

LBS Research Online

A Atmaz and [S Basak](#)

Option prices and costly short-selling

Article

This version is available in the LBS Research Online repository: <http://lbsresearch.london.edu/id/eprint/999/>

Atmaz, A and [Basak, S](#)

(2019)

Option prices and costly short-selling.

Journal of Financial Economics, 134 (1). pp. 1-28. ISSN 0304-405X

DOI: <https://doi.org/10.1016/j.jfineco.2019.04.004>

Elsevier

<https://www.sciencedirect.com/science/article/pii/...>

Users may download and/or print one copy of any article(s) in LBS Research Online for purposes of research and/or private study. Further distribution of the material, or use for any commercial gain, is not permitted.

Option prices and costly short-selling[☆]

Adem Atmaz^a, Suleyman Basak^{b,*}

^a*Krannert School of Management, Purdue University, West Lafayette, IN, USA*

^b*London Business School, London, UK*

Abstract

Much empirical evidence shows that stock short-selling costs and bans have significant effects on option prices. We reconcile these findings by providing a dynamic analysis of option prices with costly short-selling and option marketmakers. We obtain simple, closed-form, unique option bid and ask prices that represent option marketmakers' expected hedging costs, and are weighted-averages of well-known benchmark prices (Black-Scholes, Heston). Our analysis delivers rich implications that support the empirical evidence on the effects of short-selling costs and bans on option prices, as well as uncovering several novel predictions. We also apply our methodology to corporate bonds, which have option-like payoffs.

Keywords: Option prices, options marketmaking, shorting fee, partial lending, short-selling bans.

JEL classification: G12, G13, G23.

1. Introduction

Short-selling activity has much grown over the last several decades and now accounts for a significant fraction of trades.¹ A pervasive imperfection in selling a stock short is that

[☆]We thank our discussants Mike Gallmeyer, Tim Johnson, as well as the seminar participants at the 2017 Asian and North American Summer Meetings of the Econometric Society, 2017 Wabash River Finance Conference, and 2018 SFS Finance Cavalcade. We have also benefited from the valuable suggestions of Ron Kaniel (the Co-Editor) and an anonymous referee. All errors are our responsibility.

*Corresponding author

Email address: sbasak@london.edu (Suleyman Basak)

¹For instance, Hanson and Sunderam (2014) show that the average short interest ratios for NYSE and AMEX stocks have more than quadrupled from 1988 to 2011, and Diether, Lee, and Werner (2009) report

it is costly (discussed below), and growing empirical evidence shows that these costs have significant effects on option prices. The evidence includes option bid-ask spreads and put option implied volatilities being increasing in the short-selling costs (Evans, Geczy, Musto, and Reed, 2007; Lin and Lu, 2016), and apparent put-call parity violations being increasing in the short-selling costs (Lamont and Thaler, 2003; Ofek, Richardson, and Whitelaw, 2004; Evans, Geczy, Musto, and Reed, 2007). Similar evidence during the 2008 short-selling ban period shows that option bid-ask spreads and apparent put-call parity violations of banned stocks were higher than those of unbanned stocks (Battalio and Schultz, 2011; Grundy, Lim, and Verwijmeren, 2012; Lin and Lu, 2016), and marketmakers asymmetrically adjusted the banned stock options by decreasing their call bid prices but increasing their put ask prices (Battalio and Schultz, 2011). On the theory side, however, there is no existing work to reconcile these findings, nor any work, in that regard, that provides option prices incorporating short-selling costs in a straightforward manner.

In this paper, we provide a dynamic analysis of option prices in the presence of costly short-selling and option marketmakers. We obtain simple, tractable, closed-form option prices, which represent marketmakers' expected cost of hedging and are weighted-averages of well-known benchmark option prices. The model emerges rich in implications that support all the empirical evidence stated above, as well as uncovering a number of new predictions. We adopt the classic Black and Scholes (1973) option pricing framework for our baseline analysis and incorporate costly short-selling in the underlying stock following standard short-selling and stock lending market practices. Short-sellers incur a *shorting fee* to borrow shares from lenders, who lend only a part of their long position in the stock. This *partial lending* is a key feature of our analysis and follows from the fact that most stocks in reality have excess supply of lendable shares (D'Avolio, 2002; Saffi and Sigurdsson, 2010). An investor then effectively pays a different rate for short-selling a share as compared to the rate she earns from holding a share long. We then demonstrate that this cost and benefit asymmetry makes the standard no-arbitrage restrictions insufficient to determine option prices, consistently with earlier works (see below), and hence creates an economic role for option marketmakers in determining prices.

that roughly 30% of the trading volume in NYSE and NASDAQ was due to short-selling in 2005. Relatedly, Saffi and Sigurdsson (2010) report that the amount of global supply of lendable shares in December 2008 was \$15 trillion (about 20% of the total market capitalization) and \$3 trillion of this amount was lent out to short-sellers.

Accordingly, we impose more structure and introduce option marketmakers into our dynamic framework following the related literature and the marketmaking in exchange-traded option markets. Marketmakers are competitive and continuously quote bid and ask option prices that result in zero expected profit from each possible sell or buy order. To hedge the risk in each order, marketmakers form a delta-hedge portfolio, which perfectly hedges the option and is held either until the option maturity or liquidated prior to that when a subsequent offsetting order arrives. Hence, each option is hedged at its maturity in one of two ways, either via a hedge portfolio or via a subsequent offsetting order. We first obtain an intuitive representation for option prices with bid and ask prices being the marketmakers' expected cost of hedging sell and buy orders, respectively. This is a notable generalization of the Black-Scholes option prices, which are equal to the cost of their hedge portfolios, the only way to hedge them.

We then obtain unique, closed-form, option bid and ask prices that are simple weighted-averages of Black-Scholes prices, and hence preserve their well-known properties. These prices all depend on the partial lending. This is somewhat surprising as it implies that a friction associated with stock lending also affects those options whose hedge portfolios require short-selling the stock at all times. In our model this occurs due to the possibility of hedging via a subsequent offsetting order. We also show that marketmakers quote higher bid and lower ask option prices than the respective hedge portfolio proceeds and costs. This implies that investors have incentives to trade with marketmakers rather than to replicate the options themselves within our model. This is in contrast to the benchmark models in which options do not offer any cost advantages over and above their replicating alternatives. In our model, competitive marketmakers are able to offer these more favorable prices to investors because it is less costly for them to hedge their trades through offsetting orders as compared to hedge portfolios.

Looking more closely at the behavior of our option prices, we find that call and put bid-ask spreads are increasing in the shorting fee for typical options, consistent with empirical evidence (Evans, Geczy, Musto, and Reed, 2007; Lin and Lu, 2016). This is because an increase in the shorting fee not only increases short-selling costs but also partially increases the benefit of holding a share long, leading to higher bid-ask spreads for typical options. We further show that put bid and ask prices, and hence the put option implied volatilities, are increasing in the shorting fee since a higher shorting fee increases both the marketmakers'

cost of and the proceeds from the hedge portfolios as a put seller and buyer, respectively, in line with the empirical evidence (Evans, Geczy, Musto, and Reed, 2007; Lin and Lu, 2016). We also show that implied stock prices decrease in the shorting fee, and hence deviate more from the underlying stock prices which then lead to higher apparent put-call parity violations, as also empirically documented (Lamont and Thaler, 2003; Ofek, Richardson, and Whitelaw, 2004; Evans, Geczy, Musto, and Reed, 2007). This is because the implied stock is equivalent to the stock but without the short-selling costs and lending benefits.

We also provide several new testable predictions. In particular, we show that call and put bid-ask spreads are decreasing in the partial lending. This opposite effect from the shorting fee implication arises because an increase in the partial lending only increases the benefit of holding a share long but has no effect on hedge portfolios that require short-selling the stock. We also show that the option marketmakers' participation in the stock lending market is decreasing in the shorting fee for each call option sold, due to the increased lending benefits. Finally, we demonstrate that the effects of short-selling costs on option bid-ask spreads are more pronounced for relatively illiquid options with lower trading activity. This occurs because marketmakers are more likely to hedge the relatively illiquid options via hedge portfolios, through which short-selling costs affect option prices directly.

We then apply our model to the widely-studied 2008 US short-selling ban period, during which option marketmakers were still allowed to short-sell. The evidence indicates that the short-selling ban reduced (roughly halved) the short-selling activity while increasing (roughly doubling) the shorting fee of banned stocks (Boehmer, Jones, and Zhang, 2013; Harris, Namvar, and Phillips, 2013; Kolasinski, Reed, and Thornock, 2013). By adjusting our model accordingly, we show that the option bid-ask spreads of banned stocks and their apparent put-call parity violations are higher than those of unbanned stocks, consistent with empirical evidence (Battalio and Schultz, 2011; Grundy, Lim, and Verwijmeren, 2012; Lin and Lu, 2016), and the option bid and ask prices of the banned stocks are affected asymmetrically by the ban, also consistent with evidence (Battalio and Schultz, 2011).² These results arise because the short-selling ban only affects the marketmakers' hedge portfolios that are short,

²The similarity of the empirically documented effects of short-selling costs and bans on option prices suggests that the economic mechanisms driving the effects of costly short-selling were also present during the short-selling ban. Therefore, by developing a model that is also valid during periods of short-selling bans, we reconcile all these empirical findings with similar mechanisms in our model.

but does not affect the ones that are long in the stock as they earn (roughly) the same rate per share. We also quantify our baseline model and demonstrate that the effects of short-selling costs on option prices are economically significant for expensive-to-short stocks. We also apply our model to and shed light on the well-publicized event of extreme short-selling in the Palm stock in 2000, during which there were apparent violations of the law of one price.

We then extend our baseline one-factor economy to a two-factor stochastic volatility economy by adopting the Heston (1993) framework. In this setting, option marketmakers attempt to hedge the risk in each order by again forming a delta-hedge portfolio, but now this portfolio no longer perfectly hedges the option and leads to ex-post hedging errors. However, as before, we obtain explicit closed-form solutions for option bid and ask prices, which are now weighted-averages of Heston option prices, and show that all our main results and underlying economic mechanisms continue to go through in this more elaborate setting. This is because the delta-hedged portfolio's associated cumulative hedging error, which can be positive or negative ex-post, has a current value of zero, and hence option prices preserve their baseline setting structure. However, given the more elaborate stochastic volatility setting, option prices now have richer behavior. In particular, via a simple numerical exercise, we illustrate that under stochastic volatility, the effects of costly short-selling on option prices are more pronounced and a higher (lower) negative skewness in the underlying stock return leads to greater effects for call (put) options. For instance, we show that call bid-ask spreads of expensive-to-short stocks are greater under higher negative skewness.

Finally, as an another application of our methodology, we take the classic Merton (1974) setting and study the effects of costly stock short-selling on corporate bonds, which have option-like payoffs for which the firm value is treated as the underlying security. We first show that the presence of the stock shorting fee leads to an implied shorting fee in the firm value due to the (perfect) correlation between the firm value and the stock price. We then obtain explicit closed-form solutions for the corporate bond bid and ask prices that are comparable to those in our baseline setting for options, but are now weighted-averages of Merton corporate bond prices. We also demonstrate that higher stock short-selling costs lead to lower corporate bond prices (and hence to higher yields), consistent with evidence (Kecskes, Mansi, and Zhang, 2013), as well as how the presence of stock short-selling costs can generate bid-ask spreads for corporate bonds.

Related works that study the effects of shorting fees (among other frictions) on option prices are Lou (2015) and Jensen and Pedersen (2016). Lou primarily focuses on funding costs that include a shorting fee. On the other hand, Jensen and Pedersen overturns the classic result of Merton (1973) by showing that in the presence of shorting fees, as well as margin and funding costs, it may in fact be optimal to exercise an American call option early. Differently from these works, we study the effects of the shorting fee jointly with partial lending, a key feature of our analysis. Moreover, we consider a marketmaking facility facing stochastic arrivals of orders to obtain bid and ask prices for options and relate their behavior to the empirical evidence on short-selling costs.

Our paper also contributes to the theoretical literature on option market microstructure by demonstrating how costly short-selling can induce option bid-ask spreads. Security bid-ask spreads are traditionally explained by inventory considerations (Stoll, 1978; Amihud and Mendelson, 1980; Ho and Stoll, 1981), or asymmetric information (Glosten and Milgrom, 1985; Easley and O’Hara, 1987) with some featuring option marketmakers (Biais and Hillion, 1994; Easley, O’Hara, and Srinivas, 1998). Our paper contributes to this literature by providing determinants of option bid-ask spreads in a dynamic setting in which option marketmakers are not subject to inventory or asymmetric information risk.

Our work is also related to the large theoretical literature that investigates the effects of various market imperfections on option prices. These include looking at the effects of taxes (Scholes, 1976), transaction costs (Leland, 1985; Boyle and Vorst, 1992; Edirisinghe, Naik, and Uppal, 1993), trading constraints including short-selling restrictions (Broadie, Cvitanić, and Soner, 1998), different interest rates for borrowing and lending (Bergman, 1995), buy-in risk (Avellaneda and Lipkin, 2009), funding, collateral and margin requirements (Piterbarg, 2010; Leippold and Su, 2015). Effective market incompleteness implied by these imperfections in general leads to no-arbitrage ranges rather than unique option prices. This is typically addressed, if at all, by introducing a utility maximization problem which often times leads to complex option prices that depend on investor preferences. In contrast, the markets are complete in our baseline analysis in the sense that it is still possible to perfectly hedge the option payoffs by trading in the underlying stock and the bond. Hence, standard no-arbitrage restrictions along with a marketmaking function suffice to obtain unique closed-form preference-free option bid and ask prices.

The remainder of the article is organized as follows. Section 2 presents our baseline

model with costly short-selling and Section 3 introduces option marketmakers and provides the unique bid and ask prices. Section 4 investigates the behavior of the option prices, and Section 5 the 2008 short-selling ban, the quantitative analysis, and the application to Palm stock 2000. Section 6 extends our one-factor baseline model to a two-factor stochastic volatility economy, and Section 7 applies our methodology to corporate bonds. Section 8 concludes. Appendix contains the proofs.

2. Economy with costly short-selling

To study the effects of costly short-selling on option prices, in this Section, we adopt the classic Black-Scholes framework as our baseline setting and incorporate costly short-selling in the underlying stock. In this setting, the securities market includes a riskless bond and a (non-dividend paying) stock whose price processes B and S follow

$$dB_t = B_t r dt, \tag{1}$$

$$dS_t = S_t [\mu dt + \sigma d\omega_t], \tag{2}$$

where r is the constant riskless interest rate, μ and σ are the constant mean return and the return volatility of the stock, respectively, and ω is a standard Brownian motion.

We incorporate short-selling costs into this economy by following standard short-selling and stock lending market practices. Short-sellers borrow shares from investors who are long in the stock. All short-selling proceeds are kept as collateral in an account that earns the riskless interest rate r . This interest income is shared between the lender and the short-seller. The lender's account earns the *shorting fee* rate $\phi > 0$, and the short-seller's account earns the rebate rate $r - \phi$.³ On the stock lending side, the key unavoidable feature is partial lending. That is, investors who are long in the stock do not necessarily lend all their shares but only a fraction $0 \leq \alpha < 1$ of them, where henceforth, we refer to α as the *partial lending* parameter. The partial lending feature follows from the fact that most stocks in reality have excess supply of lendable shares (D'Avolio, 2002; Saffi and Sigurdsson, 2010). Hence, even

³Note that the rebate rate can be negative and the rate short-sellers are effectively paying to lenders is the shorting fee ϕ as it is the foregone interest rate for them. The exact mechanics of stock short-selling are somewhat more involved but its essentials are captured by our formulation above (see Reed, 2013 for an extensive discussion of short-selling).

if investors attempt to do so, they may not be able to successfully lend all their long stock positions.⁴ In sum, an investor effectively pays a rate ϕ for short-selling a share but only earns the rate $\alpha\phi$ from holding a stock share long in our economy with costly short-selling. We demonstrate that this asymmetry between the cost of short-selling and the benefit of holding a share long plays an important role in option prices.

We consider standard European-style call and put options written on the stock with a strike price K and a maturity date T . For the call, the buyer's payoff is $\max\{S_T - K, 0\}$ and the seller's $-\max\{S_T - K, 0\}$, while for the put, the buyer's payoff is $\max\{K - S_T, 0\}$ and the seller's $-\max\{K - S_T, 0\}$ at the maturity date. An option price is said to admit arbitrage if the investors can form a self-financing portfolio to obtain a strictly positive initial profit with zero payoff at the option maturity by trading at that option price. In the classic Black-Scholes economy without costly short-selling, the no-arbitrage restriction alone is sufficient to uniquely determine the option prices, given by the cost of the hedge portfolio, a self-financing portfolio in the underlying stock and the riskless bond which perfectly hedges (offsets) the option seller's payoff at the maturity date. Lemma 1 shows that this is not the case with costly short-selling, consistently with earlier works studying other market frictions on option prices.

Lemma 1 (Ranges of option prices). *In the economy with costly short-selling, call and put prices, C_t and P_t , satisfy*

$$C_t^{BS}(\phi) \leq C_t \leq C_t^{BS}(\alpha\phi), \quad (3)$$

$$P_t^{BS}(\alpha\phi) \leq P_t \leq P_t^{BS}(\phi), \quad (4)$$

where $C_t^{BS}(q)$ and $P_t^{BS}(q)$ denote the standard Black-Scholes call and put prices adjusted

⁴The partial lending feature can also be justified due to the standard equilibrium condition for security markets, that is, since short-sellers need to sell the shares back to other long holders, not every long position can be lent to short-sellers in equilibrium. As for the size of the excess supply, Saffi and Sigurdsson (2010) report that the amount of global supply of lendable shares in December 2008 was \$15 trillion (about 20% of the total market capitalization) and only \$3 trillion of this amount was actually lent out. Saffi and Sigurdsson also report that the average fraction of outstanding shares lent out in their sample was 8.91% for the US and 5.75% for the world, which could proxy our partial lending parameter for a typical stock.

for the constant dividend yield q , respectively, and are given by

$$C_t^{BS}(q) = S_t e^{-q(T-t)} \Phi(d_1(q)) - K e^{-r(T-t)} \Phi(d_2(q)), \quad (5)$$

$$P_t^{BS}(q) = -S_t e^{-q(T-t)} \Phi(-d_1(q)) + K e^{-r(T-t)} \Phi(-d_2(q)), \quad (6)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and

$$d_1(q) = \frac{\ln(S_t/K) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad \text{and} \quad d_2(q) = d_1(q) - \sigma\sqrt{T-t}. \quad (7)$$

Lemma 1 shows that with costly short-selling, the no-arbitrage restriction alone is not sufficient to determine option prices uniquely, as they now fall within a range. The maximum prices they can attain in (3)–(4) are the costs of the hedge portfolio (the self-financing portfolio in the underlying stock and the riskless bond which perfectly hedges the option payoff at the maturity date) for option sellers. If options are traded at prices higher than these then they admit arbitrage, because selling the option and forming the hedge portfolio at a lower cost leads to a positive initial profit with no payoff at the option maturity date. The minimum prices they can attain in (3)–(4) are the proceeds from the hedge portfolio for option buyers and differ from the maximum prices. If options are traded at prices lower than these then they admit arbitrage, since buying the option and receiving a higher amount by forming the hedge portfolio leads to a positive initial profit with no payoff at the option maturity date.

With costly short-selling, these ranges for option prices arise because the cost of short-selling (ϕ) and the benefit of holding a share long ($\alpha\phi$) are not the same. Therefore, the cost of the hedge portfolio for option sellers and the proceeds from the hedge portfolio for option buyers are different, as when one hedge portfolio is long the other is short in the underlying stock. This is in contrast to the classic Black-Scholes economy without costly short-selling, in which option prices are unique due to the fact that the cost of short-selling and the benefit of holding a share long are the same and equal to zero.

Remark 1 (Additional considerations). To highlight our results as clearly as possible, we did not consider several possible issues, but they can easily be incorporated into our analysis. First, our model can be extended to a setting in which the stock pays a constant dividend yield δ , by adding it to both the shorting fee ϕ and the lending income $\alpha\phi$. For

instance, the range for the call price becomes $C_t^{BS}(\phi + \delta) \leq C_t \leq C_t^{BS}(\alpha\phi + \delta)$. Second, in our formulation 100% of the short-selling proceeds are kept as collateral. This rate is very close to the actual practice in the US for domestic stocks, as Reed (2013) reports lenders typically require 102% of the short-selling proceeds as a collateral to help protect themselves. Our model can be generalized to any constant collateral rate κ by simply multiplying the shorting fee ϕ by κ . For instance, in this case the range for the call price becomes $C_t^{BS}(\phi\kappa) \leq C_t \leq C_t^{BS}(\alpha\phi\kappa)$. Third, in our model the lender gets all of the shorting fee ϕ upon successfully lending a share. In reality, this is true for some large institutions with internal lending departments which directly lend to short-sellers. Other lenders typically use an agent bank/brokerage and get only a fraction of the shorting fee, with the rest going to the agent bank/brokerage for providing this service. Reed (2013) reports that these lenders typically get 75% of the shorting fee. Incorporating this feature into our model is straightforward by multiplying the partial lending parameter α by a constant fraction γ capturing the lender's share of shorting fee. For instance, in this case the range for the call price becomes $C_t^{BS}(\phi) \leq C_t \leq C_t^{BS}(\alpha\gamma\phi)$.

In our analysis, we only consider standard call and put options as most of the empirical evidence on the effects of costly short-selling is on these. Our analysis, however, is equally valid for other European-style derivatives whose payoffs are monotonically either non-decreasing or non-increasing in the underlying stock, such as forward contracts. Moreover, to keep our baseline analysis comparable to the Black-Scholes economy with constant parameters, we take the shorting fee ϕ and partial lending parameter α to be constants. In reality, the levels of shorting fee and partial lending are likely time varying. Introducing time-variation in these parameters may be addressed by the methodologies employed in option pricing with stochastic dividend yields (e.g., Geske, 1978; Broadie, Detemple, Ghysels, and Torr es, 2000).

3. Option bid and ask prices with marketmakers

As the previous Section illustrates, with costly short-selling, standard no-arbitrage restrictions alone cannot determine option prices. To determine prices, one would need to impose more structure on the economy. Towards that, in this Section we introduce option marketmakers and obtain unique bid and ask option prices. We show that option prices have simple forms in terms of the familiar Black-Scholes prices, and hence inherit their well-known properties.

We incorporate option marketmakers in our framework following the marketmaking for exchange-traded option markets, such as the CBOE, as well as the related literature.⁵ There are numerous competitive option marketmakers who stand ready to buy and sell options to fulfill investor orders, and hence facilitate trading at any point in time. Marketmakers continuously quote bid and ask option prices that result in zero expected profit for each possible trade.⁶ Since fulfilling investor orders may generate arbitrary (and adverse) positions, marketmakers attempt to hedge the risk in each order. They form a delta-hedge portfolio, which perfectly hedges the option and is held either until the option maturity or liquidated prior to that when a subsequent offsetting order arrives (e.g., a current call buy order’s offsetting order is a subsequent call sell order), as the latter also perfectly hedges the incoming order at its maturity. Hence, each sell or buy order is hedged at its maturity in one of two ways, either via a hedge portfolio or via a subsequent offsetting order. The first way of hedging risk is what makes options marketmaking different from that in other markets, and as discussed in Section 2 this hedging can still be achieved perfectly in our baseline economy with costly short-selling. The second way, matching offsetting orders, is the more familiar one in marketmaking, particularly for equities.⁷

We model the arrival of offsetting (buy or sell) orders in a simple way as in Bollen, Smith, and Whaley (2004), which in turn is based on the classic microstructure model of Garman (1976). At each time t , the arrivals of offsetting trades have mutually independent exponential distributions with strictly positive parameters $\lambda_{Cs}, \lambda_{Cb}, \lambda_{Ps}, \lambda_{Pb}$, representing

⁵Options traded in the over-the-counter markets would involve other issues such as search costs and bargaining (e.g., Duffie, Gârleanu, and Pedersen, 2005, 2007) that are not the main focus of our analysis.

⁶Setting prices such that each option trade yields zero expected profit follows from the competitive marketmaking assumption and is consistent with the literature on option marketmakers (Easley, O’Hara, and Srinivas, 1998; Johnson and So, 2012).

⁷The marketmakers’ delta hedging behavior is consistent with standard market practice (e.g., see Hu, 2014 for a recent work highlighting this behavior or McDonald, 2012 for a textbook treatment). As is well-recognized in the literature, option marketmakers have a greater need for hedging their positions as compared to equity marketmakers. This is because they face far greater inventory holding costs due to higher illiquidity, and implicit leverage of options result in higher and stochastic volatilities (e.g., Jameson and Wilhelm, 1992; Cho and Engle, 1999; Muravyev, 2016). They also face far greater order imbalances as compared to those in underlying stocks (Lakonishok, Lee, Pearson, and Poteshman, 2007), which can be attributed to inventory risk (Muravyev, 2016). In our model inventory risk does not play a role since marketmakers immediately form a (perfect) hedge portfolio for each order to hedge the risk of adverse fluctuations in the underlying stock (availability of second way of hedging), a feature that is also highlighted in the literature (e.g., Cho and Engle, 1999).

the arrival rates of an offsetting call sell, call buy, put sell, put buy order, respectively.⁸ These arrival rates are inherently related to liquidity as one may argue that the more liquid options, which have higher buying/selling activity, are more likely to have higher offsetting order arrival rates (or, equivalently, lower expected arrival times for offsetting orders). We explore the effects of buying/selling activity in Section 4 (Proposition 4). The respective distribution functions of offsetting orders are denoted by $F_{Cs}, F_{Cb}, F_{Ps}, F_{Pb}$, which along with the objective of marketmakers given above are sufficient to determine the option prices as follows. First, for each possible order, the marketmakers compute the current value of profits from each case depending on whether an offsetting order arrives by the option maturity or not. Then, they find the expected profits by probability weighing these values using the distribution functions $F_{Cs}, F_{Cb}, F_{Ps}, F_{Pb}$. Finally, they set the option bid and ask prices so that the expected profit is zero.

For instance, the call ask price at time t is set by the competitive marketmaker such that selling a call option at the ask price C_t^{Ask} and forming the hedge portfolio at a cost $C_t^{BS}(\alpha\phi)$ (Lemma 1) yields zero expected profit. This is obtained by the following steps. First, the marketmaker holds the initial difference $C_t^{Ask} - C_t^{BS}(\alpha\phi)$ in the riskless bond. If there is no offsetting call sell order by the option maturity date T , then the option is perfectly hedged via the hedge portfolio and the marketmaker's profit from this trade is the initial amount in the bond, which has a current value of

$$\Pi_{T, C^{Ask}} \equiv C_t^{Ask} - C_t^{BS}(\alpha\phi). \quad (8)$$

However, if an offsetting call sell order arrives at a subsequent time u such that $t < u < T$, then the marketmaker buys the call option at a bid price C_u^{Bid} , and liquidates the hedge portfolio at a value of $C_u^{BS}(\alpha\phi)$. In this case the option is perfectly hedged via the offsetting

⁸The arrivals of offsetting orders are also independent from the Brownian motion ω in the underlying stock price dynamics (2). The exponential distribution is commonly used to model the time between the occurrence of events and arises as the distribution of the interarrival times of a Poisson process. For an exponential distribution with parameter λ , the expected arrival time is given by $1/\lambda$. The exponential distribution has the useful “memoryless property”, implying that the distribution of the arrival time is independent from the waiting time that has already occurred. In our setting, this allows us to proceed at each time t without keeping track of how long one has already waited for the offsetting order.

order, and the marketmaker's profit from this trade has a current value of

$$\Pi_{u,C^{Ask}} \equiv C_t^{Ask} - C_t^{BS}(\alpha\phi) + V_t [C_u^{BS}(\alpha\phi) - C_u^{Bid}], \quad (9)$$

where $V_t [C_u^{BS}(\alpha\phi) - C_u^{Bid}]$ is the current value of the time- u random payoff $C_u^{BS}(\alpha\phi) - C_u^{Bid}$ which is yet to be determined.⁹ By probability weighing (8)–(9) with the distribution function of the offsetting call sell order arrival time F_{C_s} , the marketmaker obtains its expected profit from selling of a call option at time t as

$$\begin{aligned} \Pi_{C^{Ask}} &\equiv \int_t^T \Pi_{u,C^{Ask}} dF_{C_s}(u) + \Pi_{T,C^{Ask}} \left(1 - \int_t^T dF_{C_s}(u)\right) \\ &= C_t^{Ask} - C_t^{BS}(\alpha\phi) + \int_t^T V_t [C_u^{BS}(\alpha\phi) - C_u^{Bid}] dF_{C_s}(u). \end{aligned} \quad (10)$$

Finally, the marketmaker sets the call ask price so that the expected profit (10) is zero, which after rearranging leads to the expected cost representation for the call ask price in Lemma 2 (call bid, put bid and ask prices follow similarly).

Lemma 2 (Expected cost of hedging representation). *In the economy with costly short-selling and marketmakers, the call bid and ask prices satisfy*

$$C_t^{Bid} = \int_t^T \left\{ C_t^{BS}(\phi) + V_t [C_u^{Ask} - C_u^{BS}(\phi)] \right\} dF_{C_b}(u) + C_t^{BS}(\phi) \left(1 - \int_t^T dF_{C_b}(u)\right), \quad (11)$$

$$C_t^{Ask} = \int_t^T \left\{ C_t^{BS}(\alpha\phi) - V_t [C_u^{BS}(\alpha\phi) - C_u^{Bid}] \right\} dF_{C_s}(u) + C_t^{BS}(\alpha\phi) \left(1 - \int_t^T dF_{C_s}(u)\right). \quad (12)$$

Similarly, the put bid and ask prices satisfy

$$P_t^{Bid} = \int_t^T \left\{ P_t^{BS}(\alpha\phi) + V_t [P_u^{Ask} - P_u^{BS}(\alpha\phi)] \right\} dF_{P_b}(u) + P_t^{BS}(\alpha\phi) \left(1 - \int_t^T dF_{P_b}(u)\right), \quad (13)$$

$$P_t^{Ask} = \int_t^T \left\{ P_t^{BS}(\phi) - V_t [P_u^{BS}(\phi) - P_u^{Bid}] \right\} dF_{P_s}(u) + P_t^{BS}(\phi) \left(1 - \int_t^T dF_{P_s}(u)\right), \quad (14)$$

⁹Note that the current value operator $V_t[X_u]$ gives the amount required at time- t to form self-financing portfolios in the underlying stock and the riskless bond to obtain the payoff X_u almost surely at time $u \geq t$. Since there is a difference between the cost of short-selling and the benefit of holding a share long, one needs to account for the sign of the payoff X_u while determining its current value as we show in the Appendix.

where $V_t[X_u]$ denotes the time- t value of the payoff X_u at time $u \geq t$, and the Black-Scholes call and put prices, $C_t^{BS}(\cdot)$ and $P_t^{BS}(\cdot)$, are as in (5)–(6) of Lemma 1.

Lemma 2 indicates that in the economy with costly short-selling and marketmakers, option bid and ask prices are given by the marketmakers' expected cost of hedging sell and buy orders, respectively. This is because the first terms in option prices (11)–(14) are the (probability weighted) current values of the subsequent offsetting orders and the second terms are the (probability weighted) current costs of the hedge portfolios. Summing these probability weighted hedging costs gives the expected cost of hedging representations, where the expectations are taken with respect to the uncertainty about the relevant offsetting order arrival times given by the distribution functions F . For instance, the quantity $C_t^{BS}(\alpha\phi) - V_t[C_u^{BS}(\alpha\phi) - C_u^{Bid}]$ in the first term in the call ask price representation in (12) is the value (cost) of the subsequent offsetting call sell order if it arrives at time $u \geq t$, and the quantity $C_t^{BS}(\alpha\phi)$ in the second term is the cost of the hedge portfolio in the underlying stock. This expected cost of hedging representation is a notable generalization of the standard Black-Scholes model in which option prices are equal to the cost of their hedge portfolios, the only way to hedge them. Even though this representation for option prices is simple and intuitive, to the best of our knowledge it has not been explored previously in the literature.

We note that the hedge portfolio proceeds and costs arise as the bid and ask prices only in the unrealistic polar case of no possibility of an offsetting trade ($\lambda = 0$). This is intuitive since then the marketmakers know that they can only hedge the options via hedge portfolios, and so they set the option prices equal to their respective hedge portfolio costs and proceeds. For instance, when there is no possibility of an offsetting call sell order ($\lambda_{Cs} = 0$), the call ask price in (12) coincides with the maximum possible price $C_t^{Ask} = C_t^{BS}(\alpha\phi)$ as this is the cost of the hedge portfolio for a call option seller (Section 2). For the realistic case with the possibility of offsetting orders ($\lambda > 0$), we need to solve the coupled systems (11)–(12) for the call option, and (13)–(14) for the put option, in which the current bid and ask prices depend on the future prices of the other. Solving the above coupled systems involve substituting conjectured (and later verified) bid and ask prices into these systems, differentiating, and solving the resulting systems of two linear first order differential equations simultaneously as shown in the Appendix. This procedure yields the closed-form solutions for the call and put option bid and ask prices, as reported in Proposition 1.

Proposition 1 (Option bid and ask prices). *In the economy with costly short-selling*

and marketmakers, the call bid and ask prices are given by

$$C_t^{Bid} = (1 - w_{t,C^{Bid}}) C_t^{BS}(\alpha\phi) + w_{t,C^{Bid}} C_t^{BS}(\phi), \quad (15)$$

$$C_t^{Ask} = w_{t,C^{Ask}} C_t^{BS}(\alpha\phi) + (1 - w_{t,C^{Ask}}) C_t^{BS}(\phi), \quad (16)$$

and the put bid and ask prices are given by

$$P_t^{Bid} = (1 - w_{t,P^{Bid}}) P_t^{BS}(\phi) + w_{t,P^{Bid}} P_t^{BS}(\alpha\phi), \quad (17)$$

$$P_t^{Ask} = w_{t,P^{Ask}} P_t^{BS}(\phi) + (1 - w_{t,P^{Ask}}) P_t^{BS}(\alpha\phi), \quad (18)$$

where the Black-Scholes call and put prices, $C_t^{BS}(\cdot)$ and $P_t^{BS}(\cdot)$, are as in (5)–(6) of Lemma 1. The weights for the call bid and ask prices $w_{t,C^{Bid}}$ and $w_{t,C^{Ask}}$ are given by

$$w_{t,C^{Bid}} = \frac{\lambda_{C_s}}{\lambda_{C_s} + \lambda_{C_b}} + \frac{\lambda_{C_b}}{\lambda_{C_s} + \lambda_{C_b}} e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)}, \quad (19)$$

$$w_{t,C^{Ask}} = \frac{\lambda_{C_b}}{\lambda_{C_s} + \lambda_{C_b}} + \frac{\lambda_{C_s}}{\lambda_{C_s} + \lambda_{C_b}} e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)}, \quad (20)$$

and the weights for the put bid and ask prices $w_{t,P^{Bid}}$ and $w_{t,P^{Ask}}$ are given by

$$w_{t,P^{Bid}} = \frac{\lambda_{P_s}}{\lambda_{P_s} + \lambda_{P_b}} + \frac{\lambda_{P_b}}{\lambda_{P_s} + \lambda_{P_b}} e^{-(\lambda_{P_s} + \lambda_{P_b})(T-t)}, \quad (21)$$

$$w_{t,P^{Ask}} = \frac{\lambda_{P_b}}{\lambda_{P_s} + \lambda_{P_b}} + \frac{\lambda_{P_s}}{\lambda_{P_s} + \lambda_{P_b}} e^{-(\lambda_{P_s} + \lambda_{P_b})(T-t)}. \quad (22)$$

Consequently, in the economy with costly short-selling, marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs, i.e. $C_t^{BS}(\phi) < C_t^{Bid} < C_t^{Ask} < C_t^{BS}(\alpha\phi)$, and $P_t^{BS}(\alpha\phi) < P_t^{Bid} < P_t^{Ask} < P_t^{BS}(\phi)$.

Proposition 1 reveals that unique option prices (15)–(18) are weighted-averages of Black-Scholes prices that represent the marketmakers' costs of and proceeds from the hedge portfolio as a seller and a buyer, respectively. Consequently, these option prices are not only easy to compute, but also preserve the well-known properties of Black-Scholes prices. In particular, option prices do not depend on investor preferences and the underlying stock mean return μ and the signs of the so-called option Greeks are the same as in the Black-Scholes model. We see that the weights for bid and ask prices (19)–(22) are driven by the arrival rates of

both the buy and sell orders, rather than only by the arrival rate of a respective offsetting order. This follows from the fact that the current bid and ask prices depend on the future ask and bid prices, respectively (see Lemma 2), and hence the arrival rates of offsetting sell and buy orders both affect the prices through these weights.

The option prices (15)–(18) also reveal that the partial lending affects all option prices. This is somewhat surprising since it implies that the partial lending, a friction that matters only for stock lending, also affects the prices of options whose hedge portfolios require short-selling the stock at all times. In our analysis, this occurs for those options because while setting prices, the marketmakers take into account of their future offsetting orders, whose hedge portfolios in turn require holding the stock long which makes their prices depend on partial lending.

Proposition 1 also uncovers that marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs. This is notable as it implies that investors now have incentives to trade with marketmakers rather than to replicate the option payoffs themselves via a portfolio in the underlying stock and the riskless bond, since this way they can sell the same payoff at a higher price and buy it at a lower price. This is in contrast to the Black-Scholes model in which options do not offer any cost advantages over and above their replicating alternatives constructed with the underlying stock and riskless bond. Competitive marketmakers are able to offer these more favorable prices to investors because it is less costly for them to hedge their trades through offsetting orders as compared to hedge portfolios. For instance, marketmakers sell the call option at its ask price (16), which is lower than the cost of its hedge portfolio, $C_t^{BS}(\alpha\phi)$. They are willing to do so as there is also the possibility to hedge the call option sold by buying a call option at a bid price in the future whose current value is less than the hedge portfolio cost $C_t^{BS}(\alpha\phi)$. This reduces the expected cost of hedging a call option sold and leads to a lower call ask price.

Remark 2 (Other features of option marketmaking). To obtain our results, we have considered only the key necessary features of option marketmaking, and have not incorporated other possible features so as to not unnecessarily confound or complicate our analysis. In our model option trades are due to market orders and occur at a fixed size as in Easley, O’Hara, and Srinivas (1998) and Muravyev (2016). Without loss of generality we normalize the trade sizes to one for convenience. Moreover, considering market orders and not additionally the more complex limit orders, which are dependent on prices, turn out to be enough

for our analysis and main message. To study the effects of costly short-selling in a simple framework that is as close as possible to the standard symmetric information option pricing models, we do not consider information asymmetry between marketmakers and investors which may also affect the option bid and ask prices as demonstrated in Easley, O'Hara, and Srinivas (1998).

Remark 3 (Further discussion of marketmakers' hedging behavior). In our setting, marketmakers form a hedge portfolio for each option immediately after its trade, which if held until option maturity perfectly hedges the option and ensures the marketmakers face no market risk arising from that trade. This hedging behavior is consistent with the traditional minimum-variance criterion, a commonly considered objective in the risk management literature (see Basak and Chabakauri, 2012 for the related discussion and references therein). Moreover, the only time the marketmakers prematurely liquidate the hedge portfolio is when a subsequent offsetting order arrives (e.g., a current call buy order's offsetting order is a subsequent call sell order with the same strike and maturity), since the offsetting order also perfectly hedges the current order at its maturity. In fact, the strategy of (not liquidating the hedge portfolio and) treating the subsequent offsetting order as a new order that needs to be hedged via another portfolio is suboptimal, since it leads to lower profits. For instance, as discussed earlier, while determining the current call ask price at time t , if an offsetting call sell order arrives at a subsequent time $u < T$, the marketmaker buys this offsetting call option at a bid price C_u^{Bid} and liquidates the hedge portfolio for the current option at a value $C_u^{BS}(\alpha\phi)$. However, if the marketmaker were to hedge the offsetting order via a new hedge portfolio by also keeping its existing hedge portfolio until maturity, this strategy would lead to a lower profit. This is because now it is not liquidating the initial hedge portfolio for $C_u^{BS}(\alpha\phi)$ but instead receiving less $C_u^{BS}(\phi)$ from the proceeds of the new hedge portfolio.

The marketmakers, however, do not consider a subsequent order as offsetting if it does not have the same option type, strike and maturity as the initial option. This is because options with different types, strikes or maturities would only partially hedge the current option at its maturity, and hence expose the marketmakers to market risk in an economy in which they could actually achieve zero exposure to market risk – a situation that would not be consistent with the minimum-variance criterion. Moreover, when setting option prices ex-ante and possibly matching each current option with a single future offsetting order, the only relevant uncertainty for the marketmakers is represented by a single distribution function

for the relevant offsetting order. However, if partial hedging with all possible subsequent orders (with different types, strikes or maturities) were also considered, the analysis would be significantly complex, since the relevant uncertainty for the marketmakers would then be represented by a high (possibly infinite) dimensional distribution function. In this case, it does not appear to be possible to obtain (ex-ante) option prices that reflect (ex-post) hedging costs using our methodology.

As the above discussion highlights, when setting option prices ex-ante, the marketmakers treat each option separately and stand ready to form the hedge portfolio immediately after their trade. However, there can be instances in which marketmakers hedge their ex-post net portfolio (which may consist of several options) rather than hedge each option separately. In our setting, this could occur in the very special case of marketmakers receiving multiple simultaneous orders at the same time. For example, suppose a marketmaker receives two simultaneous buy orders at time t , one call and one put (written on the same underlying with the same strike and maturity), and fulfills these orders at their ask prices (16) and (18), respectively. Then the hedge portfolio for this (ex-post) net portfolio is the sum of $e^{-\alpha\phi(T-t)}\Phi(d_1(\alpha\phi))$ and $-e^{-\phi(T-t)}\Phi(-d_1(\phi))$ (see, (A.3) and (A.8) in the Appendix), which may require a positive (long), negative (short), or even zero holding in the underlying stock, without affecting the option prices the marketmaker sets and quotes ex-ante.

4. Behavior of option bid and ask prices

In this Section, we investigate the behavior of the option prices obtained in Section 3. Consistent with empirical evidence, we show that a higher shorting fee leads to higher bid-ask spreads for typical options, higher put option implied volatilities, and higher apparent put-call parity violations. Furthermore, we show that call and put bid-ask spreads are decreasing in the partial lending, the option marketmakers' participation in the stock lending market is decreasing in the shorting fee for each call option sold, and the effects of short-selling costs on option bid-ask spreads are more pronounced for relatively illiquid options with lower trading activity.

In addition to presenting the effects of costly short-selling on option prices, we also present our results for the (options) implied stock prices using the well-known put-call parity relation,

which yields the implied stock bid and ask prices as

$$\tilde{S}_t^{Bid} \equiv C_t^{Bid} - P_t^{Ask} + Ke^{-r(T-t)}, \quad (23)$$

$$\tilde{S}_t^{Ask} \equiv C_t^{Ask} - P_t^{Bid} + Ke^{-r(T-t)}. \quad (24)$$

That is, an investor selling the call at the bid price C_t^{Bid} , buying the put at the ask price P_t^{Ask} , and selling the riskless bond of an amount $Ke^{-r(T-t)}$ obtains the payoff $-S_T$ at option maturity. This strategy is equivalent to selling the stock short, but without paying the shorting fee prior to the option maturity date, and yields the implied stock bid price (23). Similar reasoning leads to the implied stock ask price (24). The implied stock bid and ask prices (23)–(24) allow us to relate our results to the documented evidence on the effects of costly short-selling on apparent put-call parity violations, which are typically measured as percentage deviations of implied stock prices from the underlying stock price (e.g., Ofek, Richardson, and Whitelaw, 2004; Evans, Geczy, Musto, and Reed, 2007). Proposition 2 reports the effects of the shorting fee on the call, put, and implied stock prices, as well as on the marketmakers' participation in the stock lending market.

Proposition 2 (Effects of shorting fee). *In the economy with costly short-selling and marketmakers,*

- i) The call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the shorting fee ϕ .*
- ii) Both the call and put bid-ask spreads are increasing in the shorting fee ϕ when $\alpha e^{(1-\alpha)\phi(T-t)} < \Phi(d_1(\phi)) / \Phi(d_1(\alpha\phi))$.*
- iii) The implied stock bid and ask prices are decreasing in the shorting fee ϕ .*
- iv) The option marketmakers' stock lending for each call option sold is decreasing in the shorting fee ϕ , while their lending for each put option bought is increasing in the shorting fee ϕ when $\sigma\sqrt{T-t} < \phi(-d_1(\alpha\phi)) / \Phi(-d_1(\alpha\phi))$.*

Proposition 2 reveals that the call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the shorting fee ϕ (property (i)). This is because option prices (15)–(18) are weighted-averages of the marketmakers' costs of and proceeds from the hedge portfolio as a seller and a buyer, respectively. A higher shorting fee reduces both the cost of the hedge portfolio as a call seller as it increases the benefit of holding a share

long, and also the proceeds from the hedge portfolio as a call buyer as it increases the cost of short-selling, leading to lower call ask and bid prices. In contrast, a higher shorting fee increases both the marketmakers' cost of the hedge portfolio as a put seller as it increases the cost of short-selling, and also the proceeds from the hedge portfolio as a put buyer as it increases the benefit of holding a share long, leading to higher put bid and ask prices. Figure 1 illustrates this result by plotting the option bid and ask prices against the shorting fee. One immediate consequence of this result is that the higher the shorting fee, the lower the call implied volatility and the higher the put implied volatility, where we here employ the standard approach of inverting the Black-Scholes formula using the option prices (15)–(18) as inputs. This finding is in line with the empirical evidence in Evans, Geczy, Musto, and Reed (2007) and Lin and Lu (2016), which demonstrate that put implied volatilities are increasing in the shorting fee.

[INSERT FIGURE 1 HERE]

Even though the call bid and ask prices are decreasing, while those of the put are increasing, both the call and put bid-ask spreads are increasing in the shorting fee ϕ for typical options and realistic values of shorting fee and partial lending (property (ii)). This result can also be seen immediately in Figure 1. The condition given in the property is equivalent to a higher shorting fee reducing the marketmakers' cost of the hedge portfolio as a call seller less than the proceeds from the hedge portfolio as a call buyer (see (A.30) in the Appendix). This condition arises because an increase in the shorting fee not only increases short-selling costs but also increases the benefit of holding a share long partially, and call prices decrease convexly in these costs and benefits. Hence, for relatively low levels of short-selling costs, this condition is satisfied since the marketmakers' hedge portfolio as a call seller is affected only partially. However, for extremely high levels of short-selling costs this relation may reverse, since their hedge portfolio proceeds as a call buyer may decrease less due to convexity. As we demonstrate in our quantitative analysis in Section 5, this condition is satisfied for option contracts with typical (e.g., short) maturities and realistic (e.g., low) values of shorting fee and partial lending. We then have the result that option bid-ask spreads are increasing in the shorting fee, as empirically documented by Evans, Geczy, Musto, and Reed (2007) and Lin and Lu (2016).¹⁰

¹⁰Conversely, for this condition to not hold, the option maturity would need to be long, e.g., over a year,

Turning to the implied stock prices, we see that, the higher the shorting fee ϕ , the lower the implied stock bid and ask prices (property (iii)), and hence, the higher their deviations from the underlying stock price. This is because, as we discussed earlier, the strategy that yields the implied stock bid price (23) is equivalent to selling the stock short, but without paying the shorting fee prior to the option maturity date. Hence, by no-arbitrage, the implied stock bid price must be lower than the underlying stock price. A similar mechanism leads to the implied stock ask price being lower than the underlying stock price. A higher shorting fee being associated with higher apparent put-call parity violations is well-supported by the empirical evidence, as in Lamont and Thaler (2003), Ofek, Richardson, and Whitelaw (2004), Evans, Geczy, Musto, and Reed (2007), and also is in the spirit of Gârleanu and Pedersen (2011), who show that presence of margin costs can lead to deviations from the law of one price. However, at this point it is useful to highlight that in our economy, option bid and ask prices lie within their respective no-arbitrage ranges presented in Section 2. Therefore, the implied stock prices being less than the underlying stock price does not necessarily imply a true arbitrage.

Our model also has implications for the extent to which the option marketmakers participate in the stock lending market. This relation is of interest as there is some recent evidence of a tight link between the behavior of option marketmakers and the stock lending market (Blocher and Ringgenberg, 2018). Property (iv) reveals that the marketmakers' stock lending for each call option they sell at the ask price is decreasing in the shorting fee ϕ . This is because an increase in the shorting fee reduces the number of (long) stock shares needed for hedging, since now more is earned from lending and the call ask prices are lower. Property (iv) also shows that the marketmakers' stock lending for each put option they buy at the bid price is increasing in the shorting fee for typical put options. The condition given in the put property arises because an increase in the shorting fee not only increases the lending benefits but also increases the put bid price paid. For typical put options with relatively low levels of short-selling costs this condition is satisfied, since the increase in the lending benefits are small as compared to the additional increase in the price. However, for very

and also the shorting fees and partial lending must be unrealistically high simultaneously, e.g., higher than 40% each. However, the exchange-traded options typically have far shorter maturities and stock shorting fees are a lot lower. For instance, the median option maturity in the full sample of Ofek, Richardson, and Whitelaw (2004) is 115 days, and the typical stock in the highest shorting fee decile has a shorting fee of 6.96% in the sample of Drechsler and Drechsler (2016).

high levels of short-selling costs this relation may reverse, since the increase in the lending benefits may dominate the additional increase in the price, leading to a lower number of stock shares needed for hedging.

Proposition 3 (Effects of partial lending). *In the economy with costly short-selling and marketmakers,*

- i) The call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the partial lending α .*
- ii) Both the call and put bid-ask spreads are decreasing in the partial lending α .*

Proposition 3 reveals that the call bid and ask prices are decreasing, while those of the put are increasing in the partial lending α (property (i)). The intuition is similar to that of the shorting fee discussed in Proposition 2. Increasing partial lending reduces the marketmakers' cost of the hedge portfolio as a call seller, while increasing their proceeds from the hedge portfolio as a put buyer, since it increases the benefit of holding a share long in these hedge portfolios. However, increasing partial lending has no effect on the marketmakers' hedge portfolio as a call buyer and put seller since their hedge portfolios require short-selling the stock. This then decreases the call bid and ask prices and increases the put bid and ask prices since they are weighted-averages of the costs of and proceeds from these hedge portfolios (Proposition 1). Figure 2 illustrates this result by plotting the option bid and ask prices against the partial lending. Again, the immediate consequence of this result is that the higher the partial lending, the lower the call implied volatility and the higher the put implied volatility.

[INSERT FIGURE 2 HERE]

As we can also see from Figure 2 both the call and put bid-ask spreads are decreasing in the partial lending α (property (ii)). This is in contrast to the earlier shorting fee result that option bid-ask spreads are increasing in the shorting fee for typical options (Proposition 2(ii)). This opposite effect arises because partial lending only affects the hedge portfolio that is long in the stock and has no effect on the hedge portfolio that requires short-selling the stock, leading to an unconditional, simpler result. Therefore, a higher partial lending reduces the call ask price more than the call bid price since the hedge portfolio for a call seller requires holding a share long, resulting in a lower call bid-ask spread. Similarly, a

higher partial lending increases the put bid price more than the put ask bid price, leading to a lower put bid-ask spread.

As discussed in Section 3, the offsetting order arrival rates are inherently related to option liquidity. That is, the more liquid options with higher buying/selling activity are more likely to have higher offsetting order arrival rates (or, equivalently, lower expected arrival times). Proposition 4 investigates the effects of the offsetting order arrival rates.

Proposition 4 (Effects of offsetting order arrival rates). *In the economy with costly short-selling and marketmakers,*

- i) The call and put bid and ask prices are decreasing in their offsetting sell order arrival rates λ_{Cs} , λ_{Ps} , while they are increasing in their offsetting buy order arrival rates λ_{Cb} , λ_{Pb} , respectively.*
- ii) Both the call and put bid-ask spreads are decreasing in their offsetting order arrival rates λ_{Cs} , λ_{Cb} and λ_{Ps} , λ_{Pb} , respectively.*
- iii) The effects of the shorting fee and the partial lending on the call and put bid-ask spreads are decreasing in the offsetting order arrival rates λ_{Cs} , λ_{Cb} , λ_{Ps} , λ_{Pb} .*

Property (i) reveals that the call and put bid and ask prices are decreasing in their offsetting sell order arrival rates λ_{Cs} , λ_{Ps} , while they are increasing in their offsetting buy order arrival rates λ_{Cb} , λ_{Pb} , respectively. This is fairly intuitive since it states that option prices are decreasing in investors' selling activity, but are increasing in buying activity. In our economy, this result is due to the fact that option bid and ask prices are given by the marketmakers' expected cost of hedging a sell and a buy order, respectively, where the expectation is taken with respect to the uncertainty about the offsetting order arrival times (Lemma 2). An increase in the arrival rates increases the probability of hedging via an offsetting order, which not only costs less for hedging a buy order but also yields more proceeds from hedging a sell order as compared to the hedge portfolio.

Property (ii) shows that both the call and put bid-ask spreads are decreasing in their respective offsetting order arrival rates. This result is also intuitive as it simply says that option bid-ask spreads are decreasing in liquidity (e.g., investors' buying/selling activity). Property (iii) shows that the effects of the shorting fee and partial lending on bid-ask spreads decrease in the investors' buying/selling activity. That is, an increase in the shorting fee

increases, while an increase in the partial lending decreases the option bid-ask spreads more for relatively illiquid options with lower offsetting order arrival rates. This is intuitive as it simply says that the effects of short-selling costs are more pronounced for relatively illiquid options with lower trading activity. In our model, this occurs because marketmakers are more likely to hedge the relatively illiquid options via hedge portfolios, through which short-selling costs affect option prices directly.

5. 2008 Short-Selling ban, quantitative analysis, and palm stock 2000

In this Section, we first apply our model to the widely-studied 2008 US short-selling ban period. Consistent with empirical evidence, we show that the option bid-ask spreads of banned stocks and their apparent put-call parity violations are higher than those of unbanned stocks, and the option bid and ask prices are affected asymmetrically by the ban. We then quantify our model and demonstrate that the effects of short-selling costs on option prices are economically significant for expensive-to-short stocks. Finally, we apply our model and shed light on the behavior of option prices of the Palm stock in 2000, during which it experienced extreme short-selling and violations of the law of one price.

5.1. 2008 Short-Selling ban

We here apply our model to the September 2008 US short-selling ban period.¹¹ During this period the option marketmakers were exempt from the ban and were allowed to short-sell to provide the marketmaking facility. Therefore, our option prices of Proposition 1 are still valid for options on both the banned and unbanned stocks during this period. The main difference between the banned and unbanned stocks was the fact that the short-selling ban reduced the overall short-selling activity on banned stocks since the only short-sellers on them were the marketmakers and specialists. This meant that investors who were long in these stocks were more likely to lend a smaller fraction of their shares. In fact, the evidence indicates that the short-selling ban reduced (roughly halved) the short-selling activity but increased (roughly doubled) the shorting fee of banned stocks (Boehmer, Jones, and Zhang,

¹¹See Battalio and Schultz (2011) for more details and relevant regulatory events for this period starting from September 19, 2008 and ending on October 8, 2008 during which a short-selling ban was imposed on nearly 800 financial stocks in the US.

2013; Harris, Namvar, and Phillips, 2013; Kolasinski, Reed, and Thornock, 2013).¹² In light of this evidence, we take the shorting fee of the banned stocks, ϕ_{Ban} , to be twice the shorting fee of the unbanned stocks, denoted by ϕ as before. Moreover, we also take the partial lending for the banned stocks, α_{Ban} to be half of the partial lending of the unbanned stock, denoted by α as before.

In sum, these simple adjustments imply that the marketmakers effectively pay double the rate $\phi_{Ban} = 2\phi$ for short-selling, but earn the same rate $\alpha_{Ban}\phi_{Ban} = \alpha\phi$ for holding a stock share long in their hedge portfolios of options on banned stocks as compared to options on otherwise identical unbanned stocks during the ban period, as well as to options on them before the ban.¹³ Proposition 5 reports the effects of the short-selling ban on the option prices of banned and unbanned stocks during the ban period.

Proposition 5 (Effects of short-selling ban). *During the short-selling ban,*

- i) The call bid and ask prices of banned stocks are lower, while the put bid and ask prices of banned stocks are higher than those of unbanned stocks.*
- ii) Both the call and put bid-ask spreads of banned stocks are higher than those of unbanned stocks.*
- iii) The implied stock bid and ask prices of banned stocks are lower than those of unbanned stocks.*
- iv) The call bid price decreases more than the ask price, while the put ask price increases more than the bid price of banned stocks.*

Proposition 5 first reveals that during the short-selling ban, options on banned stocks have lower call and higher put prices as compared to those of unbanned stocks (property (i)). Proposition 5 further reveals that options on banned stocks have higher bid-ask spreads and

¹²Boehmer, Jones, and Zhang (2013) report that during the ban period, shorting activity roughly halved (decreased from 21.40% to 9.96% of trading volume), while Harris, Namvar, and Phillips (2013) also document a similar magnitude for the reduction in the short interest levels of banned stocks (a decrease from roughly 7.00% to 4.00%). On the other hand, Kolasinski, Reed, and Thornock (2013) report that during the ban period, shorting fees of banned stocks roughly doubled (increased by 113% from 0.65% to 1.38%).

¹³We note that doubling the shorting fee while halving the partial lending of the banned stocks allows us to demonstrate the effects of the ban clearly and in a simple fashion as Proposition 5 illustrates. Adjusting the shorting fee and partial lending exactly as in the evidence complicates the analysis unnecessarily but also leads to similar results that can be shown numerically.

lower implied stock prices than those of unbanned stocks (property (ii)–(iii)), consistent with empirical evidence (Battalio and Schultz, 2011; Grundy, Lim, and Verwijmeren, 2012; Lin and Lu, 2016). Proposition 5 also demonstrates an asymmetric effect of the short-selling ban for options on banned stocks in that their call bid prices decrease more than their ask prices, while their put ask prices increase more than their bid prices (property (iv)), also consistent with empirical evidence (Battalio and Schultz, 2011). These results arise because the short-selling ban only affects those hedge portfolios that are short in the stock, but does not affect the ones that are long in the stock since they earn the same rate per share. For instance, the short-selling ban reduces the proceeds from the hedge portfolios of marketmakers as call buyers but has no effect on the costs of their hedge portfolios as call sellers. Since call prices are weighted average of these costs and proceeds (Proposition 1), the short-selling ban leads to lower call bid and ask prices, higher call bid-ask spreads, and relatively higher decreases in the call bid prices. Moreover, a decrease in call prices along with an increase in put prices immediately lead to lower implied stock prices (23)–(24) which then lead to higher apparent put-call parity violations for banned stocks as compared to unbanned stocks.

5.2. Quantitative analysis

To quantify our model, we determine the parameter values as follows. The shorting fee and partial lending values are based on the comprehensive data used in Drechsler and Drechsler (2016), who sort stocks into deciles by their shorting fee and report the average shorting fee and short interest ratios, SIR (total number of shares shorted normalized by shares outstanding), for the sample period 2004-2013. We investigate the quantitative effects of short-selling costs by considering options on a typical stock in the lowest shorting fee decile (D1) and the highest shorting fee decile (D10), henceforth expensive-to-short stocks, for which Drechsler and Drechsler report the average shorting fees to be 0.02% and 6.96%, respectively. We next take the ratio of a stock’s short interest to long interest (the short interest plus outstanding shares) to be an observable proxy for its partial lending parameter α . This ratio is a plausible proxy since it gives the fraction of aggregate long position lent to short-sellers. Normalizing by the outstanding shares gives this ratio in terms of only the short interest ratio as $SIR/(1 + SIR)$. Moreover, since lenders are mainly institutions in reality (see, Reed, 2013), we further refine this measure by considering the short interest ratios normalized by institutional ownership, denoted by SIR_{IO} . These are readily provided for each decile in Drechsler and Drechsler, who report the values of 4.5% and 26.5%, for the

lowest (D1) and the highest (D10) shorting fee deciles, respectively. Applying these values to $SIR_{IO}/(1 + SIR_{IO})$, we obtain the partial lending parameter values for these deciles as 4.31% and 20.95%, respectively.¹⁴

For the securities market parameter values, we take the interest rate to be the average 3-month T-bill rate for the sample period of Drechsler and Drechsler (2016) which is 1.80%. We set the stock price as the reported average stock price of 32.20 in Ofek, Richardson, and Whitelaw (2004).¹⁵ The return volatility of the stock is set to 40% as in Jensen and Pedersen (2016). For option specific parameter values, we consider varying moneyness (e.g., the ratio of option strike to stock price, K/S_t) levels of 0.90, 1.00, 1.10 to demonstrate the varying effects of short-selling costs across option moneyness. Option time-to-maturity is taken to be 0.25 (3 months) which is well within the reported average option maturities in the samples of empirical works that we compare our results to.¹⁶ Finally, we determine the offsetting call sell order arrival rates by giving equal weights to both ways of hedging in our model, hedging via an offsetting order and via a hedge portfolio. For instance, giving a probability of 0.5 to there being no arrival of an offsetting call sell order by the maturity date, $1 - \int_t^T dF_{C_s}(u) = e^{-\lambda c_s(T-t)}$ yields the value for the offsetting call sell order arrival rate as 2.77, which is also the value of all the other arrival rates.¹⁷ This procedure leads to the parameter values in Table 1.

[INSERT TABLE 1 HERE]

Table 2 reports the quantitative effects of short-selling costs on our option prices of Proposition 1. We consider a call option (Panel A) and a put option (Panel B) of a typical stock in the lowest (D1) and the highest (D10) shorting fee deciles for three different option

¹⁴We note that in Drechsler and Drechsler (2016), the sample average of the short interest ratio is 9.02%, which would imply an average partial lending value of 8.27% for a typical stock according to our formula above. This value is comparable to the average fraction of outstanding shares actually lent out in the US, 8.91%, in the sample of Saffi and Sigurdsson (2010) (discussed in Section 2).

¹⁵Our main results do not vary much with any particular value of the stock price as we use the same value for the typical stock in D1 and D10.

¹⁶For instance, the median option maturity in the full sample of Ofek, Richardson, and Whitelaw (2004) is 115 days.

¹⁷We recognize that the arrival rates for call and put buy and sell orders may be different since they are inherently linked to option buying/selling activity, which may differ across options as shown by Lakonishok, Lee, Pearson, and Poteshman (2007). However, keeping the same value for all arrival rates allows us to more clearly compare the quantitative effects of short-selling costs.

moneyness levels. We also report the percentage differences of mid-points of bid and ask prices (denoted by C_t^{Mid} and P_t^{Mid} for call and put options, respectively) from the standard Black-Scholes in the relative change column.

Table 2 reveals that our model implies a significantly lower call and a higher put bid and ask prices for the typical stock in D10, as compared to those for the typical stock in D1. In particular, for the typical stock in D10, its at-the-money (ATM) call mid-point price is 6.80% lower (-6.82% vs -0.02%), while that of the put is 6.28% higher (6.30% vs 0.02%) than the corresponding ones in D1. We see that these effects are stronger for out-of-the-money call and put options, being 8.17% lower and 8.15% higher for option moneyness of 1.10 and 0.90, respectively. Table 2 also quantifies the relative bid-ask spread by reporting the ratio of the bid-ask spread to the mid-point prices. We see that the typical stock in D10, has a 2.35% higher ATM call bid-ask spread as compared to the spread of the typical stock in D1. For the ATM put this difference is 1.95%. Again, these effects are stronger for out-of-the-money options. We also see that the typical stock in D10 has a 2.81% lower (37.18% vs 39.99) ATM call implied volatility as compared to the implied volatility of the typical stock in D1. However, for the ATM put, the typical stock in D10 has a 2.45% higher (42.46 vs 40.01%) implied volatility compared to the implied volatility of the typical stock in D1.

[INSERT TABLE 2 HERE]

Finally, substituting the at-the-money option prices in Table 2 into (23)–(24) yields the implied stock bid and ask prices for the typical stock in D1 to be the same as the underlying stock price. However, this procedure yields the implied stock bid and ask prices to be 31.81 and 31.92, respectively, for the typical stock in D10. In terms of percentage deviation these values imply a 1.04% lower implied stock mid-price from the underlying stock price. This magnitude is within the documented range in Evans, Geczy, Musto, and Reed (2007) who report an average deviation of 0.36% and the 90th percentile deviation of 1.40% in their sample.¹⁸

¹⁸Similarly, Ofek, Richardson, and Whitelaw (2004) find that a one standard deviation (2.77%) increase in the shorting fee leads to a 0.67% lower implied stock price as compared to the underlying stock price in their sample. For this magnitude of an increase in the shorting fee, our model implies a comparable 0.60% lower implied stock price after also adjusting for their sample average maturity. We note that the implied stock prices are stable and do not vary much in option moneyness, and therefore it is sufficient to only consider the at-the-money option prices to derive the implied stock prices as we do here.

5.3. Palm stock 2000

We here apply our model to the option prices of the Palm stock during its IPO in March 2000. This event was notable since it was a prime example of apparent violations of the law of one price (Lamont and Thaler, 2003). In particular, there was a long-lasting mispricing of Palm relative to its parent company 3Com in the sense that the subsidiary Palm was worth more than its parent company 3Com. This long-lasting mispricing of 3Com/Palm was often attributed to the extreme short-selling costs of Palm. In particular, its shorting fee was reported to be around 35% during this period (D’Avolio, 2002), and its short interest after the IPO in March was 19.4%, then increased to 44.9% in April, and to 70% in May, and peaked at 147.6% in July (Lamont and Thaler, 2003).

We demonstrate the effects of short-selling costs on Palm option prices by comparing our prices to those in Lamont and Thaler, who provide prices of at-the-money Palm options on March 17, 2000 for three different maturities, May ($T - t = 0.17$), August ($T - t = 0.42$) and November ($T - t = 0.67$). We take the shorting fee to be its reported value of 35%. Then we use the (approximate) average short interest ratio for the August maturity option, 70% to back out our partial lending parameter value as before and obtain $0.70/1.70 = 41.18\%$. For the securities market parameter values, we follow Lamont and Thaler and set the interest rate as the 3-month LIBOR rate of 6.21% to price May options, and the 6-month LIBOR rate of 6.41% to price August and November options. We set the stock price as the reported Palm stock price on March 17, 2000 of 55.25, which is also the strike price for the at-the-money options considered. The return volatility of the Palm stock is set to its average realized volatility during the life of the mid-maturity option expired in August, 104.6%.¹⁹ Finally, we again give equal weights to both ways of hedging for August maturity option, hedging via an offsetting order and via a hedge portfolio. This yields the value for all the offsetting order arrival rates as 1.66, which is also kept the same for the May and November maturity options. Using these parameter values, we now quantify the effects of costly short-selling on Palm options and present our results in Table 3 for at-the-money call and put options, as well as for the implied stock prices and their percentage deviations from the underlying stock price for three different option maturity dates.²⁰

¹⁹We estimated the return volatility of Palm in a standard way using the standard CRSP data. Considering the shorter maturity May or longer maturity November also give similar very high volatility values.

²⁰Note that Table 3 differs from our earlier Table 2 as it also has data counterparts.

[INSERT TABLE 3 HERE]

Table 3 reveals that Palm option prices displayed significant apparent put-call parity violations, in the sense that put prices were higher than call prices (which should not happen for at-the-money options), and the implied stock prices were significantly lower than the underlying stock price. In particular, the evidence indicates that for the mid-maturity options expiring in August, the call ask price of 10.75 was significantly less than the put bid price of 17.25. Our model also generates this feature by yielding a lower call ask price of 12.18 than the put bid price of 15.54, a feature not possible in the standard Black-Scholes model. In terms of the deviation from the underlying stock price, we see that the implied stock bid price was 21.14% and ask price 14.81% lower than the underlying stock price. The option prices implied by our model are able to generate roughly half of this deviation as they imply 10.66% (bid) and 8.72% (ask) lower prices. As Lamont and Thaler (2003) also highlight, high levels of short-selling costs were only part of the story as there were several other extreme risks and costs for short-sellers of the Palm stock during that time (e.g., search costs, uncertainties about collateral levels, shorting fee and early recall of the shares by lenders). Nevertheless, our model demonstrates that, for mid and long maturity options, roughly half of the price deviations could be due to the costly short-selling implying that the combined effects of all the other risks, costs and considerations could amount to the remaining half.

6. Stochastic volatility economy

Our analysis so far has been presented in the context of a one-factor economy with the stock price being the only source of risk. In this Section we extend our baseline setting to a two-factor stochastic volatility economy with the stock volatility arising as an additional source of risk in which option marketmakers can no longer perfectly hedge their trades via a portfolio. We show that option bid and ask prices preserve their baseline setting structure and demonstrate that all of our main results and underlying economic mechanisms (Section 3–4) continue to go through in this more elaborate setting. Moreover, we illustrate via a simple numerical exercise that under stochastic volatility, the effects of costly short-selling on option prices are more pronounced and a higher (lower) negative skewness in the underlying stock return leads to greater effects for call (put) options.

We adopt the stochastic volatility setting of Heston (1993) and consider separate dynam-

ics for the underlying stock price and its return volatility. In this framework, the stock price S and its return variance ν follow

$$dS_t = S_t [\mu dt + \sqrt{\nu_t} d\omega_t], \quad (25)$$

$$d\nu_t = \kappa_\nu (\bar{\nu} - \nu_t) dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu t}, \quad (26)$$

where κ_ν , $\bar{\nu}$, and σ_ν are positive constants representing the mean reversion, long-run mean, and volatility of the variance, and the Brownian motions ω and ω_ν have correlation ρ . The correlation parameter ρ controls the skewness of the underlying stock return. When $\rho < 0$, the stock return distribution is negatively skewed, and as argued in the literature this feature plays an important role in capturing the observed negative slopes in option implied volatilities, the so-called smile curves (e.g., see Duffie, 2001). We note that our baseline stock price process (2) is nested as a special case in the limit by setting $\bar{\nu} = \sigma^2$ and $\sigma_\nu = 0$, since when the volatility of the variance parameter σ_ν is zero, the variance deterministically converges to its long-run mean $\bar{\nu}$. The short-selling costs in the stock are as in our baseline setting (Section 2).

In this framework, when the underlying stock is the only risky asset available for trading, markets are incomplete. Hence, to determine the prices of contingent claims one typically uses equilibrium arguments and assumes a specific form for the volatility risk premium. Heston (1993) assumes a proportional risk premium for the variance process, $\lambda_\nu \nu_t$ for some constant λ_ν , a specification we also follow. In the absence of costly short-selling, this specification is sufficient to determine unique option prices as a function of the stock price and its return variance, as Heston (1993) shows. However, with costly short-selling this specification does not lead to unique option prices. To see this, consider a delta-hedge strategy in the underlying stock and riskless bond which tracks the option value at all times and delivers the option payoff upon its maturity. In contrast to the delta-hedge portfolio in our baseline setting, the delta-hedge portfolio in this setting is not self-financing and requires capital injection/withdrawal, henceforth *hedging error* $d\epsilon_t$, over dt to ensure that the portfolio maintains the option value. Hedging errors arise simply because this portfolio now does not hedge against the changes in the stock return variance. As we demonstrate in Lemma A1 of the Appendix, the cumulative hedging error, which can be positive or negative ex-post,

turns out to have a current value of zero.²¹ This implies that the delta-hedge portfolio costs and proceeds reflect the (asymmetric) cost of short-selling and the benefit of holding a share long as in our baseline setting, and hence once again leading to a range for possible prices.

To determine unique prices, we again introduce option marketmakers who attempt to hedge the risk in each order by forming a delta-hedge portfolio, which however no longer perfectly hedges the option and leads to ex-post hedging errors. The arrival of offsetting (buy or sell) orders and the way marketmakers determine the option bid and ask prices are as in our baseline setting (Section 3). Following steps similar to the one-factor case we are able to obtain explicit closed-form solutions for the call and put option bid and ask prices in the stochastic volatility economy, as reported in Proposition 6.

Proposition 6 (Option bid and ask prices in the stochastic volatility economy).

In the stochastic volatility economy with costly short-selling, the call bid and ask prices are

$$C_t^{Bid} = (1 - w_{t,C^{Bid}}) C_t^H(\alpha\phi) + w_{t,C^{Bid}} C_t^H(\phi), \quad (27)$$

$$C_t^{Ask} = w_{t,C^{Ask}} C_t^H(\alpha\phi) + (1 - w_{t,C^{Ask}}) C_t^H(\phi), \quad (28)$$

and the put bid and ask prices are

$$P_t^{Bid} = (1 - w_{t,P^{Bid}}) P_t^H(\phi) + w_{t,P^{Bid}} P_t^H(\alpha\phi), \quad (29)$$

$$P_t^{Ask} = w_{t,P^{Ask}} P_t^H(\phi) + (1 - w_{t,P^{Ask}}) P_t^H(\alpha\phi), \quad (30)$$

where the weights for the call and put bid and ask prices $w_{t,C^{Bid}}$, $w_{t,C^{Ask}}$, $w_{t,P^{Bid}}$, and $w_{t,P^{Ask}}$ are as in Proposition 1 of Section 3, and $C_t^H(q)$ and $P_t^H(q)$ denote the Heston call and put prices adjusted for the constant dividend yield q , and are given by

$$C_t^H(q) = S_t e^{-q(T-t)} \Psi_1(q) - K e^{-r(T-t)} \Psi_2(q), \quad (31)$$

$$P_t^H(q) = -S_t e^{-q(T-t)} (1 - \Psi_1(q)) + K e^{-r(T-t)} (1 - \Psi_2(q)), \quad (32)$$

²¹This is due to the fact that delta-hedged portfolio removes all the expected changes in the hedging errors and only the residual term with mean zero remains under the risk-neutral measure. A similar result of the cumulative hedging error having zero current value also obtains implicitly in Bakshi and Kapadia (2003).

where $\Psi_j(\cdot)$, $j = 1, 2$ is the conditional probability function given by

$$\Psi_j(q) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{e^{-iz \ln K} f_j(\ln S_t, \nu_t, t; z, q)}{iz} \right] dz, \quad (33)$$

where i denotes the imaginary unit, $\operatorname{Re}[c]$ is the real part of a complex number c , and the characteristic function f_j is given by

$$f_j(x, v, t; z, q) = e^{\mathcal{C}_j(T-t; z, q) + \mathcal{D}_j(T-t; z) v + izx}, \quad (34)$$

$$\mathcal{C}_j(\tau; z, q) = iz(r - q)\tau + \frac{\kappa_\nu \bar{\nu}}{\sigma_\nu^2} \left[(b_j - \rho \sigma_\nu iz + d_j)\tau - 2 \ln \left(\frac{1 - g_j e^{d_j \tau}}{1 - g_j} \right) \right], \quad (35)$$

$$\mathcal{D}_j(\tau; z) = \frac{b_j - \rho \sigma_\nu iz + d_j}{\sigma_\nu^2} \left(\frac{1 - e^{d_j \tau}}{1 - g_j e^{d_j \tau}} \right), \quad (36)$$

with

$$g_j = \frac{b_j - \rho \sigma_\nu iz + d_j}{b_j - \rho \sigma_\nu iz - d_j}, \quad d_j = \sqrt{(\rho \sigma_\nu iz - b_j)^2 - \sigma_\nu^2 (2u_j iz - z^2)}, \quad (37)$$

$$u_1 = 1/2, \quad u_2 = -1/2, \quad b_1 = \kappa_\nu + \lambda_\nu - \rho \sigma_\nu, \quad b_2 = \kappa_\nu + \lambda_\nu. \quad (38)$$

Consequently, in the stochastic volatility economy with costly short-selling,

- i) Marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs, i.e. $C_t^H(\phi) < C_t^{\text{Bid}} < C_t^{\text{Ask}} < C_t^H(\alpha\phi)$, and $P_t^H(\alpha\phi) < P_t^{\text{Bid}} < P_t^{\text{Ask}} < P_t^H(\phi)$.
- ii) All the properties on the effects of the shorting fee, partial lending, and offsetting order arrival rates as stated in Propositions 2–4 of Section 4 in our baseline setting hold.

Proposition 6 reveals that option bid and ask prices (27)–(30) preserve their baseline setting structure, but are now weighted-averages of Heston prices, which again represent the marketmakers' costs of and proceeds from the delta-hedged portfolio as a seller and a buyer, respectively. This is because the underlying short-selling costs and price setting mechanism are as before, with the only difference being the marketmakers' delta-hedge portfolio not leading to a perfect hedge. This difference turns out to not change the price structure since the delta-hedged portfolio's associated cumulative hedging error, which can be positive or negative ex-post, has a current value of zero. However, given the more elaborate stochastic

volatility setting, the parameters of the return variance process (26) also enter into option prices, leading to richer price behavior, for example, as illustrated in Table 4.

Since the underlying economic mechanisms and the structure of option prices are as in our baseline setting, we again obtain the result that marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs (property (i)). Once again this is notable as it implies that investors have incentives to trade with marketmakers rather than attempt to replicate the option payoffs themselves via delta-hedged portfolios with hedging errors. Similarly, the behavior of option prices with respect to the shorting fee, partial lending and offsetting order arrival rates are as before (property (ii)).²² This is because the short-selling costs enter into option prices through the hedge portfolio costs and proceeds, and in both settings the hedge portfolios require similar holdings in the underlying stock. For instance, a long call option payoff is obtained by forming the delta-hedge portfolio with $e^{-\alpha\phi(T-t)}\Phi(d_1(\alpha\phi)) > 0$ units in the stock in the baseline model, and with $e^{-\alpha\phi(T-t)}\Psi_1(\alpha\phi) > 0$ units in the stock in the stochastic volatility setting.

[INSERT TABLE 4 HERE]

[INSERT TABLE 5 HERE]

We now undertake a simple numerical exercise to quantify the effects of short-selling costs on option prices in the stochastic volatility setting. Towards that, we keep the values of the parameters that are common to the baseline setting as in Table 1 of Section 5, and choose the values of the additional parameters arising from the variance process (26) following Duffie, Pan, and Singleton (2000). As is common in the stochastic volatility literature, we set the values of the parameters of the variance process under the risk-neutral dynamics, which is given by $d\nu_t = \kappa_\nu^*(\bar{\nu}^* - \nu_t) dt + \sigma_\nu\sqrt{\nu_t}d\omega_{\nu t}^*$, where $\kappa_\nu^* = \kappa_\nu + \lambda_\nu$, and $\bar{\nu}^* = \bar{\nu}\kappa_\nu/\kappa_\nu^*$. Following Duffie, Pan, and Singleton (2000) Table I, we take the skewness (correlation) parameter to be -0.70 , the mean reversion coefficient parameter 6.21, and the volatility of variance parameter 0.61. We set the long-run mean of variance to 0.16 so that it is equal to the constant stock

²²The corresponding condition for Proposition 2 property (ii) now becomes $\alpha e^{(1-\alpha)\phi(T-t)} < \Psi_1(\phi)/\Psi_1(\alpha\phi)$. This is intuitive since the conditional probability function $\Psi_1(\cdot)$ given by (33) in this setting has the same economic role of $\Phi(d_1(\cdot))$ in the baseline setting. The corresponding condition for Proposition 2 property (iv) is also similar.

return variance in our baseline quantitative analysis, and the current variance to 0.0859 so that the ratio of the long-run mean to current variance is as in Duffie, Pan, and Singleton. Tables 4–5 report the effects of short-selling costs in the stochastic volatility economy on call and put options for a typical stock in the lowest (D1) and the highest (D10) shorting fee deciles for three different option moneyness levels, when the skewness is $\rho = -0.70$ (Panel A) and $\rho = -0.35$ (Panel B).

Table 4 reveals that the effects of costly short-selling on call prices are more pronounced in the stochastic volatility economy as compared to our one-factor setting (Table 2) due to the additional fluctuations in the volatility. In particular, Panel A shows that as compared to the typical stock in D1, the typical expensive-to-short stock in D10 has at-the-money (ATM) call mid-point price 8.31% (6.80%) lower and a call relative bid-ask spread 2.91% (2.35%) higher under the stochastic volatility (baseline setting). A similar finding also obtains for put options as illustrated in Table 5. We may study the effects of skewness by comparing the corresponding quantities in Panels A and B. In Table 4, we see that a higher negative skewness (Panel A) in the underlying stock returns leads to greater effects of costly short-selling on call options. In particular, we see that call bid-ask spreads of expensive-to-short stocks are greater under higher negative skewness. This is because when skewness becomes more negative, the stock price and its variance have a tendency to move in opposite directions more, and the additional fluctuations in the volatility lead to higher discrepancy between the call option hedge portfolio costs and proceeds. The negative skewness ($\rho < 0$) plays an important role in capturing the observed negative slopes in option implied volatilities (e.g., Duffie, 2001). Here, we show that skewness can also play an important role in determining the extent to which short-selling costs affect option prices.

Remark 4 (Stochastic volatility economy with perfect hedging). We here discuss an alternative formulation of our stochastic volatility setting in which marketmakers can perfectly hedge their trades. Towards that, we assume that marketmakers are able to trade in an additional risky security, a variance swap. Given the assumed risk premium for the variance process as in Heston (1993), the price of this security is also as in the Heston framework.²³ The role of the variance swap in this formulation therefore is simply to allow marketmakers determine the option prices via a delta-hedge portfolio, which perfectly hedges

²³A typical variance swap is a financial contract that pays the difference between the realized stock return variance over a period of time $[0, T]$ and a constant k , which is often referred to as the variance strike. Hence,

the option as in our baseline setting. The only difference from the earlier formulation is that the associated ex-post hedging errors are now zero. This is because the delta-hedge portfolio is now self-financing, and hence does not require capital injection/withdrawal. Since the delta-hedge portfolio in this formulation and the delta-hedge portfolio in the main stochastic volatility formulation require the same number of units in the underlying stock to hedge an option, the cost of short-selling and the benefit of holding a share long are also the same in both formulations.²⁴ Therefore, we obtain the same option bid and ask prices as in (27)–(30) of Proposition 6, and hence all our results in the main stochastic volatility economy also obtain under this alternative formulation. We provide the details of the analysis under this alternative formulation in the Appendix.

7. Corporate bonds

In this Section, we apply our methodology to corporate bonds, which have option-like payoffs for which the firm value is treated as the underlying security. We first show that the presence of the stock shorting fee leads to an implied shorting fee in the firm value. We then solve the model and obtain explicit closed-form solutions for the corporate bond bid and ask prices that are comparable to those in our baseline setting for options. This enables us to show how higher stock short-selling costs lead to lower prices (and hence to higher yields) for corporate bonds, consistent with evidence, as well as how the presence of stock short-selling costs can generate bid-ask spreads for corporate bonds.

We take the classic Merton (1974) setting for corporate bonds and consider the firm (assets) value process A with dynamics

$$dA_t = A_t [\mu_A dt + \sigma_A d\omega_t], \quad (39)$$

where μ_A and σ_A are the constant mean return and return volatility of the assets, respectively. The firm has two classes of claims: corporate bond and stock. The corporate bond is a zero-

in this setting the payoff of a variance swap is given by $\left(\frac{1}{T} \int_0^T \nu_t dt - k\right) N$, where N is the notional amount. We provide the explicit dynamics of the variance swap price in (A.61) of the Appendix, where we can see that the variance swap price is not affected by the stock short-selling costs.

²⁴For instance, a portfolio that delivers the long call option payoff requires $\partial C^H(\alpha\phi)/\partial S_t = e^{-\alpha\phi(T-t)}\Psi_1(\alpha\phi) > 0$ units in the underlying stock in both formulations.

coupon bond with face value K and maturity date T . In this setting, the corporate bond and stock have option-like payoffs, since in the event of $A_T < K$ the firm defaults, and hence the bondholders' payoff is $\min\{A_T, K\}$ and the stockholders' $\max\{A_T - K, 0\}$ at the maturity date. To apply our methodology as in our baseline setting, we assume that available for trading are a riskless bond with price B still following (1), and the (levered) stock with price S as in the Merton (1974) model and given by (A.65) in the Appendix with dynamics

$$dS_t = S_t [\mu_t dt + \sigma_t d\omega_t], \quad (40)$$

where μ_t and σ_t are the stochastic mean return and return volatility of the stock, respectively, and given by (A.66)–(A.67).²⁵

The stock short-selling costs are as in our baseline setting, but now we allow for a possibly stochastic shorting fee rate $\phi_t > 0$, since now the stock mean return μ_t is also stochastic and both enter additively in valuation formulas (e.g., (A.2) in the Appendix). Moreover, as we show later, this generality enables us to choose the stock shorting fee appropriately and obtain tractable corporate bond prices with structures comparable to those for options in our baseline setting with constant parameters. The following Lemma reports a corresponding short-selling fee implicit in the firm value.

Lemma 3 (Implied shorting fee in firm value). *In the economy with costly short-selling in the levered stock, the shorting fee of ϕ_t in the stock leads to an implied shorting fee of $\phi_A \equiv \phi_t \sigma_A / \sigma_t < \phi_t$ in the firm value.*

Lemma 3 reveals that the presence of the shorting fee in the (levered) stock leads to an implied shorting fee in the firm value. This is because the firm value and stock price are (perfectly) correlated, and the value of any payoff that depends on the firm value, such as a corporate bond, is obtained via a hedge portfolio in the stock. Hence, the stock short-selling

²⁵As Merton (1974) shows, in the absence of stock short-selling costs, the unique corporate bond and stock prices are determined by treating them as derivatives written on the firm value and following standard no-arbitrage valuation arguments. This results in the stock price taking the form of a Black-Scholes call option price, but with the firm value (39) being treated as the underlying security price and the face value of debt K as the strike price. Moreover, due to the one-factor structure of the Merton framework, the firm value and stock price are perfectly correlated. Therefore, for our purpose of corporate bond valuation, it is sufficient to rule out trading in the firm value, and instead take the stock as the only risky security available for trading which is also more plausible.

costs must also be reflected on the (risk-neutral) firm value dynamics used for valuation (see (A.71) in the Appendix). Moreover, the implied shorting fee in the firm value is less than that of the stock, $\phi_A < \phi_t$. This is intuitive since the firm value does not entirely come from the stock, but as the stock price approaches to the firm value, the firm shorting fee approaches to the stock shorting fee.

For tractability, we take the stock shorting fee to be proportional to the stock volatility, $\phi_t \propto \sigma_t$, so that the implied firm shorting fee is constant. This specification is consistent with the empirical evidence by Drechsler and Drechsler (2016), who report a strong positive relation between shorting fees and stocks' (idiosyncratic) volatility. Following steps similar to those in our baseline setting for options, we show that corporate bond prices fall within a range by no-arbitrage and then consider marketmakers as before.²⁶ The arrival rates of offsetting buy or sell (corporate bond) orders are now denoted by λ_{Db} , λ_{Ds} , respectively. We solve the model and obtain explicit closed-form solutions for the corporate bond bid and ask prices as reported in Proposition 7.

Proposition 7 (Corporate bond bid and ask prices). *In the economy with costly short-selling in the levered stock, the corporate bond bid and ask prices are given by*

$$D_t^{Bid} = (1 - w_{t,DBid}) D_t^M(\alpha\phi_A) + w_{t,DBid} D_t^M(\phi_A), \quad (41)$$

$$D_t^{Ask} = w_{t,DAsk} D_t^M(\alpha\phi_A) + (1 - w_{t,DAsk}) D_t^M(\phi_A), \quad (42)$$

where $D_t^M(q)$ denotes the standard Merton corporate bond price adjusted for the constant total payout yield q , and is given by

$$D_t^M(q) = A_t e^{-q(T-t)} \Phi(-d_1(q)) + K e^{-r(T-t)} \Phi(d_2(q)) \quad (43)$$

²⁶We acknowledge that the corporate bond market is mainly an over-the-counter dealership market and hence it may differ from the exchange-traded options market that we consider for our baseline analysis. Nevertheless, we believe our analysis here is valuable since not only it highlights the broader applicability of our methodology, but it also enables us to demonstrate one economic channel as to how stock short-selling costs can affect corporate bond bid and ask prices, as there is some evidence of this relation in the literature (see our discussion after Proposition 7).

with

$$d_1(q) = \frac{\ln(A_t/K) + \left(r - q + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}}, \quad \text{and} \quad d_2(q) = d_1(q) - \sigma_A\sqrt{T-t}. \quad (44)$$

The weights for the corporate bond bid and ask prices $w_{t,D^{Bid}}$ and $w_{t,D^{Ask}}$ are given by

$$w_{t,D^{Bid}} = \frac{\lambda_{Ds}}{\lambda_{Ds} + \lambda_{Db}} + \frac{\lambda_{Db}}{\lambda_{Ds} + \lambda_{Db}} e^{-(\lambda_{Ds} + \lambda_{Db})(T-t)}, \quad (45)$$

$$w_{t,D^{Ask}} = \frac{\lambda_{Db}}{\lambda_{Ds} + \lambda_{Db}} + \frac{\lambda_{Ds}}{\lambda_{Ds} + \lambda_{Db}} e^{-(\lambda_{Ds} + \lambda_{Db})(T-t)}. \quad (46)$$

Consequently, in the economy with costly short-selling in the levered stock,

- i) The marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs, i.e. $D_t^M(\phi_A) < D_t^{Bid} < D_t^{Ask} < D_t^M(\alpha\phi_A)$.
- ii) The corporate bond bid and ask prices are decreasing in the shorting fee ϕ_A , the partial lending α and its offsetting sell order arrival rate λ_{Ds} , while it is increasing in its offsetting buy order arrival rate λ_{Db} .
- iii) The corporate bond bid-ask spread is increasing in the shorting fee ϕ_A when $\alpha e^{(1-\alpha)\phi_A(T-t)} < \Phi(d_1(\phi_A)) / \Phi(d_1(\alpha\phi_A))$, while it is decreasing in the partial lending α and its offsetting order arrival rates λ_{Ds} and λ_{Db} .

Proposition 7 reveals that the structure of corporate bond bid and ask prices (41)–(42) are comparable to those in our baseline setting for calls and puts, but are now weighted-averages of Merton corporate bond prices.²⁷ This is because the corporate bond has an option-like payoff with its underlying being the firm value, which also has an implied shorting fee (Lemma 3). Since the underlying economic mechanism is as before, we obtain the result that marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs (property (i)), giving investors the incentives to trade with marketmakers.

We also show that the behavior of corporate bond prices with respect to the shorting fee, partial lending and offsetting order arrival rates are similar to the behavior of call option

²⁷It turns out that one could obtain the same corporate bond bid and ask prices (41)–(42) under an alternative formulation of introducing a shorting fee of ϕ_A for trading in the firm value A directly and determine the corporate bond payoff values via hedge portfolios in the firm value and the riskless bond.

prices in our baseline setting (properties (ii)–(iii)). This is because the short-selling costs enter into corporate bond prices through the hedge portfolio costs and proceeds, and both the corporate bond and the call option in our baseline setting require a long (short) stock position at all times in their hedge portfolios to hedge their respective short (long) payoffs. In particular, we show that higher stock short-selling costs lead to lower prices (and hence to higher yields) for corporate bonds, consistent with the empirical findings of Kecskes, Mansi, and Zhang (2013). Moreover, since we show that stock short-selling costs affect corporate bond bid-ask spreads, our results here may help identify the determinants of the liquidity of corporate bonds better. For instance, recently there has been much interest in understanding how the corporate bond market liquidity has been affected by post-crises regulatory changes such as Basel III and the Dodd-Frank Act, which includes the Volcker Rule that restricts certain proprietary trading activities of banks and their affiliates (e.g., Duffie, 2012; Bao, O’Hara, and Zhou, 2018; Trebbi and Xiao, 2017; Bessembinder, Jacobsen, Maxwell, and Venkataraman, 2017; Dick-Nielsen and Rossi, 2018). In particular, the evidence in Bao, O’Hara, and Zhou (2018) and Bessembinder, Jacobsen, Maxwell, and Venkataraman (2017) suggests that the corporate bond marketmakers regulated by the Volcker Rule decreased their marketmaking activity and now hedge their trades whenever possible so as to not have inventory risk. Therefore, our analysis would suggest that the effects of costly short-selling can be more pronounced on the bid and ask prices quoted by these regulated marketmakers.

8. Conclusion

In this paper, we provide an analysis of option prices in the presence of costly short-selling. Since standard no-arbitrage restrictions alone cannot determine option prices in such a setting, we introduce option marketmakers and obtain unique closed-form option bid and ask prices, which represent the marketmakers’ expected cost of hedging each option.

Consistently with empirical evidence, we find that a higher shorting fee leads to higher bid-ask spreads for typical options, higher put option implied volatilities, and higher apparent put-call parity violations. Moreover, we apply our model to the 2008 short-selling ban period and show that the option bid-ask spreads of banned stocks and their apparent put-call parity violations are higher than those for unbanned stocks, also consistently with empirical evidence. In a further application, we demonstrate that higher stock short-selling costs lead to lower corporate bond prices (and hence to higher yields), also consistently with evidence.

In addition to above implications, our model also generates several new testable predictions. Most notably, we show that (i) the call and put bid-ask spreads are decreasing in the partial lending, (ii) the option marketmakers' participation in the stock lending market is decreasing in the shorting fee for each call option sold, (iii) the effects of short-selling costs on option bid-ask spreads are more pronounced for relatively illiquid options, and (iv) the presence of stock short-selling costs generates bid-ask spreads for corporate bonds.

We extend our baseline analysis to a richer stochastic volatility setting and find that all our main results and insights remain equally valid. However, so as to not unnecessarily confound or further complicate the analysis, we do not consider other potential features of option marketmaking as discussed in Remark 2. We leave these considerations and the empirical tests of our new predictions for future research.

Appendix. Proofs

Proof of Lemma 1. To determine the option prices that admit no-arbitrage we consider the hedge portfolio, the continuously-traded self-financing portfolio with θ_t units in the underlying stock and β_t units in the riskless bond which perfectly hedges (offsets) the option payoff at its maturity.

First, suppose that this portfolio is always long in the stock, $\theta_t > 0$ for all $t \leq T$. In this case, the fraction α of the long position is lent to short-sellers. The hedge portfolio value V_t becomes

$$V_t = \beta_t B_t + \theta_t S_t = \beta_t B_t + (1 - \alpha) \theta_t S_t + \alpha \theta_t S_t, \quad (\text{A.1})$$

where the last term $\alpha \theta_t S_t$ is the total amount lent to short sellers, which in addition to the stock capital gains also earns the shorting fee ϕ . Hence, the dynamics of the hedge portfolio is given by

$$\begin{aligned} dV_t &= \beta_t dB_t + (1 - \alpha) \theta_t dS_t + \alpha \theta_t (dS_t + \phi S_t dt) \\ &= rV_t dt + (\mu - r + \alpha \phi) \theta_t S_t dt + \sigma \theta_t S_t d\omega_t, \end{aligned} \quad (\text{A.2})$$

where the second equality follows by substituting the bond and stock dynamics (1)–(2) and $\beta_t B_t$ from (A.1), and rearranging. We observe that (A.2) is the dynamics of the hedge portfolio in the Black-Scholes economy where the underlying stock pays a continuous dividend at a constant rate $\alpha\phi$. Therefore, standard valuation arguments (e.g., McDonald, 2012) yield the cost of the hedge portfolio that delivers the long call payoff $\max\{S_T - K, 0\}$ to be $V_t = C_t^{BS}(\alpha\phi)$, and the short put payoff $-\max\{K - S_T, 0\}$ to be $V_t = -P_t^{BS}(\alpha\phi)$, where a negative cost means proceeds, and $C_t^{BS}(q)$ and $P_t^{BS}(q)$ denote the standard Black-Scholes call and put prices adjusted for the constant dividend yield q and are given by (5)–(6). We also confirm our conjecture that the hedge portfolios are always long in the underlying stock by showing in these cases

$$\theta_t = \frac{\partial}{\partial S_t} C_t^{BS}(\alpha\phi) = e^{-\alpha\phi(T-t)} \Phi(d_1(\alpha\phi)) > 0, \quad (\text{A.3})$$

$$\theta_t = \frac{\partial}{\partial S_t} (-P_t^{BS}(\alpha\phi)) = e^{-\alpha\phi(T-t)} \Phi(-d_1(\alpha\phi)) > 0, \quad (\text{A.4})$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and $d_1(q)$ is as in (7).

Next, suppose that the hedge portfolio is always short in the stock, $\theta_t < 0$ for all $t \leq T$. In this case, the hedge portfolio value V_t becomes

$$V_t = \beta_t B_t + \theta_t S_t + M_t, \quad (\text{A.5})$$

where the last term M_t denotes the total amount collateralized, and hence cannot be invested in other securities, and is given by $M_t = -\theta_t S_t > 0$. For the short-seller this account earns the rebate rate $r - \phi$, implying its dynamics as $dM_t = (r - \phi) M_t dt$. Hence, the dynamics of the hedge portfolio is given by

$$\begin{aligned} dV_t &= \beta_t dB_t + \theta_t dS_t + dM_t \\ &= rV_t dt + (\mu - r + \phi) \theta_t S_t dt + \sigma \theta_t S_t d\omega_t, \end{aligned} \quad (\text{A.6})$$

where the second equality follows by substituting the bond and stock dynamics (1)–(2) and $\beta_t B_t$ from (A.5), and rearranging. This is the dynamics of the hedge portfolio in the Black-Scholes economy where the underlying stock pays a continuous dividend at a constant rate ϕ . Therefore, standard valuation arguments yield the cost of the hedge portfolio that delivers the short call payoff $-\max\{S_T - K, 0\}$ to be $V_t = -C_t^{BS}(\phi)$, and the long put payoff $\max\{K - S_T, 0\}$ to be $V_t = P_t^{BS}(\phi)$, where again a negative cost means proceeds. We also confirm our conjecture that the hedge portfolios are always short in the underlying stock by showing in these cases

$$\theta_t = \frac{\partial}{\partial S_t} \left(-C_t^{BS}(\phi) \right) = -e^{-\phi(T-t)} \Phi(d_1(\phi)) < 0, \quad (\text{A.7})$$

$$\theta_t = \frac{\partial}{\partial S_t} P_t^{BS}(\phi) = -e^{-\phi(T-t)} \Phi(-d_1(\phi)) < 0. \quad (\text{A.8})$$

Having determined the hedge portfolio costs and proceeds, we now show that option prices admit no-arbitrage if and only if the double inequalities (3)–(4) in Lemma 1 are satisfied. To see this for the call option, suppose by contradiction that the call option is trading at a price $C_t^{BS}(\alpha\phi) < C_t$. Then selling the call option (α) at the price C_t and forming the hedge portfolio at the cost $C_t^{BS}(\alpha\phi)$ would lead to a zero payoff at the option maturity date. However, this strategy has a positive initial profit $C_t - C_t^{BS}(\alpha\phi)$, hence this option price admits arbitrage. Now, suppose by contradiction that the call option is trading at a price $C_t < C_t^{BS}(\phi)$. Then buying the call option at the price C_t and forming the hedge portfolio by receiving $C_t^{BS}(\phi)$

would lead to a zero payoff at the option maturity date. However, this strategy has a positive initial profit $C_t^{BS}(\phi) - C_t$, hence this option price also admits arbitrage. On the other hand, if the call price satisfies the double inequality (3), then it admits no-arbitrage because selling or buying the option and perfectly hedging it at its maturity can at most lead to a zero initial profit. Going through the same steps as in the call option case shows that a put option price, P_t , admits no-arbitrage if and only if the double inequality (4) in Lemma 1 is satisfied. \square

Proof of Lemma 2. We first derive the expected cost of hedging representation for the call bid price in detail following the similar steps for the call ask price as discussed in Section 3, and then for the put prices relying on similar arguments. The call bid price at time t is set by the marketmaker such that buying a call option at the bid price C_t^{Bid} and forming the hedge portfolio and receiving the proceeds $C_t^{BS}(\phi)$ (Lemma 1) yields zero expected profit. If there is no offsetting call buy order by the maturity date T , the marketmaker's profit from this trade has a current value of

$$\Pi_{T,C^{Bid}} \equiv C_t^{BS}(\phi) - C_t^{Bid}. \quad (\text{A.9})$$

If an offsetting call buy order arrives at a subsequent time $u < T$, the marketmaker sells a call option at an ask price C_u^{Ask} , and liquidates the hedge portfolio at a value of $-C_u^{BS}(\phi)$. This leads to the marketmaker's profit from this trade having a current value of

$$\Pi_{u,C^{Bid}} \equiv C_t^{BS}(\phi) - C_t^{Bid} + V_t [C_u^{Ask} - C_u^{BS}(\phi)], \quad (\text{A.10})$$

where $V_t [C_u^{Ask} - C_u^{BS}(\phi)]$ is the current value of the time- u random payoff $C_u^{Ask} - C_u^{BS}(\phi)$ which is yet to be determined. By probability weighing (A.9)–(A.10) with the distribution function of the offsetting call buy order arrival time F_{Cb} , the marketmaker obtains its expected profit from buying of a call option at time t as

$$\begin{aligned} \Pi_{C^{Bid}} &\equiv \int_t^T \Pi_{u,C^{Bid}} dF_{Cb}(u) + \Pi_{T,C^{Bid}} \left(1 - \int_t^T dF_{Cb}(u)\right) \\ &= C_t^{BS}(\phi) - C_t^{Bid} + \int_t^T V_t [C_u^{Ask} - C_u^{BS}(\phi)] dF_{Cb}(u). \end{aligned} \quad (\text{A.11})$$

Finally, the marketmaker sets the call bid price so that the expected profit (A.11) is zero:

$$C_t^{Bid} = C_t^{BS}(\phi) + \int_t^T V_t [C_u^{Ask} - C_u^{BS}(\phi)] dF_{Cb}(u). \quad (\text{A.12})$$

Adding and subtracting $\int_t^T C_t^{BS}(\phi) dF_{Cb}(u)$ to the right hand side of the above equation gives the expected cost representation for the call bid price in (11).

For the expected cost of hedging representations for the put prices, going through the same steps as in the call option case yields the put bid and ask prices as

$$\begin{aligned} P_t^{Bid} &= P_t^{BS}(\alpha\phi) + \int_t^T V_t [P_u^{Ask} - P_u^{BS}(\alpha\phi)] dF_{Pb}(u), \\ P_t^{Ask} &= P_t^{BS}(\phi) - \int_t^T V_t [P_u^{BS}(\phi) - P_u^{Bid}] dF_{Ps}(u), \end{aligned} \quad (\text{A.13})$$

and adding and subtracting $\int_t^T P_t^{BS}(\alpha\phi) dF_{Pb}(u)$ and $\int_t^T P_t^{BS}(\phi) dF_{Ps}(u)$, respectively, to the right hand sides of the above equations give the expected cost representations for the put bid and ask prices in (13)–(14). \square

Proof of Proposition 1. To determine the call bid and ask prices, we first conjecture the functional forms for them. Then, using these functional forms we determine the current values of random payoffs in the expected cost of hedging representations for the prices in Lemma 2. Finally, by solving the resulting system of equations, we obtain the option prices in closed-form and verify our conjectured functional forms.

We conjecture that the call bid and ask prices take the forms

$$C_t^{Bid} = (1 - w_{t,C^{Bid}}) C_t^{BS}(\alpha\phi) + w_{t,C^{Bid}} C_t^{BS}(\phi), \quad (\text{A.14})$$

$$C_t^{Ask} = w_{t,C^{Ask}} C_t^{BS}(\alpha\phi) + (1 - w_{t,C^{Ask}}) C_t^{BS}(\phi), \quad (\text{A.15})$$

for all $t \leq T$ and the deterministic weight processes $w_{t,C^{Bid}}$, $w_{t,C^{Ask}}$ to be identified later.

Given our conjecture, the time- u random payoff of the call bid price (11) becomes

$$C_u^{Ask} - C_u^{BS}(\phi) = w_{u,C^{Ask}} C_u^{BS}(\alpha\phi) + w_{u,C^{Ask}} (-C_u^{BS}(\phi)). \quad (\text{A.16})$$

The current value of this random payoff is given by the amount required at time- t to form a

self-financing portfolio in the underlying stock and riskless bond to obtain this payoff at time $u \geq t$. For this, we consider two positions, where one is long and the other is short in the underlying stock for all $u \geq t$. The first position consists of $w_{u,C^{Ask}}$ units in the call option seller's hedge portfolio that is long in the stock (A.3) where $w_{u,C^{Ask}}$ is a positive constant. This position has a value of $w_{u,C^{Ask}}C_u^{BS}(\alpha\phi)$ at time- u with its current value given by

$$V_t \left[w_{u,C^{Ask}} C_u^{BS}(\alpha\phi) \right] = w_{u,C^{Ask}} C_t^{BS}(\alpha\phi), \quad (\text{A.17})$$

since this is the amount required at time- t for a self-financing portfolio to obtain the payoff $w_{u,C^{Ask}}C_u^{BS}(\alpha\phi)$ at time u . Similarly, the second position consists of $w_{u,C^{Ask}}$ units in the call option buyer's hedge portfolio that is short in the stock (A.7). This position has a value of $w_{u,C^{Ask}}(-C_u^{BS}(\phi))$ at time- u with its current value given by

$$V_t \left[w_{u,C^{Ask}} \left(-C_u^{BS}(\phi) \right) \right] = w_{u,C^{Ask}} \left(-C_t^{BS}(\phi) \right). \quad (\text{A.18})$$

Summing (A.17) and (A.18) gives the current value of the random payoff (A.16) as

$$V_t \left[C_u^{Ask} - C_u^{BS}(\phi) \right] = w_{u,C^{Ask}} C_t^{BS}(\alpha\phi) + w_{u,C^{Ask}} \left(-C_t^{BS}(\phi) \right). \quad (\text{A.19})$$

Substituting this into the call bid price representation (11) and rearranging gives

$$C_t^{Bid} = \left[\int_t^T w_{u,C^{Ask}} dF_{C_b}(u) \right] C_t^{BS}(\alpha\phi) + \left[1 - \int_t^T w_{u,C^{Ask}} dF_{C_b}(u) \right] C_t^{BS}(\phi). \quad (\text{A.20})$$

Going through the same steps as above, we also obtain the call ask price as

$$C_t^{Ask} = \left[1 - \int_t^T w_{u,C^{Bid}} dF_{C_s}(u) \right] C_t^{BS}(\alpha\phi) + \left[\int_t^T w_{u,C^{Bid}} dF_{C_s}(u) \right] C_t^{BS}(\phi). \quad (\text{A.21})$$

We next match our conjectured forms (A.14)–(A.15) with the derived expressions in (A.20)–(A.21) and obtain the system for call weights as

$$w_{t,C^{Bid}} = 1 - \int_t^T w_{u,C^{Ask}} dF_{C_b}(u) = 1 - \int_t^T w_{u,C^{Ask}} \lambda_{C_b} e^{-\lambda_{C_b}(u-t)} du, \quad (\text{A.22})$$

$$w_{t,C^{Ask}} = 1 - \int_t^T w_{u,C^{Bid}} dF_{C_s}(u) = 1 - \int_t^T w_{u,C^{Bid}} \lambda_{C_s} e^{-\lambda_{C_s}(u-t)} du. \quad (\text{A.23})$$

It is straightforward to check that the weights (19)–(20) in Proposition 1 solve the above system by substituting them into (A.22)–(A.23) and integrating simple exponential functions.²⁸ The deterministic nature of the derived weights verify that the call bid and ask prices indeed are as in (15)–(16) with the weights (19)–(20).

For the put bid and ask prices, we conjecture the forms

$$\begin{aligned} P_t^{Bid} &= (1 - w_{t, PBid}) P_t^{BS}(\phi) + w_{t, PBid} P_t^{BS}(\alpha\phi), \\ P_t^{Ask} &= w_{t, PAsk} P_t^{BS}(\phi) + (1 - w_{t, PAsk}) P_t^{BS}(\alpha\phi), \end{aligned} \quad (\text{A.24})$$

for all $t \leq T$ and the deterministic weight processes $w_{t, PBid}$, $w_{t, PAsk}$. Going through the same steps as in the call option case verify that the put bid and ask prices indeed are as in (17)–(18) with the weights (21)–(22).

Property that the marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs follows immediately from the weighted average forms of prices (15)–(18) with strictly positive weights (19)–(22) that lie in the interval $(0, 1)$. \square

Proof of Proposition 2. Property (i) that the call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the shorting fee follows from the fact that these prices are weighted-averages of Black-Scholes prices, which are both decreasing (increasing) in the shorting fee for the call (put)

$$\frac{\partial}{\partial \phi} C_t^{BS}(\phi) < 0, \quad \frac{\partial}{\partial \phi} C_t^{BS}(\alpha\phi) < 0, \quad \frac{\partial}{\partial \phi} P_t^{BS}(\phi) > 0, \quad \frac{\partial}{\partial \phi} P_t^{BS}(\alpha\phi) > 0, \quad (\text{A.25})$$

which follow from the partial derivatives of the standard Black-Scholes prices with respect to the dividend yield

$$\frac{\partial}{\partial q} C_t^{BS}(q) = -(T-t) S_t e^{-q(T-t)} \Phi(d_1(q)) < 0, \quad (\text{A.26})$$

$$\frac{\partial}{\partial q} P_t^{BS}(q) = (T-t) S_t e^{-q(T-t)} \Phi(-d_1(q)) > 0, \quad (\text{A.27})$$

²⁸Alternatively, the weights (19)–(20) can also be derived directly by differentiating the system (A.22)–(A.23) using the Leibniz integral rule, and solving the resulting system of two linear first-order differential equations simultaneously.

along with the fact that their weights do not depend on the shorting fee.

To prove property (ii) that both the call and put bid-ask spreads are increasing in the shorting fee for the given condition, we first obtain the bid-ask spread using (15)–(18) as

$$C_t^{Ask} - C_t^{Bid} = (w_{t,C^{Ask}} + w_{t,C^{Bid}} - 1) [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)], \quad (\text{A.28})$$

$$P_t^{Ask} - P_t^{Bid} = (w_{t,P^{Ask}} + w_{t,P^{Bid}} - 1) [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)]. \quad (\text{A.29})$$

Since the weights do not depend on the shorting fee ϕ , the bid-ask spreads are increasing in the shorting fee if and only if

$$\frac{\partial}{\partial\phi} C_t^{BS}(\alpha\phi) > \frac{\partial}{\partial\phi} C_t^{BS}(\phi), \quad \frac{\partial}{\partial\phi} P_t^{BS}(\phi) > \frac{\partial}{\partial\phi} P_t^{BS}(\alpha\phi). \quad (\text{A.30})$$

By using (A.26)–(A.27), we obtain these conditions as

$$\alpha e^{-\alpha\phi(T-t)} \Phi(d_1(\alpha\phi)) < e^{-\phi(T-t)} \Phi(d_1(\phi)), \quad (\text{A.31})$$

$$\alpha e^{-\alpha\phi(T-t)} \Phi(-d_1(\alpha\phi)) < e^{-\phi(T-t)} \Phi(-d_1(\phi)). \quad (\text{A.32})$$

After rearranging the first condition gives the condition in property (ii) which is also a sufficient condition for the put since $\Phi(d_1(\phi))/\Phi(d_1(\alpha\phi)) < \Phi(-d_1(\phi))/\Phi(-d_1(\alpha\phi))$.

Property (iii) that the implied stock bid and ask prices are decreasing in the shorting fee follows immediately from differentiating their definitions (23)–(24) and employing the results in property (i) that the call bid and ask prices are decreasing, put bid and ask prices are increasing in the shorting fee, yielding

$$\frac{\partial}{\partial\phi} \tilde{S}_t^{Bid} = \frac{\partial}{\partial\phi} C_t^{Bid} - \frac{\partial}{\partial\phi} P_t^{Ask} < 0, \quad \frac{\partial}{\partial\phi} \tilde{S}_t^{Ask} = \frac{\partial}{\partial\phi} C_t^{Ask} - \frac{\partial}{\partial\phi} P_t^{Bid} < 0. \quad (\text{A.33})$$

Property (iv) that the option marketmakers' stock lending for each call option sold is decreasing in the shorting fee follows from differentiating the amount they lend, which is α times the number of (long) stock shares needed for hedging (A.3). The result that the option marketmakers' stock lending for each put option bought is increasing in the shorting fee for the given condition follows from differentiating the amount they lend, α times (A.4). \square

Proof of Proposition 3. Property (i) that the call bid and ask prices are decreasing,

while the put bid and ask prices are increasing in the partial lending follows from the fact that these prices are weighted-averages of Black-Scholes prices, which are either decreasing (increasing) or do not depend on the partial lending for the call (put)

$$\frac{\partial}{\partial \alpha} C_t^{BS}(\alpha\phi) < 0, \quad \frac{\partial}{\partial \alpha} P_t^{BS}(\alpha\phi) > 0, \quad \frac{\partial}{\partial \alpha} C_t^{BS}(\phi) = 0, \quad \frac{\partial}{\partial \alpha} P_t^{BS}(\phi) = 0, \quad (\text{A.34})$$

due to (A.26)–(A.27), along with the fact that their weights do not depend on the partial lending.

Property (ii) that both the call and put bid-ask spreads are decreasing in the partial lending follows immediately from differentiating the call and put bid-ask spreads (A.28)–(A.29) with respect to the partial lending α . Since the weights do not depend on the partial lending, the bid-ask spreads are decreasing in the partial lending if and only if

$$\frac{\partial}{\partial \alpha} C_t^{BS}(\alpha\phi) < \frac{\partial}{\partial \alpha} C_t^{BS}(\phi), \quad \frac{\partial}{\partial \alpha} P_t^{BS}(\phi) < \frac{\partial}{\partial \alpha} P_t^{BS}(\alpha\phi), \quad (\text{A.35})$$

which always hold as (A.34) illustrates. \square

Proof of Proposition 4. To determine the effects of the offsetting order arrival rates on option prices, we first derive their effects on the weights (19)–(22). The effects of the offsetting call sell and buy order arrival rates on the call weights are given by

$$\frac{\partial}{\partial \lambda_{C_s}} w_{t, C^{Bid}} = \frac{\lambda_{C_b}}{(\lambda_{C_s} + \lambda_{C_b})^2} \left(1 - [1 + (\lambda_{C_s} + \lambda_{C_b})(T - t)] e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} \right) > 0, \quad (\text{A.36})$$

$$\frac{\partial}{\partial \lambda_{C_s}} w_{t, C^{Ask}} = -\frac{\lambda_{C_b}}{(\lambda_{C_s} + \lambda_{C_b})^2} \left(1 - e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} \right) - \frac{\lambda_{C_s}(T-t)}{\lambda_{C_s} + \lambda_{C_b}} e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} < 0, \quad (\text{A.37})$$

$$\frac{\partial}{\partial \lambda_{C_b}} w_{t, C^{Bid}} = -\frac{\lambda_{C_s}}{(\lambda_{C_s} + \lambda_{C_b})^2} \left(1 - e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} \right) - \frac{\lambda_{C_b}(T-t)}{\lambda_{C_s} + \lambda_{C_b}} e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} < 0, \quad (\text{A.38})$$

$$\frac{\partial}{\partial \lambda_{C_b}} w_{t, C^{Ask}} = \frac{\lambda_{C_s}}{(\lambda_{C_s} + \lambda_{C_b})^2} \left(1 - [1 + (\lambda_{C_s} + \lambda_{C_b})(T - t)] e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} \right) > 0, \quad (\text{A.39})$$

where the signs of (A.36) and (A.39) follow from the fact that $1 - (1 + x)e^{-x} > 0$ for all $x > 0$. Similarly, the effects of the offsetting put buy and sell order arrival rates on the put weights are obtained immediately by substituting “put” for “call” in (A.36)–(A.39) as they

have the same forms in (19)–(22), which yields

$$\frac{\partial}{\partial \lambda_{P_s}} w_{t, P^{Bid}} > 0, \quad \frac{\partial}{\partial \lambda_{P_s}} w_{t, P^{Ask}} < 0, \quad \frac{\partial}{\partial \lambda_{P_b}} w_{t, P^{Bid}} < 0, \quad \frac{\partial}{\partial \lambda_{P_b}} w_{t, P^{Ask}} > 0. \quad (\text{A.40})$$

Hence, property (i) that the call and put bid and ask prices are decreasing in their offsetting sell order arrival rates, while they are increasing in their offsetting buy order arrival rates follows by substituting (A.36)–(A.39) into

$$\begin{aligned} \frac{\partial}{\partial \lambda_{C_s}} C_t^{Bid} &= - [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)] \frac{\partial}{\partial \lambda_{C_s}} w_{t, C^{Bid}} < 0, \\ \frac{\partial}{\partial \lambda_{C_s}} C_t^{Ask} &= [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)] \frac{\partial}{\partial \lambda_{C_s}} w_{t, C^{Ask}} < 0, \\ \frac{\partial}{\partial \lambda_{C_b}} C_t^{Bid} &= - [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)] \frac{\partial}{\partial \lambda_{C_b}} w_{t, C^{Bid}} > 0, \\ \frac{\partial}{\partial \lambda_{C_b}} C_t^{Ask} &= [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)] \frac{\partial}{\partial \lambda_{C_b}} w_{t, C^{Ask}} > 0, \end{aligned} \quad (\text{A.41})$$

and the respective inequalities in (A.40) into

$$\begin{aligned} \frac{\partial}{\partial \lambda_{P_s}} P_t^{Bid} &= - [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)] \frac{\partial}{\partial \lambda_{P_s}} w_{t, P^{Bid}} < 0, \\ \frac{\partial}{\partial \lambda_{P_s}} P_t^{Ask} &= [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)] \frac{\partial}{\partial \lambda_{P_s}} w_{t, P^{Ask}} < 0, \\ \frac{\partial}{\partial \lambda_{P_b}} P_t^{Bid} &= - [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)] \frac{\partial}{\partial \lambda_{P_b}} w_{t, P^{Bid}} > 0, \\ \frac{\partial}{\partial \lambda_{P_b}} P_t^{Ask} &= [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)] \frac{\partial}{\partial \lambda_{P_b}} w_{t, P^{Ask}} > 0. \end{aligned} \quad (\text{A.42})$$

Property (ii) that both the call and put bid-ask spreads are decreasing in the offsetting order arrival rates follows immediately from differentiating the call and put bid-ask spreads

(A.28)–(A.29) with respect to the arrival rates and obtain

$$\begin{aligned}
\frac{\partial}{\partial \lambda_{C_s}} (C_t^{Ask} - C_t^{Bid}) &= - [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)] (T-t) e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} < 0, \\
\frac{\partial}{\partial \lambda_{C_b}} (C_t^{Ask} - C_t^{Bid}) &= - [C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)] (T-t) e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)} < 0, \\
\frac{\partial}{\partial \lambda_{P_s}} (P_t^{Ask} - P_t^{Bid}) &= - [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)] (T-t) e^{-(\lambda_{P_s} + \lambda_{P_b})(T-t)} < 0, \\
\frac{\partial}{\partial \lambda_{P_b}} (P_t^{Ask} - P_t^{Bid}) &= - [P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)] (T-t) e^{-(\lambda_{P_s} + \lambda_{P_b})(T-t)} < 0. \quad (\text{A.43})
\end{aligned}$$

Property (iii) that the effects of the shorting fee on the call and put bid-ask spreads are decreasing in the offsetting order arrival rates follows immediately from differentiating the call bid-ask spreads (A.28)–(A.29) with respect to the arrival rates after substituting the fact

$$w_{t,C^{Ask}} + w_{t,C^{Bid}} - 1 = e^{-(\lambda_{C_s} + \lambda_{C_b})(T-t)}, \quad (\text{A.44})$$

and obtain

$$\begin{aligned}
\frac{\partial}{\partial \lambda_{C_s}} \frac{\partial}{\partial \phi} (C_t^{Ask} - C_t^{Bid}) &= -(T-t) \frac{\partial}{\partial \phi} (C_t^{Ask} - C_t^{Bid}), \\
\frac{\partial}{\partial \lambda_{C_b}} \frac{\partial}{\partial \phi} (C_t^{Ask} - C_t^{Bid}) &= -(T-t) \frac{\partial}{\partial \phi} (C_t^{Ask} - C_t^{Bid}), \\
\frac{\partial}{\partial \lambda_{P_s}} \frac{\partial}{\partial \phi} (P_t^{Ask} - P_t^{Bid}) &= -(T-t) \frac{\partial}{\partial \phi} (P_t^{Ask} - P_t^{Bid}), \\
\frac{\partial}{\partial \lambda_{P_b}} \frac{\partial}{\partial \phi} (P_t^{Ask} - P_t^{Bid}) &= -(T-t) \frac{\partial}{\partial \phi} (P_t^{Ask} - P_t^{Bid}). \quad (\text{A.45})
\end{aligned}$$

Going through similar steps also gives the property that the effects of the partial lending on the call and put bid-ask spreads are decreasing in the offsetting order arrival rates. \square

Proof of Proposition 5. Property (i) that the call bid and ask prices of banned stocks are lower, while the put bid and ask prices of banned stocks are higher than those of unbanned stocks follows by comparing the call bid and ask prices of banned stocks

$$C_{t,Ban}^{Bid} = (1 - w_{t,C^{Bid}}) C_t^{BS}(\alpha\phi) + w_{t,C^{Bid}} C_t^{BS}(2\phi), \quad (\text{A.46})$$

$$C_{t,Ban}^{Ask} = w_{t,C^{Ask}} C_t^{BS}(\alpha\phi) + (1 - w_{t,C^{Ask}}) C_t^{BS}(2\phi), \quad (\text{A.47})$$

and the put bid and ask prices of banned stocks

$$P_{t,Ban}^{Bid} = (1 - w_{t,P^{Bid}}) P_t^{BS}(2\phi) + w_{t,P^{Bid}} P_t^{BS}(\alpha\phi), \quad (\text{A.48})$$

$$P_{t,Ban}^{Ask} = w_{t,P^{Ask}} P_t^{BS}(2\phi) + (1 - w_{t,P^{Ask}}) P_t^{BS}(\alpha\phi), \quad (\text{A.49})$$

with the call and put bid and ask prices of unbanned stocks in (15)–(18) along with the facts that $C_t^{BS}(2\phi) < C_t^{BS}(\phi)$ and $P_t^{BS}(\phi) < P_t^{BS}(2\phi)$.

Property (ii) that both the call and put bid-ask spreads of banned stocks are higher than those of unbanned stocks follows immediately by comparing the option prices of banned stocks (A.46)–(A.49) with those of unbanned stocks (15)–(18) along with the facts that $C_t^{BS}(2\phi) < C_t^{BS}(\phi)$ and $P_t^{BS}(\phi) < P_t^{BS}(2\phi)$.

Property (iii) that the implied stock bid and ask prices of banned stocks are lower than those of unbanned stocks follows immediately from the definitions of implied stock prices (23)–(24) along with property (i) that the call bid and ask prices of banned stocks are lower, while the put bid and ask prices of banned stocks are higher than those of unbanned stocks.

Property (iv) that the call bid price decreases more than the ask price, while the put ask price increases more than the bid price of banned stocks follows by observing this property

$$C_{t,Ban}^{Bid} - C_t^{Bid} < C_{t,Ban}^{Ask} - C_t^{Ask}, \quad P_{t,Ban}^{Bid} - P_t^{Bid} < P_{t,Ban}^{Ask} - P_t^{Ask}, \quad (\text{A.50})$$

being equivalent to property (ii) that both the call and put bid-ask spreads of banned stocks are higher than those of unbanned stocks, after rearranging. \square

Lemma A1 (Hedging errors in the stochastic volatility economy). *In the stochastic volatility economy with costly short-selling, the cumulative hedging errors over the time interval $[t, u]$, $t \leq u \leq T$, of a delta-hedged portfolio, which is initiated at time t to offset an option payoff at its maturity T , has a time- t value of zero.*

Proof of Lemma A1. We first consider the hedging errors associated with a delta-hedged portfolio (in the underlying stock and riskless bond) that delivers the long call payoff at its maturity, $\max\{S_T - K, 0\}$. We begin by noting that in the stochastic volatility economy of Heston (1993), when the underlying stock pays a constant dividend yield q , the

market value of the call payoff is given by a function $C(S, v, t)$, which satisfies the PDE

$$rC - (r - q)S \frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} = [\kappa_\nu (\bar{\nu} - \nu_t) - \lambda_\nu \nu_t] \frac{\partial C}{\partial \nu} + \frac{1}{2} S^2 v \frac{\partial^2 C}{\partial S^2} + \rho \sigma_\nu S v \frac{\partial^2 C}{\partial S \partial \nu} + \frac{1}{2} \sigma_\nu^2 v \frac{\partial^2 C}{\partial \nu^2}, \quad (\text{A.51})$$

with the appropriate boundary conditions. The first square bracket term on the right hand side is the drift term of the return variance process under the risk-neutral measure:

$$d\nu_t = [\kappa_\nu (\bar{\nu} - \nu_t) - \lambda_\nu \nu_t] dt + \sigma_\nu \sqrt{\nu_t} d\omega_{\nu t}^*, \quad (\text{A.52})$$

where $d\omega_{\nu t}^* \equiv d\omega_{\nu t} + (\lambda_\nu \sqrt{\nu_t} / \sigma_\nu) dt$ is a Brownian motion under this measure.

Now, consider the delta-hedged portfolio with β_t units in the riskless bond and θ_t units in the stock, and suppose that this portfolio is always long in the stock $\theta_t > 0$. In this case, the fraction α of the long position is lent to short-sellers. The value of the hedge portfolio becomes as in (A.1), with the change in its value over dt is given by

$$dV_t = \beta_t dB_t + (1 - \alpha) \theta_t dS_t + \alpha \theta_t (dS_t + \phi S_t dt) = [r\beta_t B_t + \alpha \phi \theta_t S_t] dt + \theta_t dS_t. \quad (\text{A.53})$$

Over the same interval, the change in the value of the call payoff is given by applying Itô's Lemma to $dC(S_t, \nu_t, t) = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS_t + dQ_t$, where we have defined

$$dQ_t \equiv \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS_t dS_t + \frac{\partial^2 C}{\partial S \partial \nu} dS_t d\nu_t + \frac{\partial C}{\partial \nu} d\nu_t + \frac{1}{2} \frac{\partial^2 C}{\partial \nu^2} d\nu_t d\nu_t. \quad (\text{A.54})$$

The difference between the change in the value of the portfolio and the value of the call payoff is the hedging error given by:

$$d\epsilon_t \equiv dV_t - dC_t = \left[r\beta_t B_t + \alpha \phi \theta_t S_t - \frac{\partial C}{\partial t} \right] dt + \left[\theta_t - \frac{\partial C}{\partial S} \right] dS_t - dQ_t. \quad (\text{A.55})$$

We choose the delta-hedge portfolio as $\theta_t = \frac{\partial C}{\partial S}$ and $\beta_t B_t = C - \frac{\partial C}{\partial S} S$ so that the delta-hedge portfolio tracks the value of the call payoff at all times and delivers the option payoff at time T . However, we note that the delta-hedge portfolio is not self-financing (since $dV_t \neq dC(S_t, \nu_t, t)$) with the hedging error $d\epsilon_t$ indicating the capital injection/withdrawal over dt to ensure this portfolio maintains the value of the call payoff. Substituting the delta-hedge portfolio into (A.55) gives $d\epsilon_t = \left[rC - (r - \alpha \phi) S \frac{\partial C}{\partial S} - \frac{\partial C}{\partial t} \right] dt - dQ_t$. Moreover,

substituting the Heston PDE (A.51) with the effective dividend yield $q = \alpha\phi$ (since $\alpha\phi$ is the rate a long stock position earns over dt) for the first term yields

$$d\epsilon_t = \left[[\kappa_\nu (\bar{\nu} - \nu_t) - \lambda_\nu \nu_t] \frac{\partial C}{\partial \nu} + \frac{1}{2} S^2 v \frac{\partial^2 C}{\partial S^2} + \rho \sigma_\nu S v \frac{\partial^2 C}{\partial S \partial v} + \frac{1}{2} \sigma_\nu^2 v \frac{\partial^2 C}{\partial v^2} \right] dt - dQ_t, \quad (\text{A.56})$$

which after substituting the quadratic covariation terms $dS_t dS_t = S_t^2 \nu_t dt$, $dS_t d\nu_t = \rho \sigma_\nu S_t \nu_t dt$, and $d\nu_t d\nu_t = \sigma_\nu^2 \nu_t dt$, and using the dynamics (A.52) simply becomes $d\epsilon_t = -\sigma_\nu \sqrt{\nu_t} (\partial C / \partial \nu) d\omega_{\nu_t}^*$. Therefore, we obtain the cumulative hedging error during the time interval $[t, u]$ as

$$\int_t^u d\epsilon_s = - \int_t^u \sigma_\nu \sqrt{\nu_s} \frac{\partial}{\partial \nu} C(S_s, \nu_s, s) d\omega_{\nu_s}^*, \quad (\text{A.57})$$

which implies the expectation $E_t^* \left[\int_t^u d\epsilon_u \right] = 0$ under the risk-neutral measure, and also $V_t \left[\int_t^T d\epsilon_u \right] = 0$ after discounting.²⁹ Hence, we conclude that forming the delta-hedge portfolio with the amount $C(S_t, \nu_t, t) = C_t^H(\alpha\phi)$ delivers the long call payoff $\max\{S_T - K, 0\}$ at its maturity T , with the associated cumulative hedging error having a current value of zero, $V_t \left[\int_t^T d\epsilon_u \right] = 0$. Here, $C_t^H(q)$ denotes the standard Heston call price adjusted for the constant dividend yield q and is given by (31). Lastly, we confirm that the delta-hedge portfolio is indeed long in the stock by observing $\theta_t = \partial C_t^H(\alpha\phi) / \partial S_t = e^{-\alpha\phi(T-t)} \Psi_1(\alpha\phi) > 0$.

Similarly, by following the above steps with $q = \phi$ (since ϕ is the rate a short stock position pays over dt), one can show that the delta-hedge portfolio with the amount $-C_t^H(\phi)$ delivers the short call payoff $-\max\{S_T - K, 0\}$ at its maturity T , with the associated cumulative hedging error having a current value of zero. We again confirm that the delta-hedge portfolio is short in the stock by observing $\theta_t = \partial(-C_t^H(\phi)) / \partial S_t = -e^{-\phi(T-t)} \Psi_1(\phi) < 0$. Moreover, by following the same steps, one can also show that the delta-hedge portfolio with the amount $P_t^H(\phi)$ delivers the long put payoff, $\max\{K - S_T, 0\}$, and the amount $-P_t^H(\alpha\phi)$ delivers the short put payoff, $-\max\{K - S_T, 0\}$. Here, $P_t^H(q)$ denotes the standard Heston put price adjusted for the constant dividend yield q and is given by (32). \square

Proof of Proposition 6. The option bid and ask prices are determined by following similar steps to those for the baseline setting, and using the delta-hedge portfolio costs that

²⁹We note that the cumulative hedging error we obtain in (A.57) is identical to the “delta-hedged gains” of Bakshi and Kapadia (2003, eq. (14)) which are presented under the original measure.

are derived in the proof of Lemma A1 above. For instance, the marketmaker sets the time- t call ask price so that by selling a call option at the ask price C_t^{Ask} and forming the delta-hedge portfolio at a cost $C_t^H(\alpha\phi)$ yields zero expected profit. If there is no offsetting call sell order by option maturity T , the option is delta-hedged via the hedge portfolio that is not self-financing but with a cumulative hedging error of $\int_t^T d\epsilon_s$, which has a current value of zero (Lemma A1), and hence the marketmaker's profit from this trade has a current value of $C_t^{Ask} - C_t^H(\alpha\phi)$. However, if an offsetting call sell order arrives at a subsequent time u , then the marketmaker buys a call option at a bid price C_u^{Bid} , and liquidates the hedge portfolio at a value of $C_u^H(\alpha\phi)$ while having incurred a cumulative hedging error of $\int_t^u d\epsilon_s$. In this case the option is perfectly hedged via the offsetting order, and since the cumulative hedging error has a value of zero, the marketmaker's profit from this trade has a value of $C_t^{Ask} - C_t^H(\alpha\phi) + V_t [C_u^H(\alpha\phi) - C_u^{Bid}]$. By probability weighing these profits, the marketmaker obtains its expected profit, and by setting the expected profit to zero it backs out the call ask price as $C_t^{Ask} = C_t^H(\alpha\phi) - \int_t^T V_t [C_u^H(\alpha\phi) - C_u^{Bid}] dF_{C_s}(u)$. Following similar steps to those in the proof of Lemma 2 for the baseline setting, also leads to the call bid price as $C_t^{Bid} = C_t^H(\phi) + \int_t^T V_t [C_u^{Ask} - C_u^H(\phi)] dF_{C_b}(u)$, and put bid and ask prices as

$$P_t^{Bid} = P_t^H(\alpha\phi) + \int_t^T V_t [P_u^{Ask} - P_u^H(\alpha\phi)] dF_{P_b}(u), \quad (\text{A.58})$$

$$P_t^{Ask} = P_t^H(\phi) - \int_t^T V_t [P_u^H(\phi) - P_u^{Bid}] dF_{P_s}(u). \quad (\text{A.59})$$

Since the above representations have similar forms to those in our baseline setting (with the only difference being the terms $C_t^{BS}(\cdot)$ and $P_t^{BS}(\cdot)$ are replaced with $C_t^H(\cdot)$ and $P_t^H(\cdot)$), going through the same steps as in the proof of Proposition 1 yields the option bid and ask prices (27)–(30) in the stochastic volatility economy.

Property (i), which states that the marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs, follows immediately from the weighted average forms of prices (27)–(30) with strictly positive weights (19)–(22) that lie in the interval $(0, 1)$. Property (ii), which states that all the properties on the effects of the shorting fee, partial lending, and offsetting order arrival rates as stated in Propositions 2–4 of Section 4 in our baseline setting hold, follows from the same steps in the corresponding proofs with the fact that the signs of the partial derivatives $\partial C_t^H(q) / \partial q = -(T-t) S_t e^{-q(T-t)} \Psi_1(q) < 0$, and $\partial P_t^H(q) / \partial q = (T-t) S_t e^{-q(T-t)} (1 - \Psi_1(q)) > 0$, are as in the baseline setting. \square

Proof of statements in Remark 4. We first determine the price of the variance swap with payoff $X_T = \left(\frac{1}{T} \int_0^T \nu_t dt - k\right) N$ for the notional amount N , and the constant variance strike k that is chosen so that the variance swap has zero value at time 0. Standard valuation arguments give the time- t value of the variance swap, denoted by X_t , as $X_t = e^{-r(T-t)} \mathbb{E}_t^* [X_T]$, where the expectation is taken under the risk-neutral measure, under which the variance process ν has the dynamics (A.52). Let $\kappa_\nu^* = \kappa_\nu + \lambda_\nu$ and $\bar{\nu}^* = \bar{\nu} \kappa_\nu / \kappa_\nu^*$, then we have $\mathbb{E}_t^* [\nu_u] = \bar{\nu}^* - (\bar{\nu}^* - \nu_t) e^{-\kappa_\nu^*(u-t)}$, and the variance swap price becomes

$$X_t = N e^{-r(T-t)} \left[\frac{1}{T} \int_0^t \nu_u du - k + \bar{\nu}^* (T-t) \frac{1}{T} - (\bar{\nu}^* - \nu_t) \frac{1 - e^{-\kappa_\nu^*(T-t)}}{\kappa_\nu^* T} \right], \quad (\text{A.60})$$

implying the price dynamics for the variance swap as

$$dX_t = (rX_t + g_t \lambda_\nu \nu_t) dt + g_t \sigma_\nu \sqrt{\nu_t} d\omega_{\nu t}, \quad (\text{A.61})$$

$$g_t = N e^{-r(T-t)} \frac{1 - e^{-(\kappa_\nu + \lambda_\nu)(T-t)}}{\kappa_\nu + \lambda_\nu}. \quad (\text{A.62})$$

We next follow the similar steps to those in the proof of Lemma 1 for the baseline setting and find the cost and proceeds of the perfect hedge portfolios. We again first consider the call option seller's payoff $-\max\{S_T - K, 0\}$. The portfolio that perfectly hedges this payoff at its maturity must have $V_T = \max\{S_T - K, 0\}$. To determine the hedge portfolio cost V_t for all $t < T$, we conjecture that the hedge portfolio is always long in the underlying stock, $\theta_t > 0$ for all $t \leq T$. In this case, the fraction α of the long position is lent to short-sellers. We decompose the cost of the hedge portfolio as

$$V_t = \beta_t B_t + \gamma_t X_t + \theta_t S_t = \beta_t B_t + \gamma_t X_t + (1 - \alpha) \theta_t S_t + \alpha \theta_t S_t, \quad (\text{A.63})$$

where now γ_t is the number of units in the variance swap. The dynamics of the (self-financing) hedge portfolio is given by

$$\begin{aligned} dV_t &= \beta_t dB_t + \gamma_t dX_t + (1 - \alpha) \theta_t dS_t + \alpha \theta_t (dS_t + \phi S_t dt) \\ &= rV_t dt + \gamma_t g_t \lambda_\nu \nu_t dt + (\mu - r + \alpha \phi) \theta_t S_t dt + \theta_t \sqrt{\nu_t} S_t d\omega_t + \gamma_t g_t \sigma_\nu \sqrt{\nu_t} d\omega_{\nu t}. \end{aligned} \quad (\text{A.64})$$

After discounting, and substituting $d\omega_t^* \equiv d\omega_t + ((\mu - r + \alpha \phi) / \sqrt{\nu_t}) dt$ and $d\omega_{\nu t}^* \equiv d\omega_{\nu t} + (\lambda_\nu \sqrt{\nu_t} / \sigma_\nu) dt$, we obtain that the discounted portfolio value is a martingale under this (risk-

neutral) measure, and hence its value satisfies the formula $V_t = e^{-r(T-t)}\mathbf{E}_t^*[\max\{S_T - K, 0\}]$. Since the stock has the risk-neutral dynamics $dS_t = S_t[(r - \alpha\phi)dt + \sqrt{v_t}d\omega_t^*]$, the cost of the hedge portfolio becomes the call option value in the Heston economy where the underlying stock pays a continuous dividend at a constant rate $\alpha\phi$, that is, $V_t = C_t^H(\alpha\phi)$.

Similarly, by following the above steps, one can show that the cost of the hedge portfolio that delivers i) the short call payoff is $-C_t^H(\phi)$, ii) the long put payoff is $P_t^H(\phi)$, iii) the short put payoff is $-P_t^H(\alpha\phi)$. Finally, since we have the same hedge portfolio costs, following the similar steps to those in the proof of Proposition 6, we obtain the same option bid and ask prices as in (27)–(30) for this stochastic volatility economy with perfect hedging. \square

Proof of Lemma 3. To determine the implied shorting fee in the firm value, we derive the (risk-neutral) firm value dynamics that is used for valuation. Towards that, we note that in this economy, as also in the Merton (1974) model, the (levered) stock price S_t , its stochastic mean return μ_t and return volatility σ_t are given by

$$S_t = A_t\Phi(d_{1A}) - Ke^{-r(T-t)}\Phi(d_{2A}), \quad (\text{A.65})$$

$$\mu_t = r + (\mu_A - r)\frac{A_t}{S_t}\Phi(d_{1A}), \quad (\text{A.66})$$

$$\sigma_t = \sigma_A\frac{A_t}{S_t}\Phi(d_{1A}), \quad (\text{A.67})$$

where

$$d_{1A} = \frac{\ln(A_t/K) + \left(r + \frac{1}{2}\sigma_A^2\right)(T-t)}{\sigma_A\sqrt{T-t}}, \quad \text{and} \quad d_{2A} = d_{1A} - \sigma_A\sqrt{T-t}. \quad (\text{A.68})$$

Consider the perfect hedge portfolio, the self-financing portfolio in the (levered) stock and riskless bond, that is long in the stock, $\theta_t > 0$ for all $t \leq T$, that delivers the payoff $f(A_T)$ for some function f . Following similar steps to those for our baseline setting in the proof of Lemma 1, we obtain the dynamics of the hedge portfolio as

$$dV_t = rV_tdt + (\mu_t - r + \alpha\phi_t)\theta_tS_tdt + \sigma_t\theta_tS_t d\omega_t, \quad (\text{A.69})$$

which implies the dynamics after discounting as $d(V_t/B_t) = \sigma_t \theta_t (S_t/B_t) d\omega_t^*$, where

$$d\omega_t^* \equiv d\omega_t + \frac{\mu_t - r + \alpha \phi_t}{\sigma_t} dt, \quad (\text{A.70})$$

is a Brownian motion under the risk-neutral measure. Therefore, substituting (A.70) into the firm value dynamics (39), using the relations (A.66)–(A.67), and rearranging yields

$$dA_t = A_t \left[\left(r - \alpha \phi_t \frac{\sigma_A}{\sigma_t} \right) dt + \sigma_A d\omega_t^* \right], \quad (\text{A.71})$$

from whose drift term we infer the implied shorting fee of the firm value to be $\phi_t \sigma_A / \sigma_t$. Similarly, considering the perfect hedge portfolio that is short in the stock, $\theta_t < 0$ for all $t \leq T$, that delivers the payoff $f(A_T)$ for some function f , has the dynamics

$$dV_t = rV_t dt + (\mu_t - r + \phi_t) \theta_t S_t dt + \sigma_t \theta_t S_t d\omega_t, \quad (\text{A.72})$$

which after following similar steps as above yields the expressions as in (A.70) and (A.71) but with $\alpha = 1$. Hence, in this case too we also infer the implied shorting fee of the firm value to be $\phi_t \sigma_A / \sigma_t$. \square

Proof of Proposition 7. The corporate bond bid and ask prices are determined by following similar steps to those for the baseline setting for options. Towards that, we first note that in this setting, the costs (proceeds) of the hedge portfolio that delivers the long (short) corporate bond payoff at its maturity is given by $D_t^M(\alpha \phi_A)$ ($D_t^M(\phi_A)$), where $D_t^M(q)$ denotes the standard Merton corporate bond price adjusted for the constant total payout yield q as given by (43). This is because the hedge portfolio that delivers the long (short) corporate bond payoff at its maturity requires a long (short) stock position at all times. Hence, when the implied firm shorting fee is constant ϕ_A (see our discussion in Section 7), the relevant firm value dynamics (A.71) becomes as in the standard Merton model adjusted for the constant total payout yield $\alpha \phi_A$ for the long payoff, and ϕ_A for the short payoff, leading to these values.

Therefore, the marketmaker sets the time- t corporate bond ask price so that by selling this bond at the ask price D_t^{Ask} and forming the hedge portfolio at a cost $D_t^M(\alpha \phi_A)$ yields zero expected profit. If there is no offsetting call sell order by maturity the bond is hedged via the hedge portfolio, and hence the marketmaker's profit from this trade has a current

value of $D_t^{Ask} - D_t^M(\alpha\phi_A)$. However, if an offsetting sell order arrives at a subsequent time u , then the marketmaker buys a bond at a bid price D_u^{Bid} , and liquidates the hedge portfolio at a value of $D_u^M(\alpha\phi_A)$. In this case the bond is perfectly hedged via the offsetting order, and the marketmaker's profit from this trade has a value of $D_t^{Ask} - D_t^M(\alpha\phi_A) + V_t[D_u^M(\alpha\phi_A) - D_u^{Bid}]$. By probability weighing these profits, the marketmaker obtains its expected profit, and by setting the expected profit to zero it backs out the bond ask price as $D_t^{Ask} = D_t^M(\alpha\phi_A) - \int_t^T V_t[D_u^M(\alpha\phi_A) - D_u^{Bid}] dF_{D_s}(u)$, where F_{D_s} is the distribution function of the offsetting sell order. Following similar steps also lead to the bond bid price as $D_t^{Bid} = D_t^M(\phi_A) + \int_t^T V_t[D_u^{Ask} - D_u^M(\phi_A)] dF_{D_b}(u)$, where F_{D_b} is the distribution function of the offsetting buy order. Since these representations have similar forms to those in our baseline setting for call options, going through the same steps as in the proof of Proposition 1 yields the corporate bond bid and ask prices (41)–(42).

Property (i), which states that the marketmakers quote higher bid and lower ask prices than the respective hedge portfolio proceeds and costs, follows immediately from the weighted average forms of prices (41)–(42) with strictly positive weights (45)–(46) that lie in the interval $(0, 1)$. Properties (ii)–(iii) follow from the fact that the shorting fee ϕ_A , the partial lending α , and the offsetting sell and buy order arrival rates λ_{D_s} and λ_{D_b} all enter into corporate bond prices (41)–(42) in the same way as they enter into the call option prices (15)–(16) in our baseline setting. Therefore, following similar steps to those for the call option in the proofs of Propositions 2–4 yields these properties. \square

References

- Amihud, Y., Mendelson, H., 1980. Dealership market: Market-making with inventory. *Journal of Financial Economics* 8, 31–53.
- Avellaneda, M., Lipkin, M., 2009. A dynamic model for hard-to-borrow stocks. *Risk* 22, 92–97.
- Bakshi, G., Kapadia, N., 2003. Delta-hedged gains and the negative market volatility risk premium. *Review of Financial Studies* 16, 527–566.
- Bao, J., O’Hara, M., Zhou, X. A., 2018. The Volcker Rule and corporate bond market-making in times of stress. *Journal of Financial Economics*, Forthcoming.
- Basak, S., Chabakauri, G., 2012. Dynamic hedging in incomplete markets: A simple solution. *Review of Financial Studies* 25, 1845–1896.
- Battalio, R., Schultz, P., 2011. Regulatory uncertainty and market liquidity: The 2008 short sale ban’s impact on equity option markets. *Journal of Finance* 66, 2013–2053.
- Bergman, Y. Z., 1995. Option pricing with differential interest rates. *Review of Financial Studies* 8, 475–500.
- Bessembinder, H., Jacobsen, S. E., Maxwell, W. F., Venkataraman, K., 2017. Capital commitment and illiquidity in corporate bonds. *Journal of Finance*, Forthcoming.
- Biais, B., Hillion, P., 1994. Insider and liquidity trading in stock and options markets. *Review of Financial Studies* 7, 743–780.
- Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81, 637–654.
- Blocher, J. A., Ringgenberg, M. C., 2018. Stock options, stock loans, and the law of one price. Unpublished working paper. Vanderbilt University.
- Boehmer, E., Jones, C. M., Zhang, X., 2013. Shackling short sellers: The 2008 shorting ban. *Review of Financial Studies* 26, 1363–1400.
- Bollen, N. P. B., Smith, T., Whaley, R. E., 2004. Modeling the bid/ask spread: Measuring the inventory-holding premium. *Journal of Financial Economics* 72, 97–141.
- Boyle, P. P., Vorst, T., 1992. Option replication in discrete time with transaction costs. *Journal of Finance* 47, 271–293.
- Broadie, M., Cvitanić, J., Soner, H. M., 1998. Optimal replication of contingent claims under portfolio constraints. *Review of Financial Studies* 11, 59–79.

- Broadie, M., Detemple, J. B., Ghysels, E., Torr s, O., 2000. American options with stochastic dividends and volatility: A nonparametric investigation. *Journal of Econometrics* 94, 53–92.
- Cho, Y.-H., Engle, R. F., 1999. Modeling the impacts of market activity on bid-ask spreads in the option market. NBER working paper no. w7331.
- D’Avolio, G., 2002. The market for borrowing stock. *Journal of Financial Economics* 66, 271–306.
- Dick-Nielsen, J., Rossi, M., 2018. The cost of immediacy for corporate bonds. Unpublished working paper. Copenhagen Business School and Texas A&M University.
- Diether, K. B., Lee, K.-H., Werner, I. M., 2009. Short-sale strategies and return predictability. *Review of Financial Studies* 22, 575–607.
- Drechsler, I., Drechsler, Q. F., 2016. The shorting premium and asset pricing anomalies. NBER working paper no. w20282.
- Duffie, D., 2001. *Dynamic Asset Pricing Theory*. Princeton University Press, third ed.
- Duffie, D., 2012. Market making under the proposed Volcker Rule. Unpublished working paper. Rock Center for Corporate Governance at Stanford University.
- Duffie, D., G rleanu, N., Pedersen, L. H., 2005. Over-the-counter markets. *Econometrica* 73, 1815–1847.
- Duffie, D., G rleanu, N., Pedersen, L. H., 2007. Valuation in over-the-counter markets. *Review of Financial Studies* 20, 1865–1900.
- Duffie, D., Pan, J., Singleton, K., 2000. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* 68, 1343–1376.
- Easley, D., O’Hara, M., 1987. Price, trade size, and information in securities markets. *Journal of Financial Economics* 19, 69–90.
- Easley, D., O’Hara, M., Srinivas, P. S., 1998. Option volume and stock prices: Evidence on where informed traders trade. *Journal of Finance* 53, 431–465.
- Edirisinghe, C., Naik, V., Uppal, R., 1993. Optimal replication of options with transactions costs and trading restrictions. *Journal of Financial and Quantitative Analysis* 28, 117–138.
- Evans, R. B., Geczy, C. C., Musto, D. K., Reed, A. V., 2007. Failure is an option: Impediments to short selling and options prices. *Review of Financial Studies* 22, 1955–1980.
- G rleanu, N., Pedersen, L. H., 2011. Margin-based asset pricing and deviations from the law

- of one price. *Review of Financial Studies* 24, 1980–2022.
- Garman, M. B., 1976. Market microstructure. *Journal of Financial Economics* 3, 257–275.
- Geske, R., 1978. The pricing of options with stochastic dividend yield. *Journal of Finance* 33, 617–625.
- Glosten, L. R., Milgrom, P. R., 1985. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics* 14, 71–100.
- Grundy, B. D., Lim, B., Verwijmeren, P., 2012. Do option markets undo restrictions on short sales? Evidence from the 2008 short-sale ban. *Journal of Financial Economics* 106, 331–348.
- Hanson, S. G., Sunderam, A., 2014. The growth and limits of arbitrage: Evidence from short interest. *Review of Financial Studies* 27, 1238–1286.
- Harris, L. E., Namvar, E., Phillips, B., 2013. Price inflation and wealth transfer during the 2008 SEC short-sale ban. *Journal of Investment Management* 11, 1–58.
- Heston, S. L., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6, 327–343.
- Ho, T., Stoll, H. R., 1981. Optimal dealer pricing under transactions and return uncertainty. *Journal of Financial Economics* 9, 47–73.
- Hu, J., 2014. Does option trading convey stock price information? *Journal of Financial Economics* 111, 625–645.
- Jameson, M., Wilhelm, W., 1992. Market making in the options markets and the costs of discrete hedge rebalancing. *Journal of Finance* 47, 765–779.
- Jensen, M. V., Pedersen, L. H., 2016. Early option exercise: Never say never. *Journal of Financial Economics* 121, 278–299.
- Johnson, T. L., So, E. C., 2012. The option to stock volume ratio and future returns. *Journal of Financial Economics* 106, 262–286.
- Kecskes, A., Mansi, S. A., Zhang, A., 2013. Are short sellers informed? Evidence from the bond market. *Accounting Review* 88, 611–639.
- Kolasinski, A. C., Reed, A., Thornock, J. R., 2013. Can short restrictions actually increase informed short selling? *Financial Management* 42, 155–181.
- Lakonishok, J., Lee, I., Pearson, N. D., Poteshman, A. M., 2007. Option market activity. *Review of Financial Studies* 20, 813–857.

- Lamont, O. A., Thaler, R. H., 2003. Can the market add and subtract? Mispricing in tech stock carve-outs. *Journal of Political Economy* 111, 227–268.
- Leippold, M., Su, L., 2015. Collateral smile. *Journal of Banking & Finance* 58, 15–28.
- Leland, H. E., 1985. Option pricing and replication with transactions costs. *Journal of Finance* 40, 1283–1301.
- Lin, T.-C., Lu, X., 2016. How do short-sale costs affect put options trading? Evidence from separating hedging and speculative shorting demands. *Review of Finance* 20, 1911–1943.
- Lou, W., 2015. Funding in option pricing: The Black-Scholes framework extended. *Risk* 28, 56–61.
- McDonald, R. L., 2012. *Derivatives Markets*. Pearson, third ed.
- Merton, R. C., 1973. Theory of rational option pricing. *Bell Journal of Economics and Management Science* 4, 141–183.
- Merton, R. C., 1974. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance* 29, 449–470.
- Muravyev, D., 2016. Order flow and expected option returns. *Journal of Finance* 71, 673–708.
- Ofek, E., Richardson, M., Whitelaw, R. F., 2004. Limited arbitrage and short sales restrictions: Evidence from the options markets. *Journal of Financial Economics* 74, 305–342.
- Piterbarg, V., 2010. Funding beyond discounting: Collateral agreements and derivatives pricing. *Risk* 23, 97–102.
- Reed, A. V., 2013. Short selling. *Annual Review of Financial Economics* 5, 245–258.
- Saffi, P. A. C., Sigurdsson, K., 2010. Price efficiency and short selling. *Review of Financial Studies* 24, 821–852.
- Scholes, M., 1976. Taxes and the pricing of options. *Journal of Finance* 31, 319–332.
- Stoll, H. R., 1978. The supply of dealer services in securities markets. *Journal of Finance* 33, 1133–1151.
- Trebbi, F., Xiao, K., 2017. Regulation and market liquidity. *Management Science*, Forthcoming.

Name	Symbol	Value
Shorting fee	ϕ	{0.02%, 6.96%}
Partial lending	α	{4.31%, 20.95%}
Offsetting order arrival rates	$\lambda_{Cs}, \lambda_{Cb}, \lambda_{Cb}, \lambda_{Pb}$	2.77
Stock price	S_t	32.20
Stock return volatility	σ	40.00%
Riskless interest rate	r	1.80%
Option moneyness	K/S_t	{0.90, 1.00, 1.10}
Option time to maturity	$T - t$	0.25

Table 1: Parameter values.

This table reports the parameter values used in our quantitative analysis. The determination of these values is presented in the text.

Panel A: Call option								
Shorting fee decile	Option moneyness	BS price	Bid price	Ask price	Relative change	Bid-Ask spread	Relative spread	Implied volatility
	$\frac{K}{S_t}$	C_t^{BS}	C_t^{Bid}	C_t^{Ask}	$\frac{C_t^{Mid} - C_t^{BS}}{C_t^{BS}}$	$C_t^{Ask} - C_t^{Bid}$	$\frac{C_t^{Ask} - C_t^{Bid}}{C_t^{Mid}}$	$\tilde{\sigma}_{t, C^{Mid}}$
D1	1.10	1.43	1.43	1.43	-0.02%	0.00	0.01%	39.99%
D10		1.43	1.29	1.33	-8.19%	0.04	2.86%	38.05%
D1	1.00	2.63	2.63	2.63	-0.02%	0.00	0.01%	39.99%
D10		2.63	2.42	2.48	-6.82%	0.06	2.36%	37.18%
D1	0.90	4.46	4.46	4.46	-0.01%	0.00	0.01%	39.99%
D10		4.46	4.18	4.26	-5.50%	0.08	1.88%	35.19%

Panel B: Put option								
	$\frac{K}{S_t}$	P_t^{BS}	P_t^{Bid}	P_t^{Ask}	$\frac{P_t^{Mid} - P_t^{BS}}{P_t^{BS}}$	$P_t^{Ask} - P_t^{Bid}$	$\frac{P_t^{Ask} - P_t^{Bid}}{P_t^{Mid}}$	$\tilde{\sigma}_{t, P^{Mid}}$
D1	1.10	4.49	4.49	4.49	0.01%	0.00	0.01%	40.01%
D10		4.49	4.67	4.74	4.89%	0.07	1.53%	43.61%
D1	1.00	2.49	2.49	2.49	0.02%	0.00	0.01%	40.01%
D10		2.49	2.62	2.67	6.30%	0.05	1.96%	42.46%
D1	0.90	1.11	1.11	1.11	0.02%	0.00	0.01%	40.00%
D10		1.11	1.19	1.22	8.17%	0.03	2.51%	41.74%

Table 2: Quantitative effects of costly short-selling on option prices.

This table reports the effects of costly short-selling for a call option (Panel A) and a put option (Panel B) on a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels. C_t^{Mid} and P_t^{Mid} denote the mid-point prices of the call and put, e.g., $C_t^{Mid} = 0.5(C_t^{Ask} + C_t^{Bid})$ and $P_t^{Mid} = 0.5(P_t^{Ask} + P_t^{Bid})$. Implied volatilities in the last columns are obtained by employing the standard approach of inverting the Black-Scholes formula using the mid-point option prices as inputs. All parameter values are as in Table 1.

Option maturity		Call		Put		Implied stock			
$T - t$		Bid	Ask	Bid	Ask	Bid	Ask		
		C_t^{Bid}	C_t^{Ask}	P_t^{Bid}	P_t^{Ask}	\tilde{S}_t^{Bid}	\tilde{S}_t^{Ask}		
Evidence		5.75	7.25	10.63	12.63	47.55	-13.93%	51.05	-7.60%
Our Model	May	8.02	8.60	9.73	10.20	52.51	-4.96%	53.56	-3.06%
Black-Scholes		9.60	9.60	9.01	9.01	55.25	0.00%	55.25	0.00%
Evidence		9.25	10.75	17.25	19.25	43.57	-21.14%	47.07	-14.81%
Our Model	Aug	11.54	12.18	15.54	15.98	49.36	-10.66%	50.43	-8.72%
Black-Scholes		15.15	15.15	13.70	13.70	55.25	0.00%	55.25	0.00%
Evidence		10.00	11.50	21.63	23.63	39.12	-29.19%	42.62	-22.86%
Our Model	Nov	13.52	13.95	19.56	19.84	46.62	-15.62%	47.32	-14.35%
Black-Scholes		19.06	19.06	16.75	16.75	55.25	0.00%	55.25	0.00%

Table 3: Quantitative effects of costly short-selling on Palm options.

This table reports the effects of extreme short-selling on the Palm options and implied stock prices on March 17, 2000 for three different option maturity dates, May 20, August 19 and November 18, 2000. The values in the Evidence rows are from Lamont and Thaler (2003) (Table 6, p. 256). The parameter values used in our model and the Black-Scholes model are as discussed in text: $\phi = 35\%$, $\alpha = 41.18\%$, $r = 6.21\%$ (May), $r = 6.41\%$ (Aug, Nov), $S_t = K = 55.25$, $\sigma = 104.6\%$, and $\lambda_{Cs} = \lambda_{Cb} = \lambda_{Cb} = \lambda_{Pb} = 1.66$.

Panel A: Call option ($\rho = -0.70$)								
Shorting fee decile	Option moneyness	Heston price	Bid price	Ask price	Relative change	Bid-Ask spread	Relative spread	Implied volatility
	$\frac{K}{S_t}$	C_t^H	C_t^{Bid}	C_t^{Ask}	$\frac{C_t^{Mid} - C_t^H}{C_t^H}$	$C_t^{Ask} - C_t^{Bid}$	$\frac{C_t^{Ask} - C_t^{Bid}}{C_t^{Mid}}$	$\tilde{\sigma}_{t, C^{Mid}}$
D1	1.10	0.99	0.99	0.99	-0.03%	0.00	0.01%	32.65%
D10		0.99	0.86	0.90	-11.23%	0.04	4.03%	30.69%
D1	1.00	2.27	2.27	2.27	-0.02%	0.00	0.01%	34.38%
D10		2.27	2.05	2.12	-8.33%	0.06	2.92%	31.42%
D1	0.90	4.27	4.27	4.27	-0.02%	0.00	0.01%	36.27%
D10		4.27	3.97	4.05	-6.08%	0.08	2.10%	30.97%

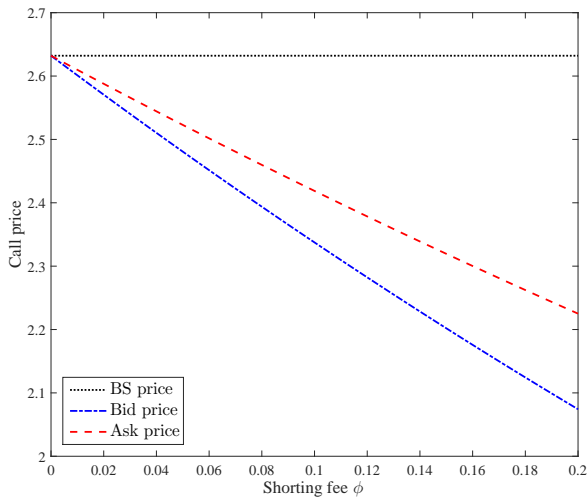
Panel B: Call option ($\rho = -0.35$)								
	$\frac{K}{S_t}$	C_t^H	C_t^{Bid}	C_t^{Ask}	$\frac{C_t^{Mid} - C_t^H}{C_t^H}$	$C_t^{Ask} - C_t^{Bid}$	$\frac{C_t^{Ask} - C_t^{Bid}}{C_t^{Mid}}$	$\tilde{\sigma}_{t, C^{Mid}}$
D1	1.10	1.06	1.06	1.06	-0.03%	0.00	0.01%	33.73%
D10		1.06	0.93	0.97	-10.23%	0.03	3.63%	31.85%
D1	1.00	2.28	2.28	2.28	-0.02%	0.00	0.01%	34.51%
D10		2.28	2.07	2.13	-8.04%	0.06	2.81%	31.64%
D1	0.90	4.24	4.24	4.24	-0.02%	0.00	0.01%	35.55%
D10		4.24	3.94	4.02	-6.07%	0.08	2.09%	30.26%

Table 4: Quantitative effects of costly short-selling on call prices in the stochastic volatility economy. This table reports the effects of costly short-selling on call option prices in the stochastic volatility economy for a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels when the skewness is $\rho = -0.70$ (Panel A) and $\rho = -0.35$ (Panel B). C_t^{Mid} denote the mid-point call prices, $C_t^{Mid} = 0.5(C_t^{Ask} + C_t^{Bid})$. Implied volatilities in the last columns are obtained by employing the standard approach of inverting the Black-Scholes formula using the mid-point option prices as inputs. The values of the parameters that are common to the baseline setting are as in Table 1, the values of the additional parameters of the stochastic volatility economy are based on Duffie, Pan, and Singleton (2000) as discussed in text: $\kappa_\nu^* = 6.21$, $\sigma_\nu = 0.61$, $\bar{\nu}^* = 0.16$, $\nu_t = 0.0859$.

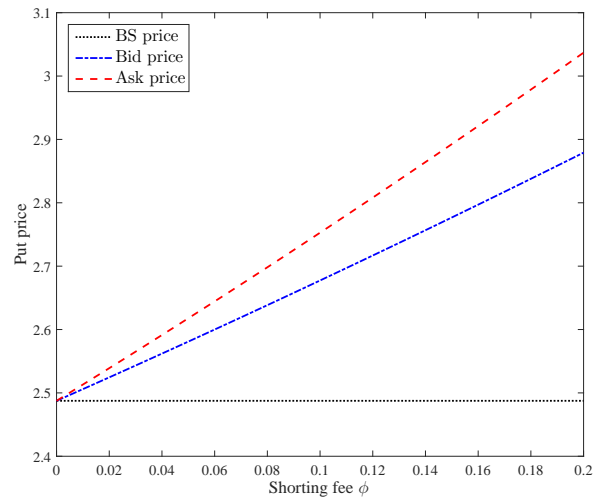
Panel A: Put option ($\rho = -0.70$)								
Shorting fee decile	Option moneyness	Heston price	Bid price	Ask price	Relative change	Bid-Ask spread	Relative spread	Implied volatility
	$\frac{K}{S_t}$	P_t^H	P_t^{Bid}	P_t^{Ask}	$\frac{P_t^{Mid} - P_t^H}{P_t^H}$	$P_t^{Ask} - P_t^{Bid}$	$\frac{P_t^{Ask} - P_t^{Bid}}{P_t^{Mid}}$	$\tilde{\sigma}_{t, PMid}$
D1	1.10	4.06	4.06	4.06	0.01%	0.00	0.01%	32.66%
D10		4.06	4.24	4.32	5.54%	0.07	1.73%	36.51%
D1	1.00	2.13	2.13	2.13	0.02%	0.00	0.01%	34.39%
D10		2.13	2.25	2.30	6.90%	0.05	2.14%	36.69%
D1	0.90	0.92	0.92	0.92	0.02%	0.00	0.01%	36.29%
D10		0.92	0.99	1.01	8.28%	0.03	2.54%	37.79%

Panel B: Put option ($\rho = -0.35$)								
	$\frac{K}{S_t}$	P_t^H	P_t^{Bid}	P_t^{Ask}	$\frac{P_t^{Mid} - P_t^H}{P_t^H}$	$P_t^{Ask} - P_t^{Bid}$	$\frac{P_t^{Ask} - P_t^{Bid}}{P_t^{Mid}}$	$\tilde{\sigma}_{t, PMid}$
D1	1.10	4.12	4.12	4.12	0.01%	0.00	0.01%	33.74%
D10		4.12	4.31	4.38	5.54%	0.08	1.73%	37.61%
D1	1.00	2.14	2.14	2.14	0.02%	0.00	0.01%	34.52%
D10		2.14	2.27	2.32	7.15%	0.05	2.21%	36.91%
D1	0.90	0.89	0.89	0.89	0.02%	0.00	0.01%	35.57%
D10		0.89	0.95	0.98	8.95%	0.03	2.74%	37.14%

Table 5: Quantitative effects of costly short-selling on put prices in the stochastic volatility economy. This table reports the effects of costly short-selling on put option prices in the stochastic volatility economy for a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels when the skewness is $\rho = -0.70$ (Panel A) and $\rho = -0.35$ (Panel B). P_t^{Mid} denote the mid-point put prices, $P_t^{Mid} = 0.5(P_t^{Ask} + P_t^{Bid})$. Implied volatilities in the last columns are obtained by employing the standard approach of inverting the Black-Scholes formula using the mid-point option prices as inputs. All parameter values are as in Table 4.



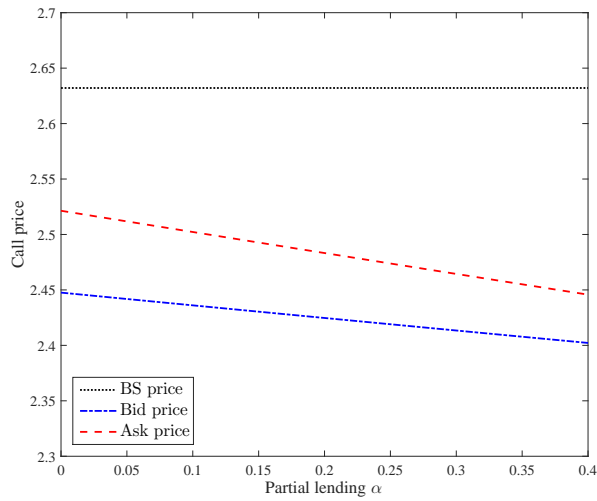
Panel A. Call price



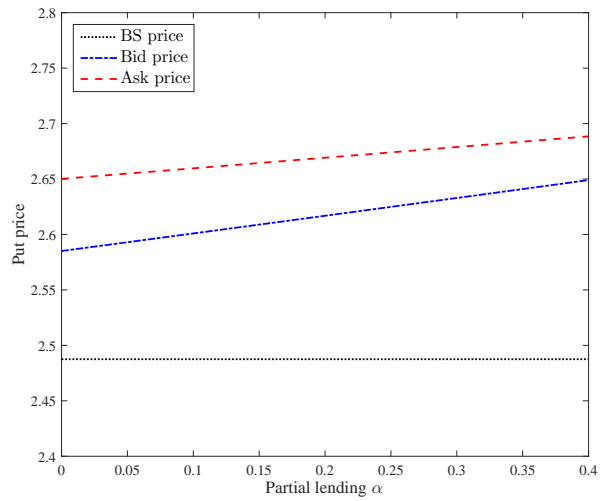
Panel B. Put price

Figure 1: Effects of shorting fee.

These panels plot the (at-the-money) option bid and ask prices for varying levels of the shorting fee ϕ . The call option prices are in Panel A and put option prices in Panel B. The horizontal dotted black lines correspond to the benchmark Black-Scholes prices. All other parameter values are as in Table 1 of Section 5 with $\alpha = 20.95\%$.



Panel A. Call price



Panel B. Put price

Figure 2: Effects of partial lending.

These panels plot the (at-the-money) option bid and ask prices for varying levels of the partial lending α . The call option prices are in Panel A and put option prices in Panel B. The horizontal dotted black lines correspond to the benchmark Black-Scholes prices. All other parameter values are as in Table 1 of Section 5 with $\phi = 6.96\%$.