bw}(p_a,p_s)=\frac{(1+\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]-\sqrt{(1-\lambda)^2\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]^2-8a\left[\frac{I-p_a\left(1-\frac{1}{k}\right)}{\beta}-\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]\right]}}{2\left[I-p_a\left(1-\frac{1}{k}\right)\right]}-1.$$

If the firm succeeds, then the optimal effort is:

$$e^l_{bw}(p_a,p_s)=\frac{(1-\lambda)\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]+\sqrt{(1-\lambda)^2\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]^2-8a\left[\frac{I-p_a\left(1-\frac{1}{k}\right)}{\beta}-\lambda\left[p_s\left(1-\frac{1}{k}\right)+m_o+\frac{1}{k}\right]\right]}}{4a}.$$

Otherwise, if $p_a<p^l_{bw}(p_s)$, the firm fails to obtain bank financing through advance selling. $\square$

Proof of Lemma A.7. From the definitions in Lemma A.5 and Lemma A.6, we have $p^l_{bb}(p_s)<p^h_{bb}(p_s)<p^h_{bw}(p_s)$ for $p_s\in(0,1]$ when $I>I_l$. Moreover, we have $p^l_{bw}(p_s)<p^h_{bw}(p_s)$. To establish the relationship between $p^l_{bw}(p_s)$, which is dependent on k, and $p^h_{bb}(p_s)/p^l_{bb}(p_s)$, we investigate the monotonic property of $p^l_{bw}(p_s)$ with respect to k. The first-order derivative of $p^l_{bw}(k)$ with respect to k is

$$\frac{dp^l_{bw}(p_s)}{dk}=-\frac{1}{k^2}\left[\frac{I-\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]}{\left(1-\frac{1}{k}\right)^2}+\frac{\beta(1-\lambda)^2(1-p_s)^2}{8a}\right]. \tag{169}$$

Let $\hat{I}_h:=\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2(1+m_o)^2}{8a}\right]$ and $\hat{I}_l:=\beta\left[\lambda(1+m_o)+\frac{(1-\lambda)^2m_o(2+m_o)}{8a}\right]$. Analyzing the properties of the first-order derivative function in (169) lead to the following results:

1. When $I>\hat{I}_h$, we have $\frac{dp^l_{bw}(p_s)}{dk}<0$. Moreover, $p^l_{bw}(p_s)\to p^l_{bb}(p_s)$ as $k\to\infty$. Together with $p^l_{bb}(p_s)<p^h_{bb}(p_s)<p^h_{bw}(p_s)$, this implies that $p^l_{bb}(p_s)<p^l_{bw}(p_s)<p^h_{bb}(p_s)<p^h_{bw}(p_s)$ as $k\to\infty$.
2. When $\hat{I}_l\leq I\leq\hat{I}_h$, let $I_t(p_s)=:\hat{I}_h-\frac{\beta(1-\lambda)^2(1-p_s)^2}{8a}$ and evidently there exists a unique $\ddot{p}_s$ such that $I_t(\ddot{p}_s)=I$. Depending on p_s, we have the following two cases:

(a) If $p_s \in (0, \ddot{p}_s)$, or equivalently $I_t(p_s) < I \le \hat{I}_h$, then $\frac{dp^l_{bw}(p_s)}{dk} > 0$ first and then $\frac{dp^l_{bw}(p_s)}{dk} < 0$. Moreover, $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ as $k \to \infty$. Thus, $p^l_{bb}(p_s) < p^l_{bw}(p_s)$ as $k \to \infty$. Therefore, $p^l_{bb}(p_s) < p^l_{bw}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ as $k \to \infty$.

(b) If $p_s \in [\ddot{p}_s, 1]$, or equivalently $\hat{I}_l < I \le I_t(p_s)$, then $\frac{dp^l_{bw}(p_s)}{dk} > 0$. Moreover, $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ as $k \to \infty$. Thus, $p^l_{bw}(p_s) < p^l_{bb}(p_s)$ as $k \to \infty$. Therefore, $p^l_{bw}(p_s) < p^l_{bb}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ as $k \to \infty$.

3. When $I < \hat{I}_l$, we have $\frac{dp^l_{bw}(p_s)}{dk} > 0$. Moreover, $p^l_{bw}(p_s) \to p^l_{bb}(p_s)$ as $k \to \infty$. This together with $p^l_{bb}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ implies that $p^l_{bw}(p_s) < p^l_{bb}(p_s) < p^h_{bb}(p_s) < p^h_{bw}(p_s)$ as $k \to \infty$.

Lemma A.7 follows immediately from the above results. □